Fourier Transform in Three Differences Spaces

Tianhao Zhang*

School of Shandong University, Shandong, China

* Corresponding Author Email: tianhao.zhang@mail.sdu.edu.cn

Abstract. This paper starts by introducing the application of the Fourier analysis and divides the Fourier analysis into four cases according to the difference between physical space and frequency space: discrete points on a circle $\leftrightarrow$ discrete points on a circle, $n \in Z \leftrightarrow \theta \in [0,2\pi]$, $\theta \in [0,2\pi] \leftrightarrow n \in Z$, and $x \in R \leftrightarrow \lambda \in R$. The four cases are then grouped into three parts: the discrete Fourier transform (DFT), the Fourier series, and the Fourier transform. For each part, this paper discusses its characteristics and basic theory, and selectively deals with the proof and application examples. This paper will promote some research related to Fourier analysis by analyzing Fourier analysis in three different spaces as a whole.

Keywords: Discrete Fourier transform, Fourier series, Fourier transform.

1. Introduction

Fourier analysis is very important in physics (heat equation), mathematics (harmonic analysis), engineering and so on. And the basic idea is that represent an "arbitrary function" as a sum of exponentials

$$F(x) = \int_{-\infty}^{\infty} \hat{f}(\lambda) e^{i\lambda x} d\lambda \tag{1}$$

For $x \in \mathbb{R}$ and $\lambda \in \mathbb{R}$. And our function $f(x)$ will be complex valued.

Fourier analysis takes place in different situations: the N roots of unity, the circle, or the real line. The first case presents no technical difficulties and is the topic in part 1. The circle has some technical considerations involving infinite series and is the topic in part 2 and 3. For the last case, the real line, we will ignore technical considerations and concentrate on applications which is contain in part 4.

Now we will show the difference between these three cases by comparing them into four cases according to physical space and frequency space.

(1) The case 1 is what will be said in part 2. In this case, with a fixed N, the domain of physical space and the domain of frequency space is both on a circle. That is, which have already defined above by $\omega := e^{\frac{2\pi}{N}i}$, $\omega^0, \omega^1, \omega^2 \ldots, \omega^{N-1}$.

Table 1 Case 4 of Fourier analysis

	Physical space	Frequency space
Domain	On a circle but discrete	On a circle but discrete
Range	$\mathbb{R}$	$\mathbb{R}$
Function	a_j	$\frac{1}{\sqrt{N}}\sum_{k=0}^{N-1} a_k\, \omega^{kj}$

(2) The second case is the independent variable $n \in \mathbb{Z}$ in physical space and θ on a circle in frequency space, which is associated with case 3.

Table 2 Case 2 of Fourier analysis

	Physical space	Frequency space
Domain	$n \in \mathbb{Z}$	θ is on a circle ($\theta \in [0,2\pi]$)
Range	$\mathbb{R}$	$\mathbb{R}$
Function	$f(n)$	$\int_0^{2\pi} \hat{f}(\theta) e^{in\theta} d\theta$

(3). Contrary to the previous case, the case 3 is the independent variable θ on a circle in physical space and $n \in \mathbb{Z}$ in frequency space. And a detailed discussion of case 2 and case 3 will appear in the unification section.

Table 3 Case 3 of Fourier analysis

	Physical space	Frequency space
Domain	θ is on a circle	$n \in \mathbb{Z}$
Range	$\mathbb{R}$	$\mathbb{R}$
Function	$f(\theta)$	$\sum_{n=-\infty}^{\infty} \hat{f}(n) e^{in\theta}$

The details of the infinite summation in this case will be said in part 3.

(4) The domain of physical space and the domain of frequency space is $\mathbb{R}$ and $\mathbb{R}$. The details of the anomalous integral in this case will be said in part 4.

Table 4 Case 4 of Fourier analysis

	Physical space	Frequency space
Domain	$x \in \mathbb{R}$	$\lambda \in \mathbb{R}$
Range	$\mathbb{R}$	$\mathbb{R}$
Function	$f(x)$	$\int_{-\infty}^{\infty} \hat{f}(\lambda) e^{i\lambda x} d\lambda$

2. The N roots of unity

Firstly, let's compare them. Fourier analysis can be done in 4 different situations. Actually, way No.2 and way No.3 is the same, that depend on what way you look at them.

2.1. Some about complex numbers

Now picking a positive integer $N = 1,2,3,4,5$ .... Define $\omega := e^{\frac{2\pi}{N}i}$ which modulus equal to 1. It satisfies $\omega^N = 1$, $\omega^{N-1} = \omega^N \omega^{N-1} = \omega^{-1}$, $\omega^{N-2} = \omega^{-2}$ and so on. Then we can get $1 + \omega^1 + \omega^2 + \cdots + \omega^{N-1} = \frac{1-\omega^N}{1-\omega} = 0$ which called n-th unit root. The following diagram will graphically show the relationship between the upper corner scale of x and its position on the circle.

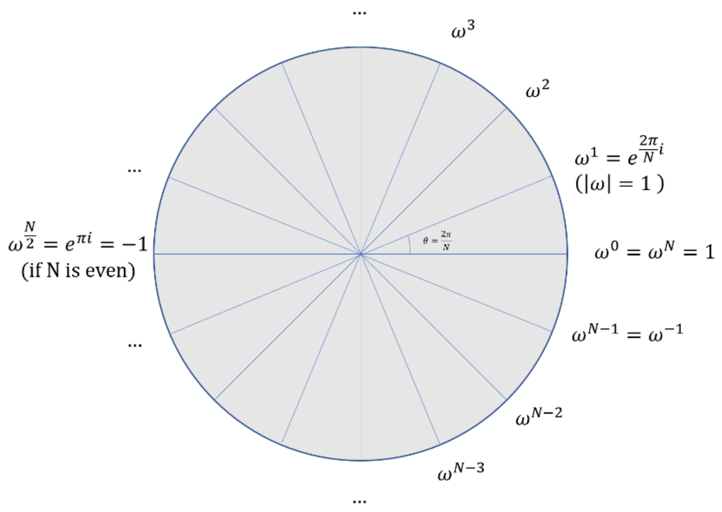

Figure 1. The graph of ω in a circle

2.2. Discrete Fourier Transform (DFT)

Suppose there is a function defined on the set $0, 1, 2, 3 \dots N-1$ corresponds to $\omega^0, \omega^1, \omega^2, \omega^3, \dots, \omega^{N-1}$ one by one. Using denote on our function $a_1, a_2, a_3 \dots, a_{N-1}$.These are functions defined on the domain of complex numbers, which is very important.

We define the DFT of this finite sequence by the following formula $\widehat{a_0}, \widehat{a_1}, \widehat{a_2}, \dots, \widehat{a_{N-1}}$ where

$$\widehat{a_j} = \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} a_k\, \omega^{kj} \tag{2}$$

for $0 \le j \le N-1$. That means $a_1, a_2, a_3 \dots, a_{N-1} \xrightarrow{DFT} \widehat{a_0}, \widehat{a_1}, \widehat{a_2}, \dots, \widehat{a_{N-1}}$. It is easy to know the inversion formula is

$$a_j = \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} \widehat{a_k}\, \omega^{-kj} \tag{3}$$

After defining the discrete Fourier transform, here are one of the main properties. Think of two set of sequences

$$a \equiv \{a_0, a_1, \dots a_{N-1}\} \tag{4}$$

$$b \equiv \{b_0, b_{1,} \dots b_{N-1}\} \tag{5}$$

where $a_j, b_k \in \mathbb{C}, j, k \in \mathbb{N}$. The $\mathbb{C}$ as a vector space over the complex numbers and $\mathbb{N}$ is the natural number field.

Define the inner product as

$$\langle a, b\rangle \coloneqq \sum_{\mathrm{k}=0}^{\mathrm{N}-1} a_j\, \overline{a_j} \tag{6}$$

Suppose we have a and b and we complete the DFT of each one of them. That is

$$\hat{a} \coloneqq (\widehat{a_0}, \widehat{a_1}, \widehat{a_2}, \dots, \widehat{a_{N-1}}) \tag{7}$$

$$\hat{b} \coloneqq (\widehat{b_0}, \widehat{b_1}, \widehat{b_2}, \dots, \widehat{b_{N-1}}) \tag{8}$$

Now let us compute $\langle \hat{a}, \hat{b}\rangle$. Since $|\omega| = 1 = \omega \cdot \overline{\omega} = \omega \cdot \omega^{-1}$, it is clear that $\overline{\omega} = \omega^{-1}$. Set $\delta(n) = \begin{cases} 1\ if\ n = 0 \\ 0\ if\ n \neq 0 \end{cases}$. As the definition, $\langle \hat{a}, \hat{b}\rangle = \sum_{j=0}^{N-1} \widehat{a_j}\, \overline{\widehat{b_j}} = \frac{1}{N} \sum_{k=0}^{N-1} \sum_{j=0}^{N-1} a_K\, \omega^{kj} \sum_{k=0}^{N-1} \overline{b_l}\, \omega^{-lj} = \frac{1}{N} \sum_{k=0}^{N-1} \sum_{l=0}^{N-1} a_k\, \overline{b_l} \sum_{j=0}^{N-1} \omega^{j(k-l)} = \frac{1}{N} \sum_{k=0}^{N-1} \sum_{l=0}^{N-1} a_K\, \overline{b_l} \cdot N \cdot \delta(k-l) = \frac{1}{N} \left(\sum_{k=0}^{N-1} a_K\, \overline{b_k}\right) \cdot N = \sum_{k=0}^{N-1} a_k\, \overline{b_k} = \langle a, b\rangle$. Now a special result Inner product

$\langle\hat{a},\hat{b}\rangle=\langle a,b\rangle$ has been proved. It is time to see another important Fourier property of DFT. The convolution is defined of the sequences $(a*b)_j \coloneqq \sum_{j=0}^{N-1} a_{j-k}b_k$. Then it is necessary to prove $(\widehat{a*b})_j = \hat{a}_j \cdot \hat{b}_j$.

Proof *Set* $m = k - l$ *We have* $\left(\widehat{a*b}\right)_j = \frac{1}{\sqrt{N}}\sum_{k=0}^{N-1}(a*b)_k\,\omega^{kj} = \frac{1}{\sqrt{N}}\sum_{k=0}^{N-1}\left(\sum_{l=0}^{N-1} a_{k-l}b_l\right)\omega^{kj} = \frac{1}{\sqrt{N}}\sum_{l=0}^{N-1} a_l \sum_{k=0}^{N-1} b_{k-l}\omega^{(k-l)j}\cdot\omega^{lj} \stackrel{\text{def}}{=} \frac{1}{\sqrt{N}}\sum_{l=0}^{N-1} a_l\omega^{lj}\sum_{m=-l}^{N-1-l} b_m\omega^{mj}$.

We have already said that $\sum_{m=-l}^{N-1-l} b_m\omega^{mj} = \sum_{m=0}^{N-1} b_m\omega^{mj}$ /

Finally, we get the conclusion $\left(\widehat{a*b}\right)_j = \frac{1}{\sqrt{N}}\sum_{l=0}^{N-1} a_l\omega^{lj}\sum_{m=0}^{N-1} b_m\omega^{mj} = \hat{a}_j\cdot\hat{b}_j$.

Here is another property of DFT. We know $\hat{a}_j = \frac{1}{\sqrt{N}}\sum_{k=0}^{N-1} a_k$. And what happen if we do the DFT of $\hat{a}_j$? Let's compute $\hat{\hat{a}}_j$. Set $\delta(n) = \begin{cases}1 \; if\; n = 0\\ 0 \; if\; n \neq 0\end{cases}$ and from the definition there is $\hat{\hat{a}}_j = \frac{1}{\sqrt{N}}\sum_{k=0}^{N-1}\hat{a}_j\,\omega^{kj} = \frac{1}{N}\sum_{k=0}^{N-1}\sum_{l=0}^{N-1} a_l\,\omega^{kl}\omega^{kj} = \frac{1}{N}\sum_{l=0}^{N-1} a_l\sum_{k=0}^{N-1}\omega^{k(l+j)} = \frac{1}{N}\sum_{l=0}^{N-1} a_l\cdot N\delta(l+j) = \sum_{l=0}^{N-1} a_l = a_{-j}$. It is an interesting conclusion. For the initial sequence $a_0, a_1, a_2, \dots a_{N-1}$, if you do the DFT twice, you change the sequence to $a_0, a_{N-1}, a_{N-2}, \dots a_1$. The following graph shows the change in the position of ω on the circle after two DFT.

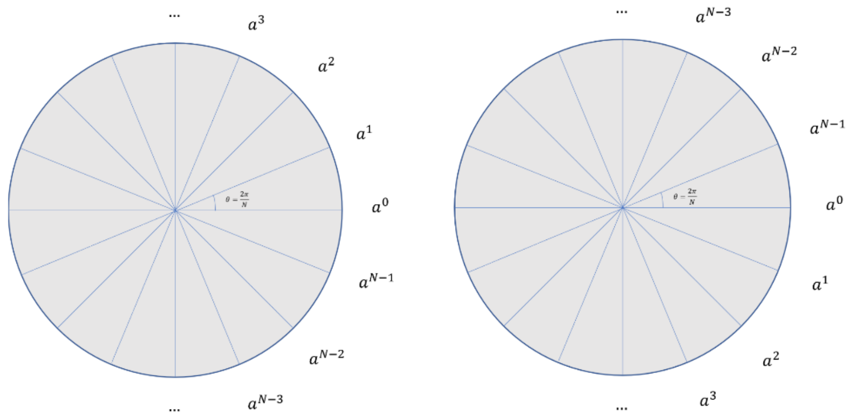

Figure 2. The circle of a_j and $\hat{\hat{a}}_j$

And if we do the DFT four time, the sequence doesn't change and still is $a_0, a_1, a_2, \dots a_{N-1}$. That is, $(DFT)^4 = Identity$.

Moral: as a linear transformation, DFT has the property that define $DFT\colon a_0, a_1, \dots a_{N-1} \to \widehat{a_0}, \widehat{a_1}, \dots, \widehat{a_{N-1}}$ ... then $DFT^4 = I$. Therefore, the only possible coefficients of DFT are $1, i, -1, -i$ ... ... This will be important in part 4 and now we know four important things:

1) inversion formula
2) $\langle\hat{a},\hat{b}\rangle=\langle a,b\rangle$
3) $(\widehat{a*b})_j = \hat{a}_j\cdot\hat{b}_j$
4) $\hat{\hat{\hat{\hat{a}}}} = a$

In addition, there is an another property: $(\widehat{c_1 a + c_2 b})_j = c_1\cdot\hat{a}_j + c_2\cdot\hat{b}_j$.

3. The θ on a circle

3.1. The changes

This is the second case, and we will replace the sequence $\{0,1,2,\dots,N-1\}$ by the circle $\theta \in [0,2\pi]$, like the red dot (line) in the figure below.

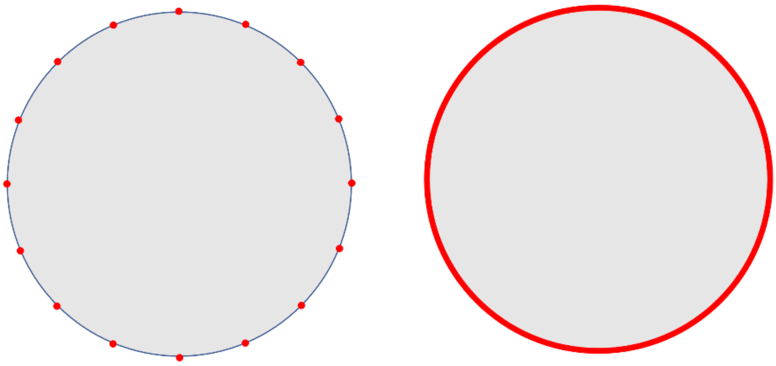

Figure 3. The point in sequence and circle

First let's see how the Fourier transform form changes. Define Fourier coefficients of $f(\theta)$:$c_j \coloneqq \frac{1}{2\pi}\int_0^{2\pi} f(\theta)\, e^{-ij\theta}\, d\theta$ where $j \in \mathbb{Z},\ \theta \in [0,2\pi]$.

This formula shows that the θ is define on a circle and the Fourier coefficients is a collection of values. And now there is a big question: can one recover $f(\theta)$ from the collection of its Fourier coefficients? The answer is YES, namely do the following thing:$f(\theta) \sim \sum_{j=-\infty}^{\infty} c_j\, e^{ij\theta}$

It is not "=" for several reasons.

1)The r.h.s is an infinite sum.

2)The formula may be easy to justify for some kind of functions $f(\theta)$ and harder to justify for others.

None of these problems arise when dealing with the DFT. So, we are talking as out Fourier Series.

3.2. Fourier Series

One difference with DFT is that physical space is different from frequency space. However, most of properties of the DFT hold in the case of Fourier Series. Now consider the inner product, the equality of $\langle \hat{a}, \hat{b} \rangle = \langle a, b \rangle$. For $f_1(\theta)$ and $f_2(\theta)$, define as before: $\langle f_1(\theta), f_2(\theta) \rangle \coloneqq \frac{1}{2\pi}\int_0^{2\pi} f_1(\theta)\, \overline{f_2(\theta)}\, d\theta = \frac{1}{2\pi}\int_0^{2\pi} \sum_{-\infty}^{\infty} c_j{}^{(1)} e^{ij\theta} \sum_{-\infty}^{\infty} \overline{c_k{}^{(2)}}\, e^{-ik\theta}\, d\theta = \frac{1}{2\pi}\sum_{j,k} c_j{}^{(1)}\, \overline{c_k{}^{(2)}} \int_0^{2\pi} e^{i(j-k)\theta}\, d\theta$

$$\text{In fact, } \frac{1}{2\pi}\int_0^{2\pi} e^{in\theta}\, d\theta = \begin{cases} 1 \text{ } if\ n = 0 \\ 0 \text{ } if\ n \neq 0 \end{cases} \tag{9}$$

where $n \in \mathbb{N}$. Applying equation (7) and set $\delta(n) = \begin{cases} 1 \text{ } if\ n = 0 \\ 0 \text{ } if\ n \neq 0 \end{cases}$, we can derive $\frac{1}{2\pi}\sum_{j,k} c_j{}^{(1)}\, \overline{c_k{}^{(2)}} \int_0^{2\pi} e^{i(j-k)\theta}\, d\theta = \sum_{j,k} c_j{}^{(1)}\, \overline{c_k{}^{(2)}}\, \delta(j-k) = \sum_{j=-\infty}^{\infty} c_j{}^{(1)}\overline{c_j{}^{(2)}} = \langle c^{(1)}, c^{(2)} \rangle$. Finally, the conclusion is $\langle f_1(\theta), f_2(\theta) \rangle = \langle c^{(1)}, c^{(2)} \rangle$. There is another special case. When $f_1(\theta) = f_2(\theta) = f(\theta)$, the inner product turns to be$\langle f_1(\theta), f_2(\theta) \rangle = \frac{1}{2\pi}\int_0^{2\pi} |f(\theta)|^2\, d\theta = \sum_{j=-\infty}^{\infty} \left|c_j\right|^2$ where $f(\theta) = \sum_{j=-\infty}^{\infty} c_j\, e^{ij\theta}$. It is not a good idea to try this integral because it is very hard to solve exactly, but now we know it is equal to the equation on the right finite summation. This famous formula is called Parseval identity.

Back to the main question we will see 3 different ways of making sense of $\sum_{j=-\infty}^{\infty} c_j\, e^{ij\theta}$. Eventually we will see a 4th way. Now let us see a little bit history.

3.3. Three ways to deal with Fourier Series

In this section, we study various other methods of Fourier series summation of function. In 1832, J.Fourier tried to solve something[1]. This will be repeatedly mentioned later in the heat equation, and we will be using the word "filters" in electrical engineering.

a) defined

$$(S_N f)(\theta) \coloneqq \sum_{j=-N}^{N} c_j\, e^{ij\theta} \tag{10}$$

and picked N=0,1,2,3…

That means, in the frequency space, we have all the coefficients c_j, and multiply all the Fourier coefficients by the function

$$Filter = \begin{cases} 1, |j| \leq N \\ 0, |j| > N \end{cases} \tag{11}$$

In another words, ignore the figure when $j > N$ or $j \leq -N$, Keeping only the middle part.

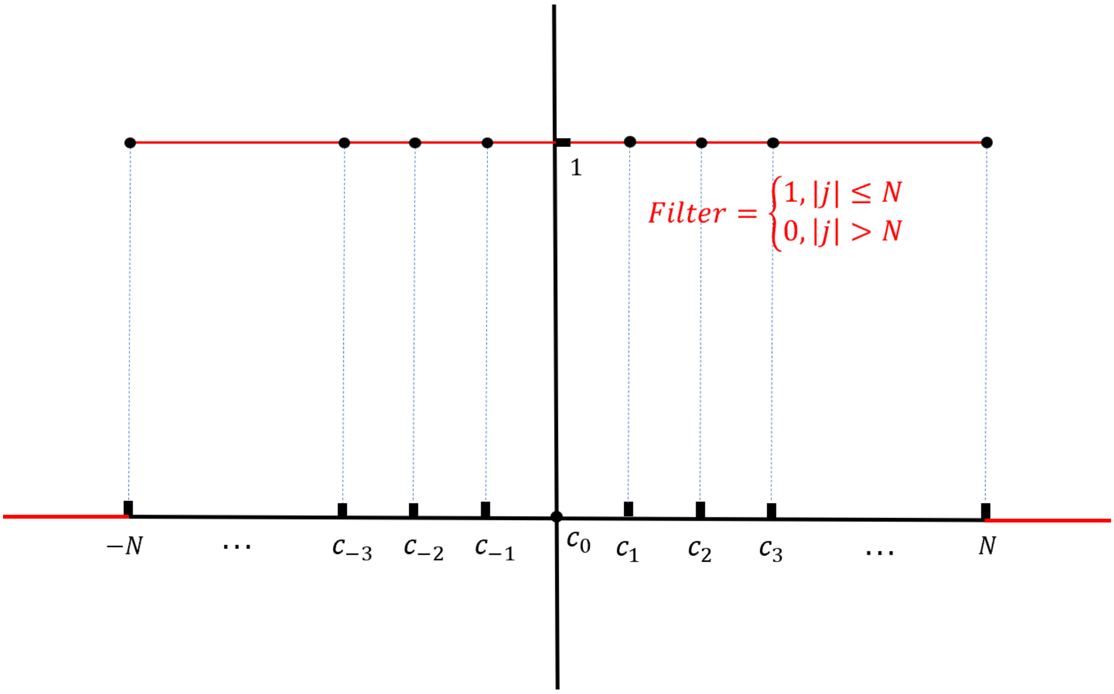

Fig. 4 The filter in frequency space in $(S_N f)(\theta)$

b) Define $(F_N f)(\theta) \coloneqq \frac{(S_0 f + S_1 f + S_2 f + \cdots S_N f)(\theta)}{N+1}$

and picked N=0,1,2,3….

This means that the averaging is done for $(S_N f)(\theta)$. By some simply calculation we can know if $j \in [-N-1, N+1]$, $filter = 1$ when $j = 0$, $filter = 0$ when $j = N+1$ and $Filter = \begin{cases} 0, j < -N-1 \\ 1 + \frac{j}{N+1}, -N-1 \leq j \leq 0 \\ 1 - \frac{j}{N+1}, 0 < j < N+1 \\ 0, j \geq N+1 \end{cases}$. Now, it is time to look how does the filter look in frequency space.

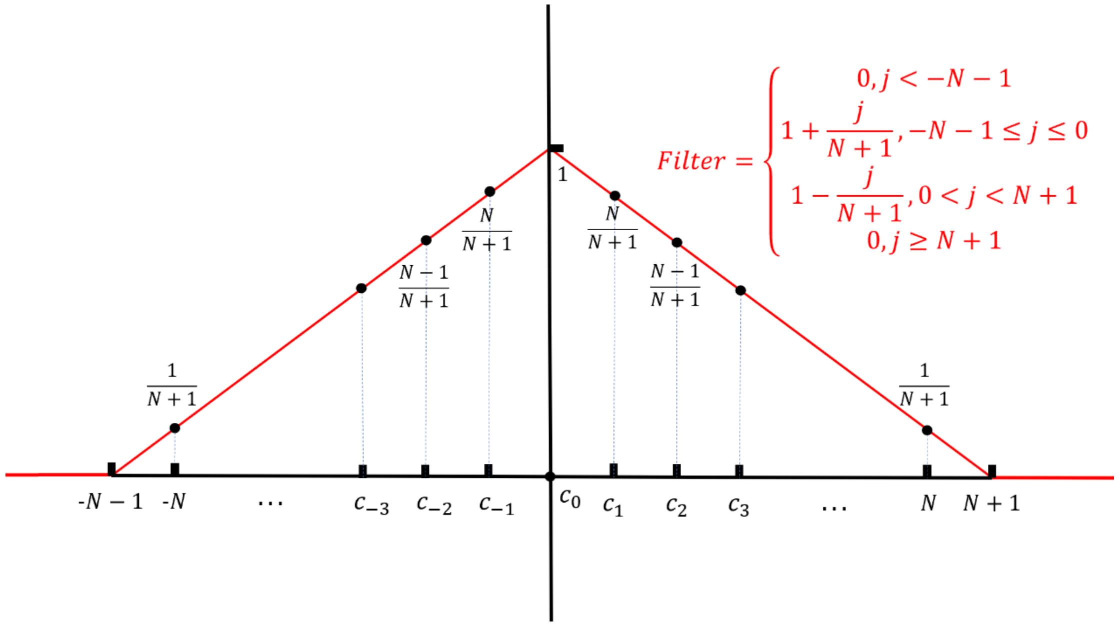

Fig. 5 The filter in frequency space in $(F_N f)(\theta)$

c) Defined $(P_r f)(\theta) := \sum_{-\infty}^{\infty} c_j r^{|j|} e^{ij\theta}$

and picked N=0,1,2,3….,$0 < r < 1$.

This equation shows that the summation term in xxx is decayed according to the exponential, with a decay factor of r. After some calculation we can know $filter = 1$ when $j = 0$, $filter = r^n$ when $|j| = n$, and $Filter = r^{|j|}$. So it is important to see how does the filter look in frequency space.

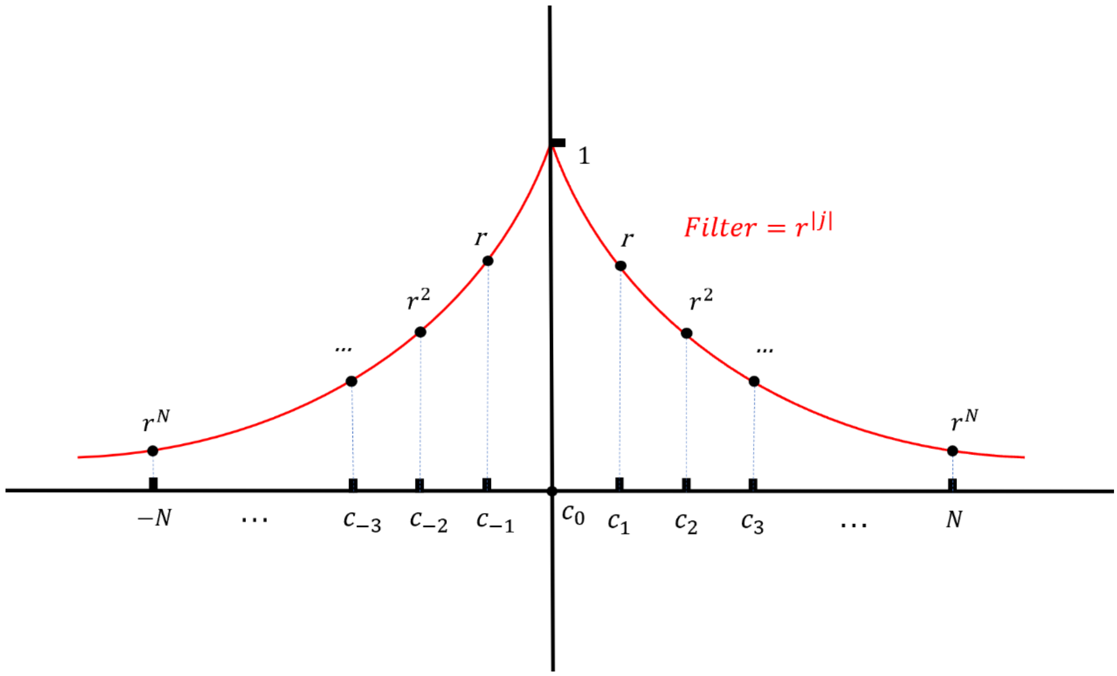

Fig. 6 The filter in frequency space in $(P_N f)(\theta)$

From the graph we can see that when $|j|$ becomes large, $Filter$ converges to 0 extremely quickly.

Now there is a very important question: what can we say about convergence either when $N \to \infty$ or $r \to 1$. Let's look at the conclusion here.

Let us start with the 3 filter $(P_r f)(\theta)$. If $f(\theta)$ is continuous as a function on the circle and in particular case $f(\theta) = f(2\pi)$, then $\lim_{r\to 1}(P_r f)(\theta) \to f(\theta)$ uniformly in θ. That means $\max_{0\le\theta\le 2\pi} |(P_r f)(\theta) - f(\theta)| \to 0$ as $r \to 1$.

The next is the 2nd filter $(F_N f)(\theta)$. Under the same condition, $f(\theta)$ is continuous, then $\max_{0\le\theta\le 2\pi} |(P_r f)(\theta) - f(\theta)| \to 0$ as $N \to \infty$.

The last is the 1st filter $(S_N f)(\theta)$. Under the same condition that $f'(\theta)$ exists at every points. One proved that $\max_{0\le\theta\le 2\pi} |(S_r f)(\theta) - f(\theta)| \to 0$ as $N \to \infty$.

The process of proving the above theorems is not very difficult, but here we omit them.

Now come back to the basic question: how to simplify $\sum_{j=-\infty}^{\infty} c_j\, e^{ij\theta}$ by these three ways.To beginning with, from Fourier Series we can know, if we have a function $f(\theta)$ where $\theta \in (0,2\pi)$, there is no need to put any other conditions and then we can get

$$f(\theta) \sim \sum_{-\infty}^{\infty} c_j e^{ij\theta} \tag{12}$$

Which the Fourier coefficient

$$c_j := \int_0^{2\pi} f(\theta)\, e^{-ij\theta} d\theta \tag{13}$$

And $i = \sqrt{-1}$. When in a circle, we can rewrite formula (11) to

$$c_j = \int_a^{a+2\pi} f(\theta) * e^{-ij\theta} d\theta \tag{14}$$

From formula (10), why we use "~" instead of "="? There are one of the reasons as follow: It is "danger" to do an infinite summation. To avoid this problem, we define

$$(S_N f)(\theta) := \sum_{-N}^{N} c_j e^{ij\theta} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \tag{15}$$

And hope that when $N \to \infty$, then $(S_N f)(\theta) \to f(\theta)$. We also define

$$(F_N f)(\theta) := \frac{(S_0 f + S_1 f + S_2 f + \cdots S_N f)(\theta)}{N+1} \ldots\ldots\ldots\ldots\ldots\ldots\ldots \tag{16}$$

And hope that when $N \to \infty$, then $(F_N f)(\theta) \to f(\theta)$. For the third way, we define

$$(P_r f)(\theta) := \sum_{-\infty}^{\infty} c_j r^{|j|} e^{ij\theta} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \tag{17}$$

And hope that when $r \to 1$, then $(P_r f)(\theta) \to f(\theta)$.In addition, there is a property of c_j:

If

$$f(\theta) \sim \sum_{-\infty}^{\infty} c_j^{(f)} e^{ij\theta} \tag{18}$$

$$g(\theta) \sim \sum_{-\infty}^{\infty} c_j^{(g)} e^{ij\theta} \tag{19}$$

And define

$$h(\theta) = \frac{1}{2\pi} \int_0^{2\pi} f(t) g(\theta - t) dt \tag{20}$$

Finally, the $h(\theta)$ will become

$$h(\theta) \sim \sum_{-\infty}^{\infty} c_j^{(f)} c_j^{(g)} e^{ij\theta} \tag{21}$$

Hence, we can get the conclusion:

If

$$h(\theta) \sim \sum_{-\infty}^{\infty} c_j^{(h)} e^{ij\theta} \tag{22}$$

Then $c_j^{(h)} = c_j^{(f)} c_j^{(g)}$.

This conclusion means that the convolution on the circle is the same as multiplication on the side of the Fourier coefficients. Now, it is time to come back to our problems. It has been said that one problem which need to be solved is a finite summation like

$$\sum_{-N}^{N} e^{in\theta} \tag{23}$$

Which equal to

$$e^{in\theta}\frac{1-e^{i\theta(2N-1)}}{1-e^{i\theta}} \tag{24}$$

since

$$\sum_{j=0}^{N} a^j = \frac{1-a^{N+1}}{1-a} \tag{25}$$

and

$$sin\alpha = \frac{e^{i\alpha}-e^{-i\alpha}}{2i} \tag{26}$$

Finally, the result is

$$\sum_{-N}^{N} e^{in\theta} = \frac{\sin(2N+1)\frac{\theta}{2}}{\sin\frac{\theta}{2}} \tag{27}$$

After finished this question, let us pay attention to the summation

$$\sum_{-\infty}^{\infty} r^{|j|}e^{ij\theta} \tag{28}$$

By some complex calculation process, including substitution method and after using formula (17)(18), we finally get

$$\sum_{-\infty}^{\infty} r^{|j|}e^{ij\theta} = \frac{1-r^2}{1-2r\cos\theta+r^2} := P_r(\theta) \tag{29}$$

It is simply to see the Fourier coefficient of $P_r(\theta)$ is $r^{|j|}$.

Now, it is time to give a conclusion. Firstly, we have three formulas.

The first is

$$f(\theta) \sim \sum_{-\infty}^{\infty} c_j e^{ij\theta} \tag{30}$$

which the Fourier coefficient is c_j.

The second is

$$P_r(\theta) = \sum_{-\infty}^{\infty} r^{|j|}e^{ij\theta} \tag{31}$$

which the Fourier coefficient is $r^{|j|}$.

The last is

$$(P_r f)(\theta) = \sum_{-\infty}^{\infty} c_j r^{|j|}e^{ij\theta} \tag{32}$$

which the Fourier coefficient is $c_j r^{|j|}$.

We have also said that “the convolution on the circle is the same as multiplication on the side of the Fourier coefficients”.

So finally, we can get

$$(P_r f)(\theta) = \frac{1}{2\pi}\int_0^{2\pi} f(t)P_r(\theta-t)dt \tag{33}$$

This means if we want to deal with $P_r f(\theta)$, we just need to focus on $P_r(\theta)$. Next, it is necessary to pay attention to $P_r(\theta)$.

To begin with, to see more clearly, the MATLAB has been used to draw the picture of $P_r(\theta)$ when r is equal to 0.3, 0.5 and 0.99 respectively. The x is limited to $(-\pi,\pi)$.

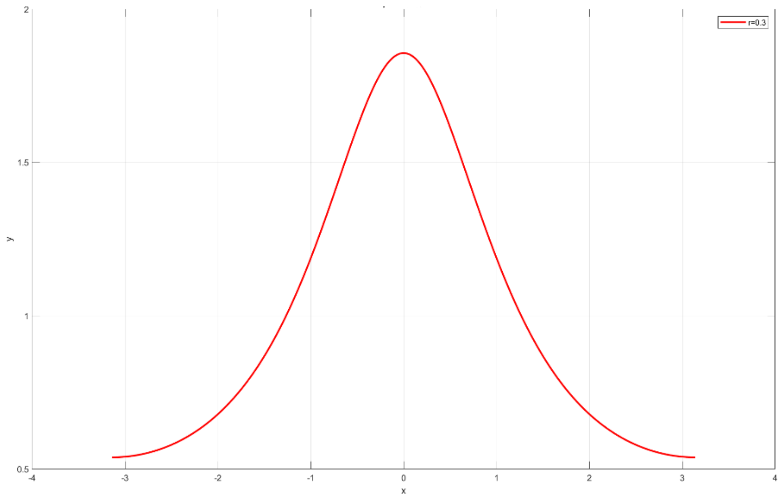

Fig. 7 r=0.3

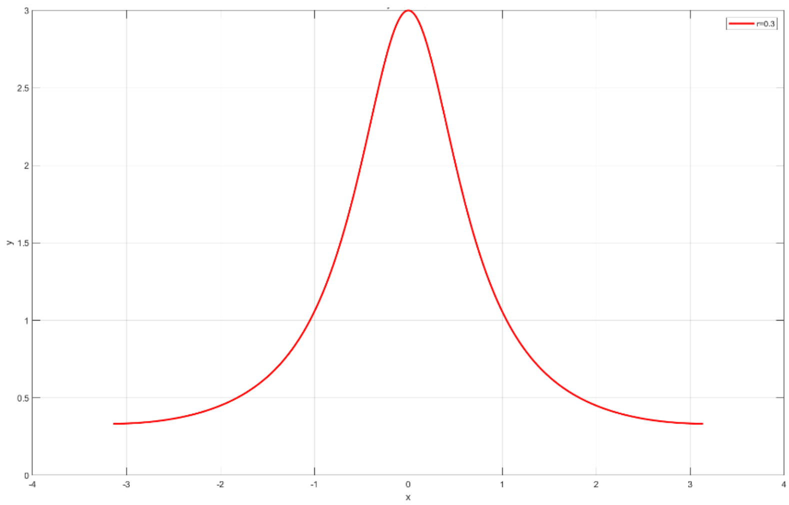

Fig. 8 r=0.5

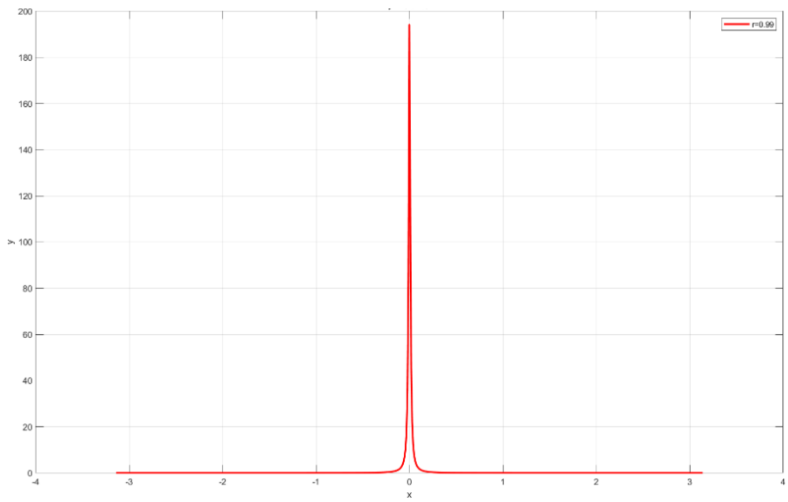

Fig. 9 r=0.99

It is clearly to see the x takes the minimum value at -π and π, and by calculation we can get the value is $\frac{1-r}{1+r}$. Similarly, we can see the x takes the maximum value at 0 where the x equal to $\frac{1+r}{1-r}$. From the formulas and these figures, or by some simply calculate, we can get some other properties of $P_r(\theta)$.

1. $P_r(\theta) > 0$.

2. $P_r(-\theta) = P_r(\theta)$.
3. $P_r(\theta) \uparrow$ for $x \in (-\pi, 0)$ and $P_r(\theta) \downarrow$ for $x \in (0, \pi)$.
4. $\frac{1}{2\pi}\int_{-\pi}^{\pi} P_r(\theta)\, d\theta = 1$.(use the series to prove)
5. For a fixed $\theta \neq 0$, $\lim_{r\to 1} p_r(\theta) = 0$

And the property 5 can be used to prove the follow formula:

$$\lim_{r\to 1} |(P_r f)(\theta) - f(\theta)| = 0 \tag{34}$$

Here we just give a simple proof. We have the conclusion

$$(P_r f)(\theta) = \frac{1}{2\pi}\int_0^{2\pi} f(t) P_r(\theta - t) dt \tag{35}$$

and

$$f(\theta) = \frac{1}{2\pi}\int_0^{2\pi} f(\theta) d\theta \tag{36}$$

So

$$\begin{aligned}(P_r f)(\theta) - f(\theta) &= \frac{1}{2\pi}\int_0^{2\pi} f(t) P_r(\theta - t) dt - \frac{1}{2\pi}\int_0^{2\pi} f(\theta) d\theta \\ &= \frac{1}{2\pi}\int_0^{2\pi} [f(t) - f(\theta)] P_r(\theta - t) dt \\ &= \frac{1}{2\pi}\int_{\theta-\pi}^{\theta+\pi} [f(t) - f(\theta)] P_r(\theta - t) dt\end{aligned} \tag{37}$$

As $f(\theta)$ is continuous, for any $\varepsilon > 0$, we can find $\delta = \delta(\varepsilon) > 0$ such that $|f(t) - f(\theta)| < \frac{\varepsilon}{2}$ if $|t - \theta| < \delta$. Then

$$|(P_r f)(\theta) - f(\theta)| = |\frac{1}{2\pi}\int_{\theta \in circle} [f(t) - f(\theta)] P_r(\theta - t) dt|$$

$$= \left|\frac{1}{2\pi}\left\{\int_{\theta \in circle, |t-\theta| \le \delta(\varepsilon)} [f(t) - f(\theta)] P_r(\theta - t) dt + \int_{\theta \in circle, |t-\theta| > \delta(\varepsilon)} [f(t) - f(\theta)] P_r(\theta - t) dt\right\}\right|$$

$$\le \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \tag{38}$$

Finally, it showed that for any given $\varepsilon > 0$, we can find r close enough to 1 which means $p_r(\theta)$ close enough to 0 such that $|(P_r f)(\theta) - f(\theta)| \le \varepsilon$. Then the formula $\lim_{r\to 1} |(P_r f)(\theta) - f(\theta)| = 0$ has been proved with the conditions that θ is fixed and $f(\theta)$ is continuous. And in the last, there is a formula without proof:

$$\lim_{r\to 1} P_r(\theta) = f(\theta) \tag{39}$$

4. The θ on real line (Fourier Transforml)

First topic is Gibbs phenomena. Last part we discussed that if there is a function $f(\theta)$ where $\theta \in (0, 2\pi)$, and $f(\theta)$ is continuous and differentiable, at a point θ_0 we can prove that $\lim_{r\to 1} (S_r f)(\theta_0) = f(\theta)$. Then the question is, what will happen when $f(\theta)$ is discontinuous at θ_0? We define a function which is not continuous at $\theta_0 = 0$:

$$y = f(\theta) = \begin{cases} -1, x \in (-\pi, 0) \\ 0, x = 0 \\ 1, x \in (0, \pi) \end{cases} \tag{40}$$

The MATLAB is used to draw the picture of the function.

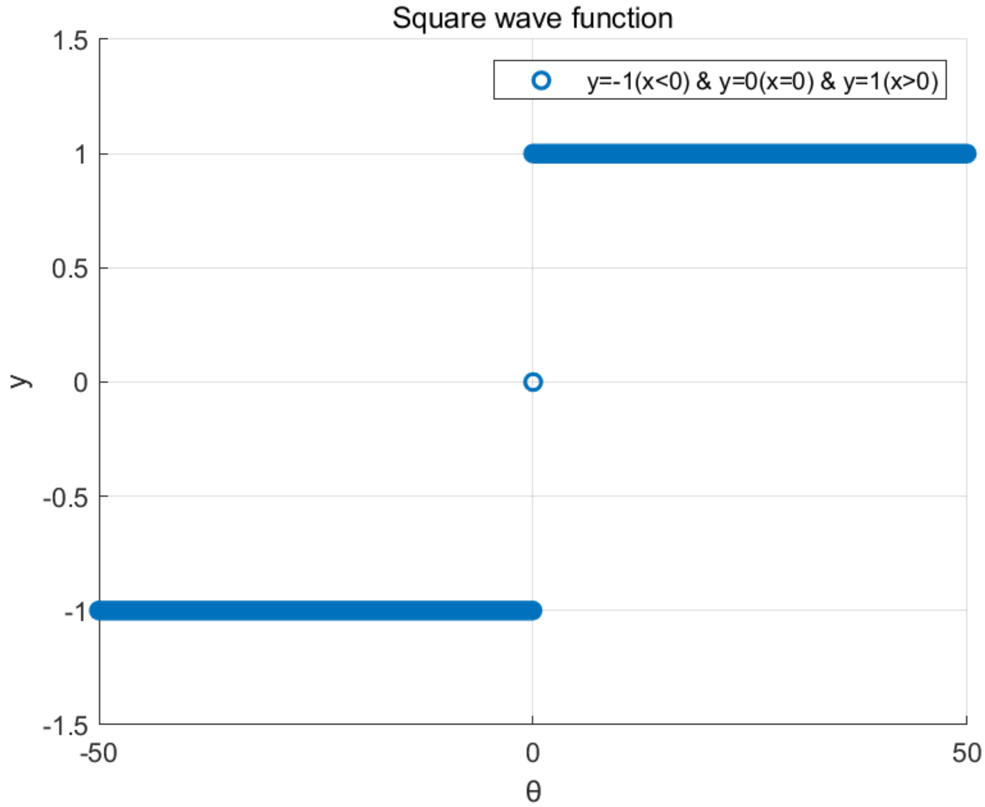

Fig. 10 Square wave function

Now the Fourier coefficient is

$$c_n = \int_{-\pi}^{\pi} f(\theta)\, e^{-ij\theta} d\theta \tag{41}$$

and

$$(S_N f)(\theta) = \sum_{-N}^{N} c_n e^{in\theta} \tag{42}$$

With some complex calculation process, we finally can get

$$(S_N f)(\theta) = \sum_{-N}^{N} c_n e^{in\theta} = \frac{2}{\pi} \int_0^{\frac{2N+1}{2}\theta} \frac{\sin\tau}{\tau} d\tau \tag{43}$$

To see more clearly, we use the MATLAB to get the picture when N is equal to 50 respectively which includes original function, integral function and objective function image.

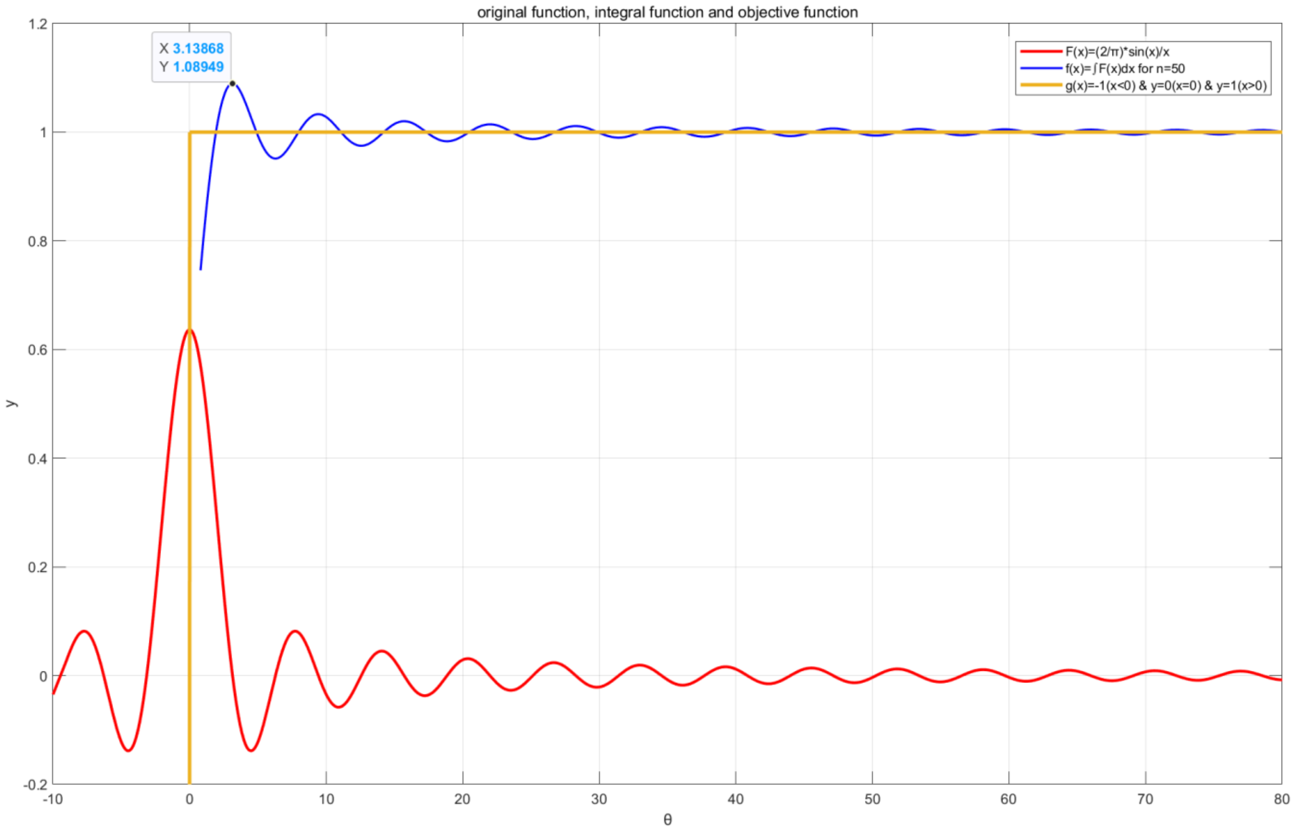

Fig. 11 three functions

By MATLAB we can know the maximum of $(S_N f)(\theta)$ which exactly the maximum of the integral function of $f(\theta)$ equal to 1.089490…It is meaning that there always exist a 9% error. When $f(\theta)$ take the maximum value, θ is close to 0. Actually, from the function (5) we can see the bigger N is, the closer to 0 θ is. It is easy to know if $N \to \infty$, $\theta \to 0$. Hence, $(S_N f)(\theta) \to f(\theta)$ is true. That is all about Gibbs phenomena.

Now let us start the part of Fourier Integral.

If we have a function $f(\theta)$ where $\theta \in R$, define

$$\hat{f}(\lambda) \coloneqq \int_{-\infty}^{\infty} f(x) e^{-i\lambda x} dx \tag{44}$$

and it is simply to know

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{f}(\lambda) e^{i\lambda x} dx \tag{45}$$

It can also be written by

$$\begin{cases} \hat{f}(\lambda) = \int_{-\infty}^{\infty} f(x) e^{-2\pi i\lambda x} dx \\ f(x) = \int_{-\infty}^{\infty} \hat{f}(\lambda) e2\pi^{i\lambda x} dx \end{cases} \tag{46}$$

There are two vector spaces should be stated.

1. First is $L^2(R)$. If $f(x) \in L^2(R)$, it means

$$\int_{-\infty}^{\infty} |f(x)|^2 dx < \infty \tag{47}$$

And $f(x) \in L^2(R) \Leftrightarrow \hat{f}(x) \in L^2(R)$.

2. Second is $C_{\downarrow}^{\infty}$. $f(x) \in C_{\downarrow}^{\infty}$ if and only if $f(x)$ satisfies following two conditions:

(1) $f^{(n)}(x)$ exist for $n = 1,2,3 \ldots$

(2) $\lim\limits_{x \to \pm\infty} |x^p f^{(n)}(x)| = 0$ when $n = 1,2,3 \ldots, p = 1,2,3 \ldots$

e.g., $e^{-x^2} \in C_{\downarrow}^{\infty}$ and $e^{-|x|} \notin C_{\downarrow}^{\infty}$.

Similarly, $f(x) \in C_{\downarrow}^{\infty} \Leftrightarrow \hat{f}(x) \in C_{\downarrow}^{\infty}$. Actually, $C_{\downarrow}^{\infty} \subset L^2(R)$.

If there is a map from $L^2(R)$ to $L^2(R)$(or from $C_{\downarrow}^{\infty}$ to $C_{\downarrow}^{\infty}$)by Fourier Transform, then it is one to one and onto.After have known these two spaces, let see some details of Fourier Transform.

1. What is the F.T. of f'?

With a simply integral we can get the answer: If $f \in C_{\downarrow}^{\infty}$, then $\widehat{f^{(n)}} = (i\lambda)^n \hat{f}$. This property can help us solve some problems. Here is an example.

Question: *Show that the solution to the differential equation*

$$U''(x) - U(x) = -f(x) \tag{48}$$

is given by

$$U(x) = \frac{1}{2}\int_{-\infty}^{\infty} e^{-|x-y|} f(y)dy \tag{49}$$

Proof Set $g(x) = e^{-|x|}$, *then*

$$U(x) = \frac{1}{2}\int_{-\infty}^{\infty} e^{-|x-y|} f(y)dy = \frac{1}{2} \cdot f(x) * g(x) \tag{50}$$

Because $\widehat{f(x) * g(x)} = \widehat{f(x)} \cdot \widehat{g(x)}$, *it is simply to see*

$$\widehat{U(x)} = \frac{1}{2} \cdot \widehat{f(x) * g(x)} = \frac{1}{2} \cdot \widehat{f(x)} \cdot \widehat{g(x)} = \frac{1}{2} \cdot \widehat{f(x)} \cdot \widehat{e^{-|x|}} \tag{51}$$

Since

$$\widehat{e^{-|x|}} = \hat{f}(\lambda) := \int_{-\infty}^{\infty} e^{-|x|} e^{-i\lambda x} dx = \int_{-\infty}^{0} e^{-x-i\lambda x} dx + \int_{-\infty}^{0} e^{x-i\lambda x} dx$$
$$= \frac{1}{1-\lambda i}(1-0) + \frac{1}{-1-\lambda i}(0-1) = \frac{2}{1+\lambda^2} \tag{52}$$

Substitute equation (66), the result will be $\widehat{U(x)} = \frac{\widehat{f(x)}}{1+\lambda^2}$. *i.e.,* $\lambda^2 \cdot \widehat{U(x)} + \widehat{U(x)} = f(x)$. *Now the formula* $\widehat{f^{(n)}} = (i\lambda)^n \hat{f}$ *can be used where* $n = 2$, *i.e.,* $\widehat{U''(x)} = -\lambda^2 \widehat{U(x)}$. *Finally, the above equation turns to be* $U''(x) - U(x) = -f(x)$ *which is the original equation. That result tells us that* $U(x) = \frac{1}{2}\int_{-\infty}^{\infty} e^{-|x-y|} f(y)dy$ *is the solution of* $U''(x) - U(x) = -f(x)$.

From this example we can see that the use of this property in the Fourier transform can be used wisely to simplify the calculation。

2. Parseval Identity:

$$\int_{-\infty}^{\infty} \widehat{f_1}(\lambda)\overline{\widehat{f_2}(\lambda)} d\lambda = \int_{-\infty}^{\infty} f_1(\lambda)\overline{f_2(\lambda)} d\lambda \tag{53}$$

by the help of $\delta(a) := \int_{-\infty}^{\infty} e^{-i\lambda a} d\lambda$ and

$$\int_{-\infty}^{\infty} g(a)[\int_{-\infty}^{\infty} e^{-i\lambda a} d\lambda]\, da = g(0) \tag{54}$$

It is means that $< \widehat{f_1}(\lambda), \widehat{f_2}(\lambda) > = < f_1(\lambda), f_2(\lambda) >$. i.e., Parseval Identity.

3. Some important examples:

(1) Set $f(x) = e^{-\frac{x^2}{\sigma}}$ where $\sigma > 0$. It is easy to know $\hat{f}(x) = \sqrt{\pi}\sigma \cdot e^{-\frac{\lambda^2\sigma^2}{4}}$ The MATLAB has been used to draw the picture of $f(x)$ when σ is equal to 0.25 and 25 respectively. The x is limited to $(-50,50)$.

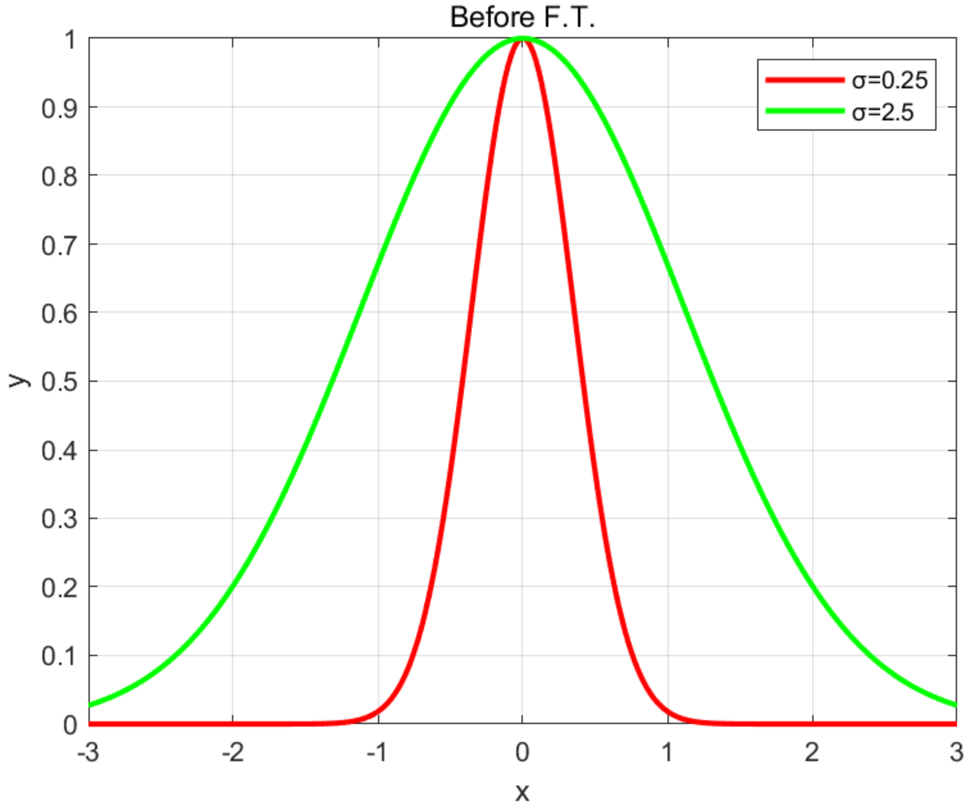

Figure. 12.before F.T.

After the F.T., the image turned to be the opposite result:

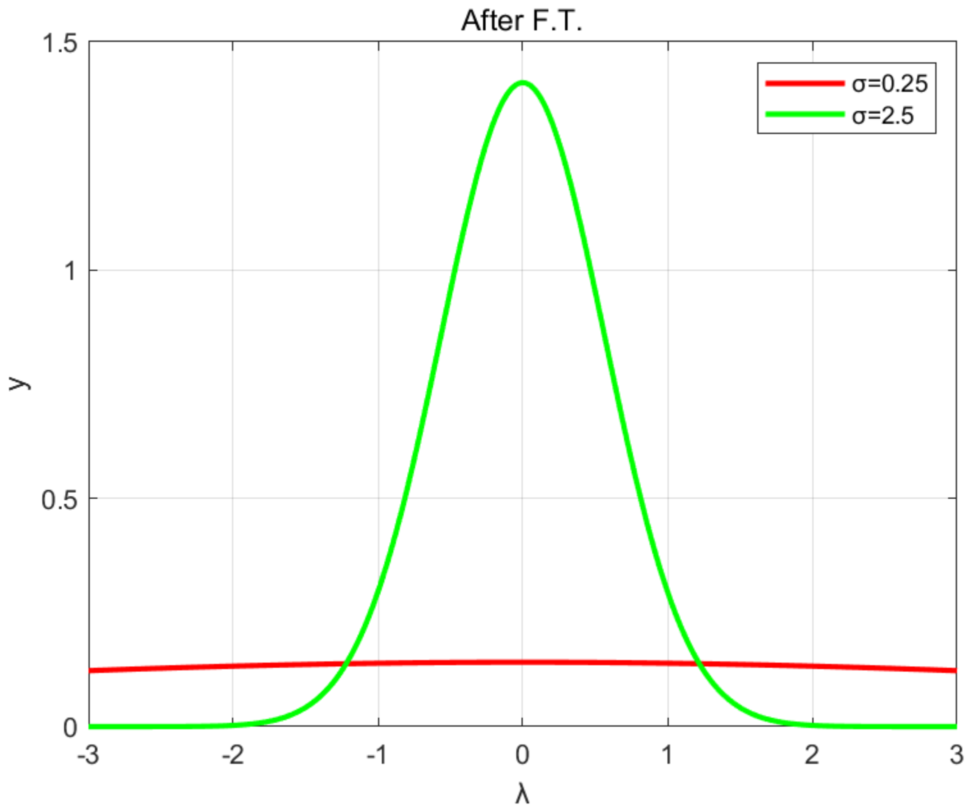

Figure. 13.after F.T.

(2) Heisenberg Principle:

For any $f(x) \in C_{\downarrow}^{\infty}$, the product

$$\left(\int_{-\infty}^{\infty} x^2|f(x)|^2 dx\right) * \left(\int_{-\infty}^{\infty} \lambda^2\left|\hat{f}(\lambda)\right|^2 d\lambda\right) \geq \frac{1}{4} \tag{55}$$

(3) Schwarz inequality

$$< f, g \geq \int_{-\infty}^{\infty} f(x)\overline{g(x)}dx$$
$$\leq \left(\int_{-\infty}^{\infty} |f(x)|^2 dx\right)^{\prime\prime 2} * \left(\int_{-\infty}^{\infty} |g(x)|^2 dx\right)^{\prime\prime 2}$$
$$= \left||f|\right| * \left||g|\right| \tag{56}$$

4. Heat equation

If we have a Heat equation $U(x,t)$ where $z \in R,\ t > 0$, and it satisfies $U_t = U_{xx}$. Define

$$\widehat{U}(\lambda, t) = \int_{-\infty}^{\infty} U(x,t)e^{-ijx}dx \tag{57}$$

And

$$\widehat{U}_t = \widehat{U_t} = \int_{-\infty}^{\infty} U_t(x,t)e^{-ijx}dx$$
$$= \int_{-\infty}^{\infty} U_{xx}(x,t)e^{-ijx}dx = (i\lambda)^2\widehat{U} \tag{58}$$

This is an ODE

$$\frac{d}{dt}\widehat{U}(\lambda,t) = -\lambda^2\widehat{U}(\lambda,t) \tag{59}$$

Just solved it and the answer is $\widehat{U}(\lambda,t) = e^{-\lambda^2 t}\widehat{U}(\lambda,0)$, and it is fact that $\widehat{f_1 * f_2} = \widehat{f_1}\widehat{f_2}$ where "*" means convolution. By the calculation above, the result is

$$U(\lambda,t) = \widetilde{e^{-\lambda^2 t}} * \ U(x,0) \tag{60}$$

i.e.

$$U(x,t) = \frac{1}{\sqrt{4\pi t}}\int_{-\infty}^{\infty} e^{\frac{-\lambda^2}{2t}}U(x,0)dy \tag{61}$$

5. Conclusion

In this study, the Fourier analysis was divided into three interrelated parts. In the first part, DFT, the basic transformation laws are described, the properties of inner product and convolution are derived and extended, and the properties of even-order DFT, namely "symmetry" and "reduction", are explored. In the second part, we explain that the use of "~" instead of "=" is related to infinite summation, compare the Fourier series with the inner product and convolution in DFT, and finally derive the Parseval identity, and introduce the three treatments of J Fourier's three treatments. After deeply studying their fitting properties, it is concluded that they converge to the objective function and are compared through images. In the Fourier transform, the study looks at its application and analyzes the error of its simulation of discontinuous functions, obtaining through detailed calculations that the simulation of a square wave will show an error of about 9%. Some other properties of the Fourier transform are illustrated by examples, such as a demonstration of its use for solving heat conduction problems, showing its power to simplify operations.

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