Fourier series and its property

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Abstract. Fourier analysis appeared to solve some partial differential equations at first. In 1747, D' Alembert solved the vibrating string equation using the method of traveling waves. In 1753, D. Bernoulli proposed the solution which for all intents and purposes is the Fourier series, but Euler did not convince of its full generality entirely since he could not make sure whether any function could be expanded in the Fourier seriesFourier analysis is an important theorem of modern mathematics. It deeply influences the development of partial differential equations and information science. This paper will talk about some basic propositions of the Fourier series and some simple theorems of the Fourier transform. It will use an algebraic view to talk about convolution and good kernels. It will refer to some methods of Hilbert space and complex analysis.

Keywords: Fourier series, Lebesgue space, Fourier transform, convolution, kernel.

1. Introduction

Fourier analysis appeared to solve some partial differential equations at first. In 1747, D' Alembert solved the vibrating string equation using the method of traveling waves. In 1753, D. Bernoulli proposed the solution which for all intents and purposes is the Fourier series, but Euler did not convince of its full generality entirely since he could not make sure whether any function could be expanded in the Fourier series[1-20]. This was finally solved by Fourier. In 1807, Baron Jean Baptiste Joseph Fourier submitted a paper about heat conduction. However, it was rejected. Then in 1811, He recomposed his paper and submitted it again. Though his paper was lack of preciseness and was rejected again, he put it into his classic book written in 1822 called The Analytic Theory of Heat, which is the main provenance of the conviction of Fourier analysis [1-20]. After 2 years, when Fourier became the secretary of the academy of sciences, he put the paper into the report of the academy [1-20]. Inside an object absorbing or releasing heat, there is no well-distributed temperature distribution. Every point of the object was changed when time changed. Thus, the temperature T is the function of time and space. Fourier first thinks of the problem of the object which is uniform and isotropy in the 3-dimension space. It used x, y, and z 3 directions to be 3 dimensions and t to be time and makes the first partial differential equation.

$$\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}+\frac{\partial^2 T}{\partial z^2}=k^2\frac{\partial T}{\partial t} \tag{1}$$

Which k is a constant value depending on the material of the object.

For a special case, only consider a cylinder of the object and suppose that there is no effect with the other place of the object. Then the equation can be simplified as

$$\frac{\partial^2 T}{\partial x^2}=k^2\frac{\partial T}{\partial t} \tag{2}$$

The boundary value condition is

$$\mathrm{T}(0,\mathrm{t})=0,\mathrm{T}(\mathrm{l},\mathrm{t})=0,\mathrm{t}>0 \tag{3}$$

And the initial value condition is

$$\mathrm{T}(\mathrm{x},0)=\mathrm{f}(\mathrm{x}),0<\mathrm{x}<\mathrm{l} \tag{4}$$

To solve this problem, Fourier separated the variables. He made

$$\mathrm{T}(\mathrm{x},\mathrm{t})=\varphi(\mathrm{x})\psi(\mathrm{t}) \tag{5}$$

And then put it into a partial differential equation.

$$\frac{\varphi''(x)}{k^2\varphi(x)} = \frac{\psi'(t)}{\psi(t)} \tag{6}$$

And the ratio must be a constant value.[2]

As it is known to all, only trigonometric functions and exponential functions can make the derivative with constant equals to themselves. Meanwhile, there is Euler formula

$$e^{ix} = \cos x + i\sin x \tag{7}$$

To associate them. Thus, it is very important to research the trigonometric series when researching the heat equation. Trigonometric series with integral can be called Fourier series. It will be told in the following place.

However, though the trigonometric series is proven to represent any function, it has some problems with convergence. Thus, the theorem of the Fourier series is not rigorous, which is characteristic of the academic circles at that time. Dirichlet's first research was the rigor of the theorem. His paper gives first sufficient conditions to make the Fourier series of $f(x)$ convergent in $f(x)$. Then Riemann and some people continued his work and strengthen his theorem.

Hilbert's and his successor's work axiomatizes the Fourier analysis. In 1927, John von Neumann used the theorem of the operator and Hilbert space. He defined a kind of Hilbert space that is a Lebesgue measurable and square-integrable complex function, called L^2 space, L is the abbreviation of Lebesgue. Its norm is

$$\|f\| =< f,f >^{\frac{1}{2}}= (\int_a^b |f|^2)^{\frac{1}{2}} \quad] \tag{8}$$

The measurement of the space is defined as $(f,g) = \|f - g\|$

Its inner product is

$$< f_n,f_m >= \int_a^b f_n\overline{f_m} \quad] \tag{9}$$

Where $\bar{f}$ is the conjugate of the f. There is some basic theorem of the inner product that

(i)$< af,g >= a < f,g >, a$ is a complex value.

(ii)$< f_1 + f_2,g >=< f_1,g > +< f_2,g >$

(iii)$< f,g >=\overline{< g,f >}$

$$\text{(iv)}< f,f >\geq 0 \text{ and } < f,f >= 0 \text{ only if } f = 0 \tag{10}$$

Then he selected orthonormal basis for the space. The orthonormal basis is a family $\{f\}$ that each two elements f_n, f_m has a relationship$< f_n,f_m >= \delta_{mn} = \begin{cases}1, m = n\\0, m \neq n\end{cases}$. In L^2 space, $f_n = e^{\frac{2\pi inx}{L}}$(b-a=L). There is some special theorem that $\overline{e^{\frac{2\pi\imath nx}{L}}}$=$e^{-\frac{2\pi inx}{L}}$ and $|e^{\frac{2\pi inx}{L}}|$=$|e^{-\frac{2\pi inx}{L}}|$=1.

There is some basic theorem in Hilbert space which will be used in the following proof. The Pythagorean theorem is that

$$\|f_1\|^2 + \|f_2\|^2+\ldots+\|f_n\|^2 = \|f_1 + f_2+\ldots+f_n\|^2 \tag{11}$$

If they are in an orthogonal family. Meanwhile, there is Schwartz inequality written as

$$| < f,g > | \leq \|f\|\|g\| \tag{12}$$

And triangle inequality

$$\|f + g\| \leq \|f\| + \|g\| \tag{13}$$

Periodic functions are a kind of functions that $f(x)$=$f(x + \omega)$ and ω is called period. In Fourier analysis, there is a rule that when researching the integral $\int_a^b f(x)dx$, there are always conditions b-a=ω and $f(a)$=$f(b)$. There is a formula

$$\int_a^b f(x)dx = \int_{a+\varepsilon}^{b+\varepsilon} f(x)dx = \int_a^b f(x-\varepsilon)dx = \int_a^b f(-x)dx \tag{14}$$

Proof: $\int_{a+\varepsilon}^{b+\varepsilon} f(x)dx = \int_{a+\varepsilon}^{b} f(x)dx + \int_{b}^{b+\varepsilon} f(x)dx = \int_{a+\varepsilon}^{b} f(x)dx + \int_{a}^{a+\varepsilon} f(x+\omega)dx = \int_{a+\varepsilon}^{b} f(x)dx + \int_{a}^{a+\varepsilon} f(x)dx = \int_a^b f(x)dx$ .Suppose $x = -y,$ then $\int_{-a}^{-b} f(-y)d-y = \int_{-b}^{-a} f(-y)dy = \int_a^b f(-y)dy = \int_a^b f(-x)dx$

2. Fourier series

As it is said before, Fourier series is set on the L^2 space. It will first set some elements which is called Fourier coefficient. With them it can constitute the Fourier series.

Definition

If f is a Lebesgue integrable function defined on an interval $[a, b]$ whose length is L (that is, $b - a = L$), then the n^{th} Fourier coefficient of f is defined by

$$f(n)^{\wedge} = \frac{1}{\mathrm{L}} < f, e^{-\frac{2\pi inx}{L}} >= \frac{1}{\mathrm{L}} \int_a^b f(x) e^{\frac{-2\pi inx}{L}} dx \tag{15}$$

Its Fourier series is defined by

$$f(x) \sim \sum_{n=-\infty}^{\infty} f(n)^{\wedge} e^{\frac{2\pi inx}{L}} \tag{16}$$

The partial sum of the Fourier series is written as

$$S_N(f) = \sum_{n=-N}^{N} f(n)^{\wedge} e^{\frac{2\pi inx}{L}} \tag{17}$$

It will be proved in the following that

$$\lim_{N\to\infty} S_N(f) = f \tag{18}$$

As a function in a small interval, f can be extended to be a L-period function in the real line. For it is easy to write, it will be still written as f. Every period function satisfies the (14), so

$$\frac{1}{\mathrm{L}} \int_a^b f(x) e^{\frac{-2\pi inx}{L}} dx = \frac{1}{\mathrm{L}} \int_{-\frac{L}{2}}^{\frac{L}{2}} f(x) e^{\frac{-2\pi inx}{L}} dx \tag{19}$$

In this paper, it will only research for a simple function $\frac{1}{2\mathrm{\pi}} \int_{-\pi}^{\pi} f(x) e^{-inx} dx$ whose period is 2π as a representative. Others have the similar proposition with this function.

There are some basic propositions of the Fourier coefficient. The first one is linearity

(i)$a\, f(x)^{\wedge} = [af(x)]^{\wedge}, a$ is a complex value.

(ii)$f(x)^{\wedge} + f(y)^{\wedge} = [f(x) + f(y)]^{\wedge}$

and a proposition about conjugate

(iii) $\overline{f(-x)^{\wedge}} = (\overline{f(x)})^{\wedge}$

and also, a proposition of continuous

$$\text{(iv)If } f \text{ is continuous, } f^{\wedge} \text{ is continuous} \tag{20}$$

Actually, it is an example of (10). If changing g of (10) into e^{-inx}, there will be equivalent. So if (10) is proved, (17) is proved.

Proof of (10): $< af, g >= \int_a^b af\bar{g} = a\int_a^b f\bar{g} = a < f, g > < f_1 + f_2, g >= \int_a^b (f_1 + f_2)\bar{g} = \int_a^b f_1\bar{g} + \int_a^b f_2\bar{g} =< f_1, g > +< f_2, g >$. Suppose $f = a + bi, g = c + di$, as the integral is on the real line, it can get $< f, g >= \int_a^b f\bar{g} = \int_a^b (a + bi)(c - di) = \int_a^b ac + bd + (bc - ad)i$,

$\overline{< g,f >} = \overline{\int_a^b g\bar{f}} = \overline{\int_a^b (c+d\imath)(a-b\imath)} = \overline{\int_a^b ac+bd-(bc-ad)\imath} = \int_a^b ac+bd+(bc-ad)i = < f,g >$.

As f is continuous, $f(n)^{\wedge} = \frac{1}{\mathrm{L}}\int_a^b f(x)e^{\frac{-2\pi inx}{L}}dx$ and the product of two continuous functions is continuous. The integral of a continuous function is continuous. Hence $f^{\wedge}$

These are important propositions which will be used in the following. The (iv) proposition of (10) is not used in this place because the f of (10) is defined by e^{-inx} and it is no need to constitute a proposition about that. Then there are also some propositions represented by Fourier series. First giving a lemma.

$$f(\theta) \sim f(0)^{\wedge} + \sum_{n=1}^{\infty}(f(n)^{\wedge} + f(-n)^{\wedge})cosn\theta + i(f(n)^{\wedge} - f(-n)^{\wedge})sinn\theta \tag{21}$$

Proof:

$f(\theta) \sim \sum_{n=-\infty}^{\infty} f(n)^{\wedge}e^{in\theta} = f(0)^{\wedge} + \sum_{n=1}^{\infty} f(n)^{\wedge}e^{in\theta} + \sum_{n=-\infty}^{-1} f(n)^{\wedge}e^{in\theta} = f(0)^{\wedge} + \sum_{n=1}^{\infty} f(n)^{\wedge}cosn\theta + if(n)^{\wedge}sinn\theta + \sum_{n=1}^{\infty} f(-n)^{\wedge}cos(-n\theta) + if(-n)^{\wedge}sin(-n\theta) = f(0)^{\wedge} + \sum_{n=1}^{\infty}(f(n)^{\wedge} + f(-n)^{\wedge})cosn\theta + i(f(n)^{\wedge} - f(-n)^{\wedge})sinn\theta$

With this lemma, the following theorem can be proved.

If f is an even function, then $f(n)^{\wedge} = f(-n)^{\wedge}$, and it is a cosine series.

If f is an odd function, then $f(n)^{\wedge} = -f(-n)^{\wedge}$, and it is a sine series.

If $f(\theta+\pi) = f(\theta)$ for all $\theta \epsilon \mathrm{R}$, $f(n)^{\wedge} = 0$ for all odd n.

$$\text{(iv)}f \text{ is real-valued iff } \overline{f(n)^{\wedge}} = f(-n)^{\wedge} \text{ for all n.} \tag{22}$$

Proof:

(i)$f(\theta) \sim f(0)^{\wedge} + \sum_{n=1}^{\infty}(f(n)^{\wedge} + f(-n)^{\wedge})cosn\theta + i(f(n)^{\wedge} - f(-n)^{\wedge})sinn\theta$

$$f(-\theta) \sim f(0)^{\wedge} + \sum_{n=1}^{\infty}(f(n)^{\wedge} + f(-n)^{\wedge})cos(-n\theta) + i(f(n)^{\wedge} - f(-n)^{\wedge})sin(-n\theta)\, f(\theta) \sim f(0)^{\wedge}$$
$$+ \sum_{n=1}^{\infty}(f(n)^{\wedge} + f(-n)^{\wedge})cosn\theta + i(f(n)^{\wedge} - f(-n)^{\wedge})sin(-n\theta)\, i(f(n)^{\wedge}$$
$$- f(-n)^{\wedge})sinn\theta = i(f(n)^{\wedge} - f(-n)^{\wedge})sin(-n\theta)$$
$$f(n)^{\wedge} = f(-n); f(\theta) \sim f(0)^{\wedge} + \sum_{n=1}^{\infty}(f(n)^{\wedge} + f(-n)^{\wedge})cosn\theta$$

(ii)$f(\theta) \sim f(0)^{\wedge} + \sum_{n=1}^{\infty}(f(n)^{\wedge} + f(-n)^{\wedge})cosn\theta + i(f(n)^{\wedge} - f(-n)^{\wedge})sinn\theta$

$$f(-\theta) \sim f(0)^{\wedge} + \sum_{n=1}^{\infty}(f(n)^{\wedge} + f(-n)^{\wedge})cos(-n\theta) + i(f(n)^{\wedge} - f(-n)^{\wedge})sin(-n\theta)$$
$$-f(\theta) \sim f(0)^{\wedge} + \sum_{n=1}^{\infty}(f(n)^{\wedge} + f(-n)^{\wedge})cosn\theta + i(f(n)^{\wedge} - f(-n)^{\wedge})sin(-n\theta)$$

$-(f(n)^{\wedge} + f(-n)^{\wedge})cosn\theta) = (f(n)^{\wedge} + f(-n)^{\wedge})cosn\theta$

$f(n)^{\wedge} = -f(-n)^{\wedge}$

$f(\theta) \sim f(0)^{\wedge} + \sum_{n=1}^{\infty} i(f(n)^{\wedge} - f(-n)^{\wedge})sinn\theta$

(iii)$f(\theta) \sim f(0)^{\wedge} + \sum_{n=1}^{\infty}(f(n)^{\wedge} + f(-n)^{\wedge})cosn\theta + i(f(n)^{\wedge} - f(-n)^{\wedge})sinn\theta$

$$f(\theta+\pi) \sim f(0)^{\wedge} + \sum_{n=1}^{\infty}(f(n)^{\wedge} + f(-n)^{\wedge})cosn(\theta+\pi) + i(f(n)^{\wedge} - f(-n)^{\wedge})sinn(\theta+\pi)$$
$$f(\theta) \sim f(0)^{\wedge} + \sum_{n=1}^{\infty}(-1)^{\mathrm{n}}(f(n)^{\wedge} + f(-n)^{\wedge})cosn\theta + (-1)^{\mathrm{n}}i(f(n)^{\wedge} - f(-n)^{\wedge})sinn\theta$$

As n is odd, $f(\theta) \sim f(0)^{\wedge} - \sum_{n=1}^{\infty}(f(n)^{\wedge} + f(-n)^{\wedge})cosn\theta + i(f(n)^{\wedge} - f(-n)^{\wedge})sinn\theta$

$-\sum_{n=1}^{\infty}(f(n)^{\wedge}+f(-n)^{\wedge})cosn\theta-i(f(n)^{\wedge}-f(-n)^{\wedge})sinn\theta=f(0)^{\wedge}-2\sum_{n=1}^{\infty}(f(n)^{\wedge}+f(-n)^{\wedge})cosn\theta+i(f(n)^{\wedge}-f(-n)^{\wedge})sinn\theta=f(0)^{\wedge}+\sum_{n=1}^{\infty}(f(n)^{\wedge}+f(-n)^{\wedge})cosn\theta+i(f(n)^{\wedge}-f(-n)^{\wedge})sinn\theta$
$f(n)^{\wedge}+f(-n)^{\wedge}=0$ and $f(n)^{\wedge}-f(-n)^{\wedge}=0$
$f(n)^{\wedge}=0$
(iv)$f(\theta)\sim f(0)^{\wedge}+\sum_{n=1}^{\infty}(f(n)^{\wedge}+f(-n)^{\wedge})cosn\theta+i(f(n)^{\wedge}-f(-n)^{\wedge})sinn\theta$

Suppose $f(n)^{\wedge}=g(\text{n})+\text{i}h(\text{n})$

$f(\theta)\sim g(0)+\text{i}h(0)+\sum_{n=1}^{\infty}(g(\text{n})+\text{i}h(\text{n})+g(-\text{n})+\text{i}h(-\text{n}))cosn\theta+i(g(\text{n})+\text{i}h(\text{n})-g(-\text{n})-\text{i}h(-\text{n}))sinn\theta$

$\overline{f(\theta)}\sim g(0)-\text{i}h(0)+\sum_{n=1}^{\infty}(g(\text{n})+\text{i}h(\text{n})+g(-\text{n})+\text{i}h(-\text{n}))cosn\theta+i(-g(\text{n})-\text{i}h(\text{n})+g(-\text{n})+\text{i}h(-\text{n}))sinn\theta\sim f(\theta)$ as f is real-valued,
$g(\text{n})+\text{i}h(\text{n})-g(-\text{n})-\text{i}h(-\text{n})=-g(\text{n})-\text{i}h(\text{n})+g(-\text{n})+\text{i}h(-\text{n})$

$g(\text{n})-g(-\text{n})=0;\ \text{i}h(\text{n})+\text{i}h(-\text{n})=0.$ Hence $\overline{f(n)^{\wedge}}=f(-n)^{\wedge}$

Inversely, if there is $\overline{f(n)^{\wedge}}=f(-n)^{\wedge}$. It can be found that $g(\text{n})-g(-\text{n})=0\text{i}h(\text{n})+\text{i}h(-\text{n})=0$. It can get $g(\text{n})+\text{i}h(\text{n})-g(-\text{n})-\text{i}h(-\text{n})=-g(\text{n})-\text{i}h(\text{n})+g(-\text{n})+\text{i}h(-\text{n})$

$\overline{f(\theta)}\sim g(0)-\text{i}h(0)+\sum_{n=1}^{\infty}(g(\text{n})+\text{i}h(\text{n})+g(-\text{n})+\text{i}h(-\text{n}))cosn\theta+i(-g(\text{n})-\text{i}h(\text{n})+g(-\text{n})+\text{i}h(-\text{n}))sinn\theta=g(0)+\text{i}h(0)+\sum_{n=1}^{\infty}(g(\text{n})+\text{i}h(\text{n})+g(-\text{n})+\text{i}h(-\text{n}))cosn\theta+i(g(\text{n})+\text{i}h(\text{n})-g(-\text{n})-\text{i}h(-\text{n}))sinn\theta\sim f(\theta)$

So, the inversion is also right.

3. Convolutions

As the linearity talks about the relationship with functions and its Fourier coefficient for addition, scalar-multiplication and conjugate (or the proposition of commutation), it is necessary to talk about the multiplication with them. There is actually a homomorphism between functions and its Fourier coefficient, and the operator is written as $*$(That is, it can be supposed that $F(f)=f^{\wedge}$, $F(f)*F(g)=F(f\cdot g)$) . $f^{\wedge}g^{\wedge}=(f*g)^{\wedge}$ is our hypothesis. This kind of multiplication is called convolution.There is a basic theorem of convolution that

$$f(x)*g(x)=\frac{1}{2\pi}\int_{-\pi}^{\pi}f(y)g(x-y)dy \tag{23}$$

Proof:

Through (13), it is obvious that $\int_{-\pi}^{\pi}f(x)dx=\int_{-\pi}^{\pi}f(y-x)dx$. as $f(n)^{\wedge}=\frac{1}{2\pi}\int_{-\pi}^{\pi}f(x)e^{-inx}dx$, $f^{\wedge}g^{\wedge}=\frac{1}{2\pi}\int_{-\pi}^{\pi}f(x)e^{-inx}dx\frac{1}{2\pi}\int_{-\pi}^{\pi}g(y)e^{-iny}dy=\frac{1}{2\pi}\int_{-\pi}^{\pi}[f(x)*g(x)]e^{-inx}dx$.

The left side:$\frac{1}{2\pi}\int_{-\pi}^{\pi}f(x)e^{-inx}dx\frac{1}{2\pi}\int_{-\pi}^{\pi}g(y)e^{-iny}dy=\frac{1}{2\pi}\int_{-\pi}^{\pi}f(x)e^{-inx}\frac{1}{2\pi}(\int_{-\pi}^{\pi}g(y)e^{-iny}dy)dx=\frac{1}{2\pi}\int_{-\pi}^{\pi}f(x)e^{-inx}\frac{1}{2\pi}(\int_{-\pi}^{\pi}g(y-x)e^{-in(y-x)}dy)dx$

$$=\frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{1}{2\pi}(\int_{-\pi}^{\pi}f(x)g(y-x)dx)e^{-iny}dy=\frac{1}{2\pi}\int_{-\pi}^{\pi}[f(x)*g(x)]e^{-inx}dx$$

$\frac{1}{2\pi}\int_{-\pi}^{\pi}\text{f(x)g(y}-\text{x)dx}=\text{f(x)}*\text{g(x)}$ After changing x with y, $f(x)*g(x)=\frac{1}{2\pi}\int_{-\pi}^{\pi}f(y)g(x-y)dy$.There are also some propositions of convolution. Certainly, the first one is the linearity

(i)$(af)*g=a(f*g)=f*(ag)$, a is a complex value.
(ii)$f*(g+h)=f*g+f*h$
It also has law of commutation and association
(iii)$f*g=g*f$
(iv)$(f*g)*h=f*(g*h)$
Then there is a proposition about continuous.
(v)If f and g is continuous, $f*g$ is continuous.
Proof:

The linearity of convolution can be solved with the linearity of Fourier coefficient

$$(af)*g=(af)^{\wedge}g^{\wedge}=a(f^{\wedge}g^{\wedge})=f^{\wedge}(ag)^{\wedge}=a(f*g)=f*(ag) \tag{24}$$

$$f*(g+h)=f^{\wedge}(g+h)^{\wedge}=f^{\wedge}g^{\wedge}+f^{\wedge}h^{\wedge}=f*g+f*h \tag{25}$$

The law of commutation and association of convolution can be solved with the law of commutation and association of functions' multiplication.

(iii)$f*g=f^{\wedge}g^{\wedge}=g^{\wedge}f^{\wedge}=g*f$

(iv) $(f*g)*h=(f*g)h^{\wedge}=f^{\wedge}g^{\wedge}h^{\wedge}=f*(g*h)$

The continuous of convolution is also be solved by the linearity of Fourier coefficient

$f*g=f^{\wedge}g^{\wedge}$. As f and g is continuous, $f^{\wedge}$ and $g^{\wedge}$ is continuous, so $f^{\wedge}g^{\wedge}$ is continuous, so $f*g$ is continuous. It is easy to find that every proposition of convolution is formed by functions' multiplication.

4. Good kernels

As there is a homomorphism between f and $f^{\wedge}$, there can be an identity on $f^{\wedge}$. That is, there is $1\cdot\mathrm{f}=\mathrm{f}$ and it can be defined that $\mathrm{e}*\mathrm{f}=\mathrm{f}$, so a set called good kernels is created, whose element is written as K_n. It needs a basic proposition that $\lim_{n\to\infty}(f*K_n)(x)=f(x)$. It is necessary to set a theory to make it established. it is defined with 3 propositions

(i)$\frac{1}{2\pi}\int_{-\pi}^{\pi}\mathrm{K_n}(\mathrm{x})\mathrm{dx}=1$

(ii)$\int_{-\pi}^{\pi}|\mathrm{K_n}(\mathrm{x})|\mathrm{dx}$ is bounded

(iii)$\int_{\delta\le|x|\le\pi}|\mathrm{K_n}(\mathrm{x})|\mathrm{dx}\to0$, as n→∞ whenever δ>0.

Actually, it needs three proposition to prove the basic proposition.

Proof:

$$\frac{1}{2\pi}(f*K_n)(x)-f(x)=\frac{1}{2\pi}\int_{-\pi}^{\pi}K_n(y)f(x-y)dy-f(x) \tag{26}$$

It is known that $\frac{1}{2\pi}\int_{-\pi}^{\pi}\mathrm{K_n}(\mathrm{x})\mathrm{dx}=1$, so $f(x)=\frac{1}{2\pi}\int_{-\pi}^{\pi}K_n(y)f(x)dy$.

So it is equal to $\frac{1}{2\pi}\int_{-\pi}^{\pi}K_n(y)[f(x-y)-f(x)]dy\le\int_{\delta\le|y|\le\pi}|K_n(y)||f(x-y)-f(x)|dy+\int_{|y|<\delta}|K_n(y)||f(x-y)-f(x)|dy$

The first integral $\int_{\delta\le|y|\le\pi}|K_n(y)||f(x-y)-f(x)|dy$ is near to 0 as definition. The second integral $\int_{|y|<\delta}|K_n(y)||f(x-y)-f(x)|dy$ is also near to 0 as $\lim_{y\to0}f(x-y)-f(x)$ and $\int_{|y|<\delta}|K_n(y)|dy$ is bounded. So, the proof is over.

In the proof it is obvious the second property is not a strong property to proof it. If it is not bounded, the second integral still possible to be near to 0. D_N is an example that does not satisfy the second condition but satisfy the basic proposition, hence it is not a good kernel. Its element is written as D_N.

$$D_N(x)=\sum_{n=-N}^{N}e^{inx} \tag{27}$$

It is easy to find that

$$f*D_N=S_N(f) \tag{28}$$

Proof:

$f*D_N=\frac{1}{\mathrm{L}}\int_a^b f(y)\sum_{n=-N}^{N}e^{in(x-y)}dy=\sum_{n=-N}^{N}(\frac{1}{\mathrm{L}}\int_a^b f(y)e^{iny}dy\ )\ e^{inx}=\sum_{n=-N}^{N}f(n)^{\wedge}e^{inx}=S_N(f)$. With the formula (17), it is obvious that it has the convolution approaching to the identity. $\lim_{n\to\infty}(f*\ D_N)(x)=f(x)$. There are some propositions similar with good kernel that

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}D_N(x)dx=1 \tag{29}$$

But (ii) $\int_{-\pi}^{\pi} |D_N(x)|dx \geq c\ln N$ as N→∞

There is a basic formula of D_N that

$$D_N = \frac{\sin((N+\frac{1}{2})x)}{\sin(\frac{x}{2})} \tag{30}$$

Proof:

Suppose $e^{ix} = \omega$ (this hypothesis will be used in the following proof). It can be divided to be two part, $\sum_{n=0}^{N}\omega$ and $\sum_{n=-N}^{-1}\omega$. With the Taylor formula, $\sum_{n=0}^{N}\omega = \frac{1-\omega^{N+1}}{1-\omega}$, $\sum_{n=-N}^{-1}\omega = \frac{\omega^{-N}-1}{1-\omega}$. The sum is $\frac{\omega^{-N}-\omega^{N+1}}{1-\omega} = \frac{\omega^{-N-1/2}-\omega^{N+1/2}}{\omega^{-1/2}-\omega^{1/2}} = \frac{\sin((N+\frac{1}{2})x)}{\sin(\frac{x}{2})}$

Through this formula, the (28) can be proved.

Proof:

The first proposition is quite simple as $\int_{-\pi}^{\pi} e^{inx}dx = \frac{(-1)^n}{in}$ if $n \neq 0$ and 2π if $n = 0$. $\int_{-\pi}^{\pi} e^{inx}dx + \int_{-\pi}^{\pi} e^{i(-n)x}dx = 0$ if $n \neq 0$, so at last it only has a 2π, and it multiply $\frac{1}{2\pi}$ is 0.

The second proposition uses the proposition written before, $D_N = \frac{\sin((N+\frac{1}{2})x)}{\sin(\frac{x}{2})}$.

It can be known as the formula that $|D_N| \geq c\frac{|\sin((N+\frac{1}{2})x)|}{|x|}$ which is obvious for some c>0.

$$\int_{-\pi}^{\pi} |D_N(x)|dx \geq \int_{-\pi}^{\pi} c\frac{|\sin((N+\frac{1}{2})x)|}{|x|}dx = 2c\int_{0}^{\pi}\frac{|\sin((N+\frac{1}{2})x)|}{|x|}dx = 2c\int_{0}^{N\pi}\frac{|\sin((1+\frac{1}{2N})x)|}{|x/N|}dx/N = 2c\int_{0}^{N\pi}\frac{|\sin x|}{|x|}dx \geq kc\int_{\pi}^{N\pi}\frac{1}{|x|}dx \geq kc\ln N$$

For the second to last step, there must be a real value $\frac{2}{k}$>0 that $\int_{(k-1)\pi}^{k\pi}\frac{2|\sin x|}{k|x|}dx > \int_{(k-1)\pi}^{k\pi}\frac{1}{|x|}dx$ and it makes the area larger. So, it is proven.

5. Means

Though D_N is not a good kernel, it also has some propositions like a good kernel. So, it is possible to set a kernel made by D_N and that it will be a good kernel. Actually, it should change the second proposition.

For D_N is not a good kernel and $\int_{-\pi}^{\pi} |D_N(x)|dx \geq c\ln N$ as $N \to \infty$, and it is obvious that x is much larger than $\ln x$ when x→∞. It can constitute a kernel

$$F_N = \frac{D_0+D_1+\ldots+D_{N-1}}{N} \tag{31}$$

And the (ii) proposition of (24) may be established. It is called Fejer kernel. This kind of mean is called Cesaro mean. $\frac{D_0+D_1+\ldots+D_{N-1}}{N}$ is called Cesaro summable to F_N. If it is a good kernel, it has to satisfy the basic proposition. First, it is obvious that

$$\lim_{n\to\infty}(f * F_N)(x) = f(x) \tag{32}$$

through (27) as

$$(f * F_N)(x) = \frac{S_0+S_1+\ldots+S_{N-1}}{N} \tag{33}$$

The left formula of (32) is defined by

$$\sigma_N = \frac{S_0+S_1+\ldots+S_{N-1}}{N} \tag{34}$$

Meanwhile the (i) proposition of good kernel is satisfied obviously. So it should only proof

$$\int_{\delta\le|x|\le\pi}|F_N(x)|dx \to 0, \text{ as } N\to\infty \text{ whenever } \delta>0 \tag{35}$$

$$F_N=\frac{1}{N}\frac{sin^2(Nx/2)}{sin^2(x/2)} \tag{36}$$

There is also a basic formula that

Proof:

The same as D_N ,$\mathrm{F_N}=\frac{1}{\mathrm{N}}\sum_{n=0}^{N-1}\frac{\omega^{-n}-\omega^{n+1}}{1-\omega}=\frac{1}{\mathrm{N}}\frac{\frac{1-\omega^{-N+1}}{1-\omega^{-1}}+\frac{\omega^{N+1}-1}{1-\omega}}{1-\omega}=\frac{1}{\mathrm{N}}\frac{\omega^N+\omega^{-N}-2}{(\omega^{-1/2}-\omega^{1/2})^2}=\frac{1}{N}\frac{sin^2(Nx/2)}{sin^2(x/2)}$.Through this formula, it can proof that $\mathrm{F_N}$ is a good kernel.

Proof:

$\int_{\delta\le|x|\le\pi}|\mathrm{F_N}(x)|dx \le \int_{\delta\le|x|\le\pi}\frac{1}{|Nsin^2(x/2)|}dx \le k\int_{\delta\le|x|\le\pi}\frac{1}{N}dx$ → 0, as N→∞ for some k>0.

Though sometimes it can use Cesaro mean to make series converge, some series are not Cesaro summable. For example,

$$1-2+3-4\ldots=\sum_{k=0}^{\infty}(-1)^k\,(k+1) \tag{37}$$

which is not Cesaro summable. It then sets an Abel mean which is defined that

$$A_r=\sum_{n=0}^{\infty}c_n r^n \tag{38}$$

For $0\le \mathrm{r}<1$. Which satisfy it is called Abel summable. Then defining a simple Fourier series

$$f(\mathrm{x})\sim\sum_{n=-\infty}^{\infty}a_n e^{inx} \tag{39}$$

whose Abel mean is

$$A_r(f)(\theta)=\sum_{\mathrm{n}=-\infty}^{\infty}r^{|n|}\;a_n e^{in\theta} \tag{40}$$

as $c_0=a_0$ and $c_n=\;a_n e^{in\theta}+\;a_{-n}e^{-in\theta}$.

Like σ_N, it makes a convolution again that

$$A_r=f*P_r \tag{41}$$

which is called Poisson kernel and that

$$P_r(\theta)=\sum_{n=-\infty}^{\infty}r^{|n|}e^{in\theta} \tag{42}$$

Through this definition, it is easy to know that it satisfies the (i) and (ii) proposition of good kernel.

Proof:

$\frac{1}{2\pi}\int_{-\pi}^{\pi}P_r(\theta)d\theta=\frac{1}{2\pi}\int_{-\pi}^{\pi}\sum_{n=-\infty}^{\infty}r^{|n|}e^{in\theta}\,d\theta=\frac{1}{2\pi}\int_{-\pi}^{\pi}\sum_{n=1}^{\infty}r^{|n|}e^{in\theta}\,d\theta+\frac{1}{2\pi}\int_{-\pi}^{\pi}\sum_{n=-1}^{\infty}r^{|-n|}e^{-in\theta}\,d\theta+1=1$

$\int_{-\pi}^{\pi}|P_r(\theta)d\theta|=\int_{-\pi}^{\pi}\sum_{n=-\infty}^{\infty}r^{|n|}=2\pi\sum_{n=-\infty}^{\infty}r^{|n|}$ which is converge obviously.

There is also a basic formula for $\mathrm{P_r}$ that

$$P_r(\theta)=\frac{1-r^2}{1-2rcos\theta+r^2} \tag{43}$$

Proof:

Suppose $re^{i\theta}=\omega$. $\lim\limits_{\mathrm{n}\to\infty}\sum_{n=0}^{N}\omega=\frac{1}{1-\omega}$, $\lim\limits_{\mathrm{n}\to\infty}\sum_{n=-N}^{-1}\omega=\frac{\omega^{-1}}{1-\omega^{-1}}$. The sum is $\frac{1}{1-\omega}+\frac{\omega^{-1}}{1-\omega^{-1}}=\frac{1-|\omega|^2}{|1-\omega|^2}=\frac{1-r^2}{1-2rcos\theta+r^2}$.$P_r$ is also a good kernel.

Proof:

The (i) (ii)proposition of good kernel has already be proved. Then, $A_r=f*P_r$ and $\lim\limits_{r\to1}A_r=\lim\limits_{n\to\infty}S_N$. Hence it satisfies the basic proposition of good kernel. It is known by (44) that $1-\mathrm{r}^2$ is approached to 0 but $1-2\mathrm{rcos}\theta+\mathrm{r}^2$ is not approached to 0 as $r\to1$ and $\delta\le|\theta|\le\pi$. So, the (iii) proposition is proved.

6. Convergence of Fourier series

When talking about the convergence of the good kernels, it is also important to research the convergence of the Fourier series. At the beginning it gives an assumption that S_N will approach to f if N approaches to infinite number. There is a more rigorous and basic formula to proof this result which is called "mean square convergence".

$$\|f - S_N(f)\|^2 = \frac{1}{\mathrm{L}}\int_a^b |f(\theta) - S_N(f)(\theta)|^2 d\theta \rightarrow 0 \text{ as N} \rightarrow \infty \tag{44}$$

Proof: $\mathrm{e}^{\mathrm{inx}}$ is written as e_n $< f, e_n >= f_n{}^{\wedge}$ with the linearity $S_N(f) = \sum_{n=-N}^{N} f_n{}^{\wedge} e_n$

$$< f - S_N(f), S_N(f) >=< f, S_N(f) > - < S_N(f), S_N(f) >= \sum_{n=-N}^{N} |f_n|^{\wedge 2} - \sum_{n=-N}^{N} |f_n{}^{\wedge}|^2 \, |e_n|^2 = 0$$

So $f - S_N(f) \perp S_N(f)$ and the Pythagorean theorem can be used. $\|f\| = \|f - S_N(f)\| + \|S_N(f)\|$ As $\|S_N(f)\| \left\|\sum_{n=-N}^{N} f_n{}^{\wedge} e_n\right\| = \sum_{n=-N}^{N} |f_n{}^{\wedge}|^2$ which is proved obvious, $\|f\| = \|f - S_N(f)\| + \sum_{n=-N}^{N} |f_n{}^{\wedge}|^2$. There is an obvious deduction with orthogonal that $\|f - S_N(f)\| \leq \left\|f - \sum_{n=-N}^{N} c_n e^{\frac{2\pi i n x}{L}}\right\|$ for any complex value c_n as N → ∞. As the c_n is arbitrary, $\|f - S_N(f)\| \leq \varepsilon$ for any $\varepsilon > 0$ as $N \rightarrow \infty$. $\|f - S_N(f)\|^2 \leq \varepsilon^2$ as N → ∞, so it is proved.

By this formula, an identity which can be easily proved is

$$\sum_{n=-N}^{N} |f_n|^{\wedge 2} = \|f\|^2 \text{ when N} \rightarrow \infty \tag{45}$$

which is called Parseval's identity.

7. Some basic theorems of Fourier transform

In the previous researching, it only talks about the function in a small interval or like some books' saying: function in the circle. However, most of the basic function is on the real line. In this time, it is easy to find that the period is infinite, and L will be infinite. Hence it cannot set a function like this. If f is not a periodic function in the whole real line, its Fourier transform will be made

As there is only a coefficient value L left from the Fourier series, it is easy to find that it has the property similar to a periodic function.

$$f(\gamma)^{\wedge} = \int_{-\infty}^{\infty} f(x) e^{-2\pi i \gamma x} dx \tag{46}$$

It is represented by a notation

$$f(x) \rightarrow f(\gamma)^{\wedge} \tag{47}$$

There are some propositions for the Fourier transform

(i) $f(x+h) \rightarrow f(\gamma)^{\wedge} e^{2\pi i \gamma h}$
(ii) $f(x) e^{-2\pi i x h} \rightarrow f(\gamma + h)^{\wedge}$
(iii)$\delta f(\delta x) \rightarrow f(\delta^{-1}\gamma)^{\wedge}$
(iv)$f'(x) \rightarrow 2\pi i \gamma f(\gamma)^{\wedge}$
(v)$-2\pi i x f(x) \rightarrow [f(\gamma)^{\wedge}]'$
(vi)$f(x)^{\wedge} \rightarrow f(-\gamma)$

Sketch of proof:(i)(ii) is obvious,(iii) it can put δ into dx and then obvious, (iv) is proved before, and (v) can be proved by (iv) and the following inversion of f(x), (vi) is the inversion.

As there is Fourier transform, it is obvious that there is an inversion for it. The inversion is

$$f(x) = \int_{-\infty}^{\infty} f(\gamma)^{\wedge} e^{2\pi i \gamma x} d\gamma \tag{48}$$

An invariable subspace is a key point of research. As the(iv) and (v) properties, it is easy to find that there is some relationship between differential and the unknown value x multiplied in the Fourier transform. So supposing that S is a subspace that for any f(x) it has $|\mathrm{x}|^k|f^{(l)}(x)|$ is bounded for every $k, l > 0$.

Proof: if $f \in$ S, any order of derivative of $f(\gamma)^\wedge$ is bounded as any power of x multiplies f(x) is bounded with the inversion.any power of x multiplies $f(\gamma)^\wedge$ is any order of derivative of $f(x)$. So f^∈ S. S is an invariable subspace for the Fourier transform.

S is called Schwartz space and also called rapidly decreasing space .

It is important to research when there will be $f(\gamma)^\wedge$=$f(\gamma)$ as an "invariable element"

There is a Gaussian function $e^{-\pi x^2}$.

Proof: $-2\pi\mathrm{i}xf(x) \rightarrow [f(\gamma)^\wedge]$'$f'(x) = -2\pi x f(x) \rightarrow 2\pi\mathrm{i}\gamma f(\gamma)^\wedge[f(\gamma)^\wedge]' = -2\pi\gamma f(\gamma)$

It will be solved obviously that

$$f(\gamma) = e^{-\pi\gamma^2} \tag{49}$$

$e^{-\pi x^2}$ is also a good kernel.

There is an important deduction that

$$e^{-\delta x} \rightarrow \frac{\delta}{\delta^2+\gamma^2} \tag{50}$$

Fourier transform has the norm preserving mapping called Plancherel formula about the norm of the Lebesgue space.

$$\|f\| = \|f^\wedge\| \tag{51}$$

Proof:

Suppose $g(x) = \overline{f(-x)}$ $g(x)^\wedge = \overline{f(x)^\wedge}$ Suppose $h = f * g$. $h^\wedge = |f^\wedge|^2$ and $h(0) = \int_{-\infty}^{\infty}|f|^2$

$h(0) = \int_{-\infty}^{\infty} h^\wedge = \int_{-\infty}^{\infty}|f^\wedge|^2$ Hence $\int_{-\infty}^{\infty}|f|^2 = \int_{-\infty}^{\infty}|f^\wedge|^2$

So $\|f\| = \|f^\wedge\|$.This formula can be used in the following application

8. An application for Fourier transform

Heisenberg inequality on Schwartz space

$$\int_{-\infty}^{\infty} x^2|f(x)|^2dx * \int_{-\infty}^{\infty}\gamma^2\left|f(\gamma)^\wedge\right|^2 d\gamma \geq (16\pi^2)^{-1}\|f\|^4 \tag{52}$$

Proof:

$4\pi\int_{-\infty}^{\infty} x^2|f(x)|^2dx * \int_{-\infty}^{\infty}\gamma^2\left|f(\gamma)^\wedge\right|^2 d\gamma = \int_{-\infty}^{\infty}|xf(x)|^2dx * \int_{-\infty}^{\infty}\left|2\pi i\gamma f(\gamma)^\wedge\right|^2 d\gamma =$ $\int_{-\infty}^{\infty}|xf(x)|^2dx * \int_{-\infty}^{\infty}|f(\gamma)'|^2d\gamma$ (by Plancherel's identity) $= \int_{-\infty}^{\infty}\left|x\overline{f(x)}\right|^2dx * \int_{-\infty}^{\infty}|f(\gamma)'|^2d\gamma \geq$ $[\int_{-\infty}^{\infty}\left|xf(x)'\overline{f(x)}\right|dx\,]^2$ (by Schwartz inequality) $\geq [\int_{-\infty}^{\infty} x\frac{1}{2}\left|f(x)'\overline{f(x)} + f(x)\overline{f(x)'}\right|dx\,]^2 =$ $\frac{1}{4}[\int_{-\infty}^{\infty} x(|f(x)|^2)'dx\,]^2 = \frac{1}{4}[x(|f(x)|^2)|_{-\infty}^{+\infty} - \int_{-\infty}^{\infty}|f(x)|^2dx\,]^2$. As it is in the Schwartz space, $x|f(x)|^2|_{-\infty}^{+\infty} = 0$. Hence $\frac{1}{4}[x(|f(x)|^2)|_{-\infty}^{+\infty} - \int_{-\infty}^{\infty}|f(x)|^2dx\,]^2 = \frac{1}{4}[\int_{-\infty}^{\infty}|f(x)|^2dx\,]^2 = \frac{1}{4}\|f\|^4$

So, it is proven.

9. Conclusion

At first, several partial differential equations appeared to be solved using Fourier analysis. D' Alembert used the traveling wave approach to resolve the vibrating string problem in 1747. The Fourier series was first presented by D. Bernoulli in 1753, but Euler was unable to fully persuade himself of its applicability since he was unsure if any function could be enlarged in the Fourier series.

One of the key theorems in contemporary mathematics is the Fourier analysis. It has a significant impact on the advancement of information science and partial differential equations. This essay will discuss some fundamental Fourier series. It will discuss convolution and effective kernels from an algebraic perspective. Fourier series has a lot of good properties of linearity and convergence. It is a good example of researching partial differential equations and functional analysis. Except for these, there are also many things for researchers to research.

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