Coupling analysis of different variables in airfoil lift formula

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Abstract. In the design process of an aircraft, there is an important criterion to consider when designing a wing: how much lift the wing can provide. The magnitude of the lift directly determines the stability of the aircraft during take-off and whether it can take off smoothly, so it has received extensive attention from scholars at home and abroad. This paper uses the Kutta-Joukowski theorem to quantify the factors that affect wing lift and enable the calculation of lift, and engineers need to analyze how these factors affect wing lift when considering this criterion. So how can engineers analyze the effect of these factors on lift, especially in multivariate situations? Need to calculate complex multivariate equations? are issues worth exploring. In this paper, a coupled analysis method is proposed to analyze the wing lift of an aircraft affected by multiple variables, which will greatly simplify the original analysis process and provide convenience for the calculation of wing lift.

Keywords: wing, Kutta-Joukowski theorem, multivariate situations, coupled analysis method.

1. Introduction

The wing is the most important part of an airplane. It provides most of the lift for takeoff. During the slow advance of the wing, the air flow above and below the wing meets at the trailing edge, and the air velocity at the top surface faster than the velocity of the air below [1]. The difference in velocity causes the pressure on the upper and lower surfaces to be different, the airflow below is slower than the airflow above, the pressure on the surface under the wing [2-4]. The force is higher than the upper surface, creating upward pressure. This upward pressure is the lift of an airplane wing. Therefore, in engineering, the core theme of wing design is how do you maximize lift from the wing.

To increase lift on a wing, it is necessary to understand what affects lift on an aircraft wing. According to the Kutta–Joukowski theorem

$$L = -\rho U \tau \tag{1}$$

This theorem can be used to calculate the lift of an airfoil or a two-dimensional object (such as a cylinder) in a homogeneous fluid. In general, the differential form of the theorem is more used

$$F_L = C_L \times \frac{1}{2}\rho U^2 \times A_S \tag{2}$$

The above formula is also known as the lift formula. The lift formula shows that velocity, density, wing area and lift coefficient all affect the lift. In previous studies, by estimating the relationship between speed, wing area and lift force, it was found that the lift force of a wing is proportional to the square of flight speed and wing area, and also changes with the angle of attack of the wing [5]. Therefore, angle of attack is also an important factor affecting wing lift. But why the angle of attack is not reflected in the lift formula will be explained in the following theory. As can be seen from the above instructions, the lift formula clearly reflects each factor that affects the lift of an airfoil [6]. This also provides a key idea for wing design - changing the lift force on the wing by changing one of the variables of the lift formula. For example, the pressure difference between the upper and lower surfaces of the wing can be adjusted appropriately by some methods. The usual method is to change the airfoil shape, angle of attack, chord size and so on. The influence of a single factor on wing lift can be found through control variables. For example, the greater the speed, the greater the lift of the wing [7]. The greater the angle of attack, the greater the lift of the aircraft, without exceeding the critical angle. There have also been studies of this idea- as the angle of attack increases, the lift of the wing rises almost linearly. However, in the real situation, it cannot make other variables constant. In engineering design, many variables change at the same time. In this case, the lift formula becomes a multivariable equation. Calculating the value of multivariable equation is often a complicated process. Therefore, this paper proposes a method of coupling analysis to avoid solving multivariable equations and analyze the influence of two variables on the lift of aircraft wings [8].

Coupling indicates the degree of association between two subsystems. It is a measure of the interaction between multiple variables [8]. For a multivariate equation, coupling can find the connection between each independent variable. Coupling is widely used in engineering and mechanics.

There is an article have explored the unidirectional fluid-structure coupling of windsurfers under different angles of attack. It shows that when the angle of attack is in the range of 20°-25°, the lift coefficient increases gradually with the increase of the angle of attack, and the lift coefficient reaches the maximum value when the angle of attack is 25°. When the angle of attack is in the range of 25°-90°, the lift coefficient of sail decreases with the increase of the angle of attack [3]. Compared with the sailboard of a boat, the wing has the same property. There is an article took NACA0012 airfoil as the research object to study the aerodynamic characteristics, flow field characteristics and fluid-structure coupling mechanism of elastic airfoil with fixed ends. It found that in the range of angle of attack from 0 ° to 8°, the lift coefficient increases with the increase of angle of attack, and reaches the peak of this interval at 8°. After the angle of attack of 8°, the lift curve drops sharply, because the wing stall phenomenon occurs here, and the influence range of stall lasts until the angle of attack of 14° [7]. However, most of the articles studied the relationship between angle of attack and lift force or lift coefficient by coupling analysis. This is essentially a study of a single variable that affects the lift of an aircraft wing. It is not proposed how to use coupling analysis to solve the lift of the wing under the influence of multiple variables. There is a paper put forward that different flapping wings can be used to change lift force at different speeds [9]. But it is a pity that this paper only presents the idea without conducting experiments or simulations.

In conclusion, most of the previous papers studied the influence of angle of attack on wing lift. However, it did not consider other variables in the study of angle of attack. Therefore, it is necessary to propose a coupling method to analyze the influence of multiple variables on wing lift. In this paper, the influence of Angle of attack and velocity on lift is analyzed by coupling method. Angle of attack and speed are chosen because these two variables can be controlled manually during the flight of the aircraft. The pilot can change these two variables at any time so that the wing provides maximum lift.

2. Proof of Principle

This section will prove a core theorem used in coupling analysis-Kutta-Joukowski theorem. It is also named lift formula. The formula will be proved in both physical and mathematical view. Finally, it will show how to use this formula in following coupling analysis [10].

2.1. The physical point of view of the intuitive derivation of the formula

This section will show how to derive the Kutta–Joukowski theorem visually from a physical point of view. Consider a wing with chord c that has an angle of attack with the airflow such that the flow velocity is U on one side of the wing and (U+v) on the other side.

Therefore, its circulation is

$$\tau = Uc - (U + v)c = -vc \tag{3}$$

The pressure difference between the wings can be found by Bernoulli's equation:

$$\frac{\rho}{2}U^2 + (P + \Delta P) = \frac{\rho}{2}(U + v)^2 + P \tag{4}$$

Expand the above formula further

$$\frac{\rho}{2}U^2 + \Delta P = \frac{\rho}{2}(U^2 + 2Uv + v^2) \tag{5}$$

Then, simplify the above formula:

$$\Delta P = \rho U v \tag{6}$$

Therefore, the lift of the wing is

$$F_L = c\Delta P = \rho U v c = -\rho U \tau \tag{7}$$

In a word, that is the physical intuition of the proof of the Kutta–Joukowski theorem. Obviously, this proof is based on physical intuition and lacks mathematical rigor. Therefore, the next part is going to prove it mathematically

2.2. Two mathematical methods

This section will use mathematical methods to prove the lift formula. Most of the mathematical proofs given in books on aerodynamics are integrals. The following two mathematical proofs will be different. The complex potential method is used and showing how to derive the Kutta–Joukowski theorem as the lift formula.

2.1.1 The complex potential method

Complex analysis plays an important role in fluid mechanics. The introduction of complex numbers simplifies many complicated calculations in fluid mechanics. Complex analysis can also be used in the process of proving the Kutta–Joukowski theorem. There is a book named an introduction to theoretical and computational aerodynamics has referred this kind of proof.

First of all, the complex force, complex potential and complex velocity of the flow field are obtained by complex analysis.

The complex potential:

$$W = Uz + \frac{i\tau}{2\pi} lnz \tag{8}$$

Therefore, the complex velocity:

$$\frac{dW}{dz} = U + \frac{i\tau}{2\pi z} \tag{9}$$

$$\frac{dW}{dz} = U + \frac{i\tau}{2\pi z}$$

Hence, the complex velocity squared can be calculated:

$$(\frac{dW}{dz})^2 = U^2[1 + \frac{i\tau}{\pi U z} - \frac{\tau^2}{4(\pi U z)^2}] \tag{10}$$

The complex number in polar coordinates is denoted by $z = e^{i\theta}$

Hence, the complex force is

$$X - iY = \int_0^{2\pi} \frac{i\rho}{2}\left(\frac{dW}{dz}\right)^2 dz = \frac{i\rho U^2}{2}\int_0^{2\pi}\left[1 + \frac{i\tau}{\pi U}e^{-i\theta} - \frac{\tau^2}{4(\pi U)^2}e^{-2i\theta}\right] ie^{-i\theta} d\theta = -\frac{\rho U^2}{2}\left[-ie^{-i\theta} + \frac{i\tau\theta}{\pi U} - \frac{\tau^2}{4(\pi U)^2}\right]_0^{2\pi} = -i\rho U\tau \tag{11}$$

Thus, $X = 0$ and in complex force, Y is the lift force. As a result, $F_L = Y = -\rho U\tau$

In a word, that is the proof using the complex potential method. Complex potentials are simpler and easier to understand than integrals. However, the results can not reflect the influence of different factors on lift. Therefore, the following will introduce another proof method: the differential method.

2.1.2 The differential method

The complex potential method above shows the procedure for proving the Kutta–Joukowski theorem. Then, it will be shown to be derived into the following lift formula

$$F_L = C_L \times \frac{1}{2}\rho U^2 \times A_S \tag{12}$$

The proof starts with an experiment. The experiment is shown in the figure (figure 1) below. The experimental setup measures the pressure above and below the wing. Through the experiment, it is found that the pressure on the upper and lower surfaces of the wing is different, that is, the upper surface produces negative pressure, the lower surface produces positive pressure, and the comprehensive calculation is the vertical upward lift force.

According to the above experiment, writing down the Bernoulli's equation for the wing and infinity:

$$P+\frac{1}{2}\rho_\infty U^2=P_\infty+\frac{1}{2}\rho_\infty U_\infty^2 \tag{13}$$

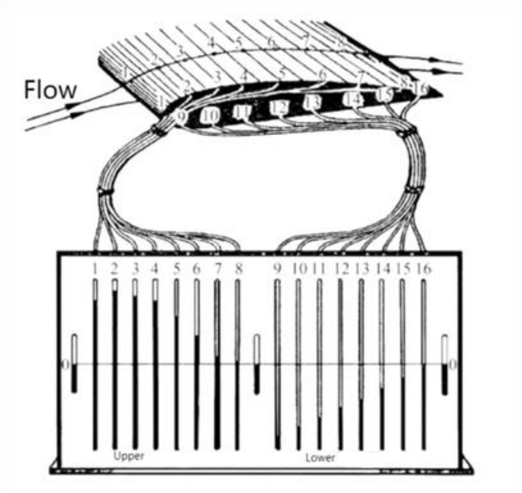

Fig. 1 Experimental device for measuring pressure difference between upper and lower wings

Solving the Bernoulli's equation and getting following result:

$$\Delta P=P-P_\infty=\frac{1}{2}\rho_\infty(U_\infty^2-U^2) \tag{14}$$

After that, the pressure coefficient is obtained by simplifying the result:

$$C_P=\frac{\Delta P}{\frac{1}{2}\rho_\infty U_\infty^2} \tag{15}$$

Then, divide the wing into smaller segments. Select one of the segments and calculate the pressure above and below it

$$\Delta P_{upper}=C_{P-upper}\frac{1}{2}\rho_\infty U_\infty^2 \tag{16}$$

$$\Delta P_{lower}=C_{P-lower}\frac{1}{2}\rho_\infty U_\infty^2 \tag{17}$$

Lift is the pressure difference between the upper and lower sides. Hence,

$$dF_L=\Delta P_{lower}-\Delta P_{upper}=\left(C_{P-lower}-C_{P-upper}\right)\frac{1}{2}\rho_\infty U_\infty^2\cos\alpha\,dx \tag{18}$$

Where α is the angle of attack. The wing has a wing span of b
Integrate both sides:

$$F_L=\frac{1}{2}\rho_\infty U_\infty^2\int_0^b\left(C_{P-lower}-C_{P-upper}\right)\cos\alpha\,dx \tag{19}$$

Assume that

$$C_L=\int_0^1\left(C_{P-lower}-C_{P-upper}\right)\cos\alpha\,dx \tag{20}$$

Therefore, the lift formula is proved:

$$F_L=C_L\times\frac{1}{2}\rho U^2\times b\times 1 \tag{21}$$

Consider that the wing span is the area of the wing. Hence,

$$F_L = C_L \times \frac{1}{2}\rho U^2 \times A_S \tag{22}$$

The above proof shows that the lift coefficient C_L is actually a function of the angle of attack α. It also explains the point made in the research purpose: when the angle of attack of the wing is larger, the lift coefficient is also larger. Therefore, the lift coefficient will directly replace the angle of attack of the wing in the coupling analysis later in this paper. In this way, the influence of angle of attack and velocity on lift force can be directly reflected.

3. Coupling analysis

In this part, the data will be used for simulation experiments. The formula derived in the previous section clearly shows the relationship between angle of attack, velocity and the magnitude of lift, which is the lift formula. According to the definition of coupling, angle of attack and velocity can be regarded as two subsystems that affect the magnitude of lift. Then you can use coupling analysis to figure out the relationship between these two subsystems and how they affect lift.

3.1. The relationship between the selected variables and lift

Before studying the relationship between the two subsystems, velocity and angle of attack, and their combined effect on lift, it is necessary to find out what effect each of them has on lift. This is easy to get with control variables.

3.1.1 The relationship between velocity and lift

The lift on the wing versus the relative speed of the aircraft against the air. The greater the relative velocity, the greater the lift. According to the formula for lift that has been derived

$$F_L = C_L \times \frac{1}{2}\rho U^2 \times A_S \tag{23}$$

The lift formula shows that when the density, wing area and lift coefficient are fixed, the lift is proportional to the square of the velocity. This shows that the relationship between lift and velocity is essentially a quadratic function of one variable. There is a sort of linear relationship between them.

3.1.2 The relationship between lift coefficient (angle of attack) and lift

Many studies have been done on the effect of angle of attack on lift. Here is an image (figure 2) from another article [4].

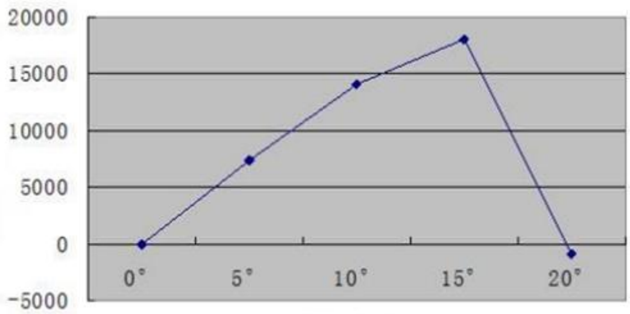

Fig. 2 Relationship between angle of attack and wing lift

The first half of the image clearly shows that when the angle of attack increases, the lift of the wing also increases. However, when the angle of attack is greater than the critical angle, the lift of the wing decreases rapidly, a phenomenon known as stalling. Focusing only on the first half of the image, when the angle of attack is not greater than the critical angle, the angle of attack and lift are also linear. The result is also the same as a classical image-Plot (figure 3) of lift coefficient and angle of attack.

The image also shows that the higher the angle of attack, the higher the lift coefficient, and the higher the lift. When the angle of attack is greater than the critical angle into the stall state, the lift coefficient decreases, lift also decreases.

Therefore, the effect of angle of attack on lift force can be transformed into the effect of lift coefficient on lift. The proof of the lift formula can also show that the lift coefficient is essentially a function of angle of attack

$$C_L = \int_0^1 \left(C_{P-lower} - C_{P-upper}\right) \cos\alpha \, dx \tag{24}$$

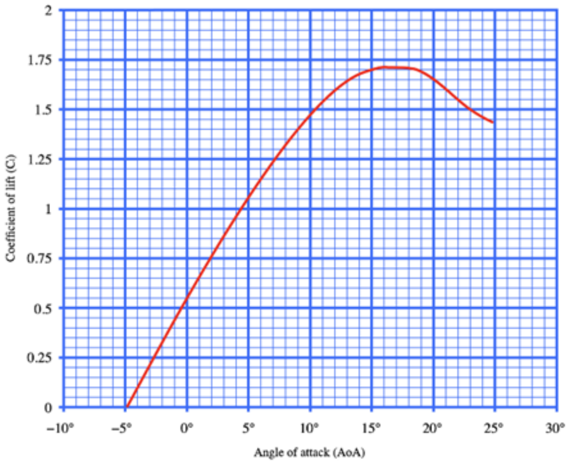

Fig. 3 Relationship between angle of attack and lift coefficient

3.2. Simulation model and data

The effect of each subsystem on lift has been explained. As explained above, the lift coefficient will be used instead of the angle of attack in the following simulation. The simulation model is the lift formula already derived above. The required data is listed below

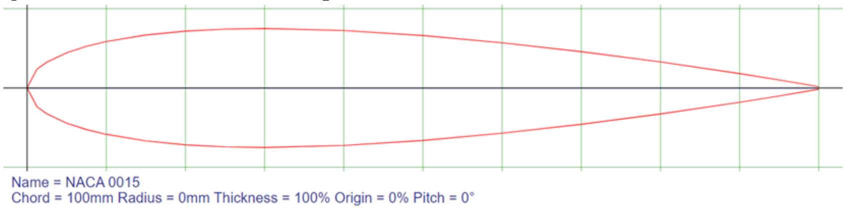

Fig. 4 NACA0015 airfoil

Take the NACA0015 airfoil (figure 4) as an example

A is the wing area fixed constant, is 124.5 square meters (the area of this airfoil on A Boeing 737) The air density is 1.29 kg per cubic meter. The following data (figure 5) are velocity (m/s), Angle of attack (α) and lift coefficient at the same Reynolds number (velocity in column 1, Angle of attack in column 2, lift coefficient in column 3.

0	0	0
1.67	1	0.11
3.33	2	0.22
5.00	3	0.33
6.67	4	0.44
8.33	5	0.55
10.00	6	0.66
11.67	7	0.7483
13.33	8	0.8442
15.00	9	0.926
16.67	10	0.9937
18.33	11	1.0363
20.00	12	1.0508
21.67	13	1.0302
23.33	14	0.9801
25.00	15	0.9119
26.67	16	0.8401
28.33	17	0.7722
30.00	18	0.7305
31.67	19	0.7041
33.33	20	0.699
35.00	21	0.7097
36.67	22	0.7298
38.33	23	0.7593
40.00	24	0.7961
41.67	25	0.8353
43.33	26	0.8838
45.00	27	0.9473
46.67	30	0.855
48.33	35	0.98
50.00	40	1.035
55.56	45	1.05
75.00	50	1.02

Fig. 5 Data system

Finally, Matlab will be used for simulation.

3.3. Coupling analysis of lift, velocity and lift coefficient (angle of attack)

By using Matlab for simulation experiments and coupling analysis of the obtained data, the following image (figure 6) can be obtained:

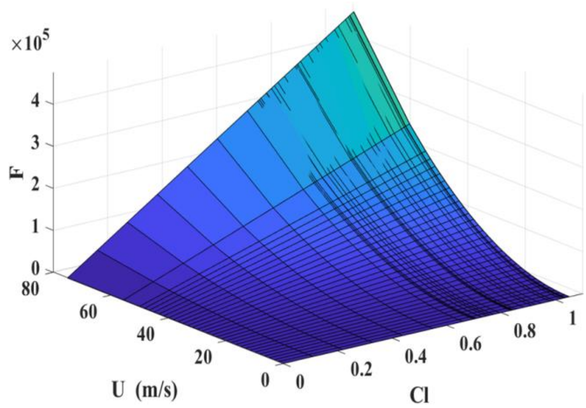

Fig. 6 Result of coupling analysis

4. Conclusion

According to the definition of coupling, velocity and lift coefficient (angle of attack) can be regarded as two subsystems of lift. The following conclusions can be obtained from coupling analysis

Firstly, the two subsystems, velocity and angle of attack, interact with each other. Through the analysis of the two-dimensional image obtained in the previous part, it is found that when the lift force keeps constant, the angle of attack should be reduced if the velocity is increased, and the lift coefficient also decreases. Therefore, there is a negative effect between the two subsystems, velocity and angle of attack.

Secondly, coupling analysis can be used to find out the influence of the simultaneous change of the two subsystems, velocity and angle of attack, on the magnitude of lift. By analyzing the image, it is found that the influence of velocity and lift coefficient (angle of attack) on lift can be drawn as a two-dimensional surface. Therefore, the optimal solution of coupling analysis is the vertex of this surface, which is the maximum value of the function representing this surface. By analyzing the image, the maximum value of this surface function is obtained when the speed is 80 m/s and the lift coefficient is 1.02 (angle of attack is 50°). Hence, at a speed of 80 m/s and a lift coefficient of 1.02 (angle of attack 50°), the wing can provide the maximum lift.

Thirdly, the coupling analysis can find the optimal solution when multiple variables affect the lift. Therefore, the optimal solution can be used in the design process to design the wing that can provide the maximum lift for the aircraft. This can save energy and improve the efficiency of the plane.

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