The Introductory Analysis on Fourier Series

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Abstract. Numerical analysis benefits from spectral methods. The Fourier series and Chebyshev polynomials were initially used to solve ordinary differential equations by Cornelius Lanczos. For the first time, spectral approaches were employed by Kreiss, Oliger, Orszag, and other scholars in the 1970s to resolve the partial differential equations of fluid mechanics. In Fourier analysis, general functions are expressed as the sums of a variety of fundamental function options, such as sines and cosines.This paper gives an introductory analysis on Fourier Series. Starting with the basic definition of Fourier coefficients and Fourier Series, the paper introduces the concept and properties of convolution. It also talks about different kernels and good kernel family, analyzing the relationship between these kernels and the original function. Then the paper will combine these ideas and prove the uniqueness and convergence of Fourier Series.

Keywords: Fourier coefficient; Fourier Series; good kernel; Dirichlet kernel; Fejer kernel; Poisson kernel; convolution; Cesaro mean; Abel mean.

1. Introduction

1.1. History and application of Fourier Transform

Have you ever noticed that pressing the number keys on your phone during a call results in a distinct sound for every phone model? This is due to the fact that each button is made up of two distinct sinusoids, which may be used to distinguish one button from another. The other person can determine what keys you hit when using your phone to navigate a menu by doing a Fourier transform on the input and analyzing the frequencies. Physics problems frequently include oscillations and vibrations. Usually, the oscillatory motion is straightforward and may be expressed as a single sine or cosine function (e.g., weights on springs, pendulums, harmonic waves, etc.). However, the wave forms are not always straightforward and, unlike sines and cosines, can be challenging to address analytically in many situations (electromagnetism, heat conduction, quantum theory, etc.) A wave varies its speed as it passes through a heterogeneous medium in response to variations in the rate of wave propagation in the medium. So one may determine the extra time delay, which in turn informs you how much the wave speed has changed in the medium, by detecting a shift in phase between what is predicted and what is measured. Of course, this is a pretty basic explanation for laypeople, yet it serves as the foundation for tomography.Scientists have a set of effective tools for describing any periodic function as the sum of sines and cosines thanks to Fourier techniques. The idea of Fourier Series was first introduced in a classical Physics problem to solve the heat equation. Discussing about the solution of this problem, Mathematician Joseph Fourier pointed out that every function can be written as combinations of trigonometric functions. And from then on, the study on Fourier analysis began [1-21]. Fourier Analysis is a useful tool to solve the differential equation. It is also broadly applied in the fields of signal processing, image processing .etc.

1.2. Constrains on functions

Before talking about Fourier Series, it is necessary to put some constrains on functions that one might analyze later. Below are some types of functions that will basically talk about:

Everywhere continuous function: continuous at every point on the interval.

Piecewise continuous function: functions only have finite discontinuities on the interval

Riemann integrable function: a function on the interval [0, L] is Riemann integrable if it is bounded and the upper and lower sum of function converge to some number F, that is, for every $\varepsilon > 0$, there is $0 = x_0 < x_1 < \cdots < x_N = L$, so that the lower sum: $U = \sum_{j=1}^{N}[sup_{xj-1 \leq x \leq xj} f(x)](x_j - x_{j-1})$, and the upper sum: $\mathcal{L} = \sum_{j=1}^{N}[inf_{xj-1 \leq x \leq xj} f(x)](x_j - x_{j-1})$ satisfy that $U - \mathcal{L} < \varepsilon$. After talking about types of functions that will discuss later, some properties of function on circle are also necessary to be introduced. Each point on a unit circle can be represented as $e^{i\theta}$, if F is a function on the circle, define function f on the real line such that: $f(\theta) = F(e^{i\theta})$, one can get $f(\theta + 2\pi) = f(\theta)$(periodicity). Notice that functions on a real line that is 2π periodic and the functions on a real line that has interval [0, 2π] which have same value at two end-points, are the same description to illustrate functions on a circle. With these understandings, the formal definition can be given:If f is an integrable function on interval [a, b] of length L, then Fourier coefficients of nth element is: $a_n = \frac{1}{L}\int_a^b f(x)e^{-\frac{2\pi inx}{L}}dx,\ n \in \mathbb{Z}$. The function f is: $f(x) \sim \sum_{n=-\infty}^{\infty} a_n\, e^{\frac{2\pi inx}{L}}$. Specifically, if the function is on the interval [-π,π], then $a_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-inx}dx,\ n \in \mathbb{Z}$ and the Fourier Series is $f(x) \sim \sum_{n=-\infty}^{\infty} a_n\, e^{inx}$. Notice that Fourier Series is a member of trigonometric series which satisfy the form $\sum_{n=-\infty}^{\infty} c_n\, e^{\frac{2\pi inx}{L}},\ c \in \mathbb{C}$.

2. Uniqueness of Fourier Series

One meaningful discussion about Fourier Series is its uniqueness, that is, whether a function can be uniquely determined by its Fourier coefficients. Elaborating on this topic, below are some insightful conclusions that can be drawn. If f is an integrable function on the circle with $\hat{f}(n) = 0$ for all $n \in \mathbb{Z}$, then $f(\theta_0) = 0$ whenever f is continuous at θ_0 [1]. To proof this theorem, suppose there is a function f that satisfies the assumption of $\hat{f}(n) = 0$ for all $n \in \mathbb{Z}$ on the interval $[-\pi, \pi]$. Also, when $\theta_0 = 0,\ f(\theta_0) > 0$. If this type of function cannot exist, then the theorem is proved by contradiction. By assumption, f is continuous at $\theta = 0$. Thus, one can pick $0 < \delta < \pi/2$ satisfying the condition: $f(\theta) > \frac{f(0)}{2}$ if $|\theta| < \delta$. Set $p(\theta) = \epsilon + \cos\theta$. Choose positive ϵ to be small enough that $|p(\theta)| < 1 - \frac{\epsilon}{2}$ when $\delta \leq |\theta| \leq \pi$. Then choose positive $\eta, \eta < \delta$ such that $p(\theta) \geq 1 + \frac{\epsilon}{2}$ for $\eta < |\theta|$. Since f is integrable, it is also bounded. One can choose B such that $|f(\theta)| \leq B$ for all θ. Set $p_k(\theta) = [\,p(\theta)]^k$, then $p_k(\theta)$ is trigonometric polynomial. Because $\hat{f}(n) = 0$ for $n \in \mathbb{Z}$, $\int_{-\pi}^{\pi} f(\theta)\,[\,p(\theta)]^k d\theta = 0$ for all k.

But, $\left|\int_{\delta \leq |\theta|} f(\theta)\,[\,p(\theta)]^k d\theta\right| \leq 2\pi B(1 - \frac{\epsilon}{2})^k$. $\int_{\eta \leq |\theta| < \delta} f(\theta)\,[\,p(\theta)]^k d\theta \geq 0$,

$\int_{|\theta| < \eta} f(\theta)\,p_k(\theta)d\theta \geq 2\eta\frac{f(0)}{2}\ (1 + \frac{\epsilon}{2})^k$. Thus, $\int f(\theta)\,p_k(\theta)d\theta \to \infty\ as\ k \to \infty$, contradiction.

Above is when f is real-valued function. When f is complexed-valued function, one can write $f(\theta) = u(\theta) + v(\theta), where\ u(\theta) = \frac{f(\theta) + \overline{f(\theta)}}{2}, v(\theta) = \frac{f(\theta) - \overline{f(\theta)}}{2i}$, since both $u(\theta)$ and $,v(\theta)$ vanish, so does $f(\theta)$. The theorem above also implies that if f is continuous on the circle, then $\hat{f}(n) = 0$ for all $n \in \mathbb{Z}$ suggests that $f = 0$ [1]. The next corollary states that if f is continuous on the circle the Fourier Transform of f satisfied that $\sum_{n=-\infty}^{\infty}|\hat{f}\,(n)| < \infty$ (absolutely convergent), then, the Fourier Series converge uniformly to original function f : $\lim_{N \to \infty} S_N(f)(\theta) = f(\theta)$ uniformly in θ [1]. To prove this corollary, notice that since $\sum_{n=-\infty}^{\infty}|\hat{f}\,(n)| < \infty$, $S_N(f)(\theta)$ also converge uniformly and absolutely. Thus, $g(\theta) = \lim_{N \to \infty}\sum_{n=-N}^{N}\hat{f}(n)e^{in\theta}$ is a continuous function on the circle, and the Fourier coefficient for g is just $\hat{f}(n)$. Since $\hat{f}(n) = \hat{g}(n)$, based on the previous property, $g = f$. Before talk about the third corollary, notation O is introduced. For some constant C, if it is true that $|f(x)| \leq C|g(x)|$ as x approach a, then it can be represented as below: $f(x) = O(g(x))$ as $x \to a$. If $g(x) = 1$, this means the function the function is bounded. Knowing this notation, here is another corollary: If function f is twice differentiable on the circle, then $\hat{f}(n) = O(1/|n|^2)$ as $n \to \infty$

So Fourier series of f converges absolutely and uniformly [1].The proof process is integrating by part, and find that $2\pi\hat{f}(n) = \int_0^{2\pi} f(\theta)e^{-in\theta}d\theta = \frac{-1}{n^2}\int_0^{2\pi} f''(\theta)\, e^{-in\theta}d\theta$. $2\pi|n|^2|\hat{f}(n)| \leq \left|\int_0^{2\pi} f''(\theta)\, e^{-in\theta}d\theta\right| \leq \int_0^{2\pi} |f''(\theta)|\, d\theta \leq C\, |\hat{f}(n)| \leq C|1/|n|^2|, C = 2\pi B$ (B is the bound of f'') as $n \to \infty$. One important property one can notice is that: $\widehat{f'(n)} = in\hat{f}(n)$.

3. Kernels and Convolution

3.1. Important Kernels in Fourier Analysis

Below are some important kernels that will be useful when discuss about Fourier Series.

The first one is Dirichlet Kernel: $D_n(x) = \sum_{|k|\leq n} e^{ikx} = e^{-inx}\sum_{k=0}^{2n} e^{ikx} = e^{-inx}\frac{e^{(2n+1)ix}-1}{e^{ix}-1} = \frac{e^{-inx-\frac{1}{2}ix}}{e^{-\frac{1}{2}ix}} \cdot \frac{e^{(2n+1)ix}-1}{e^{ix}-1} = \frac{e^{ix\left(n+\frac{1}{2}\right)}-e^{-ix\left(n+\frac{1}{2}\right)}}{e^{\frac{1}{2}ix}-e^{-\frac{1}{2}ix}} = \frac{2i\sin\left(n+\frac{1}{2}\right)x}{2i\sin\frac{x}{2}} = \frac{\sin\frac{(2n+1)}{2}x}{\sin\frac{x}{2}}$. The second one is Poisson Kernel: $P_r(\theta) = \sum_{n=-\infty}^{\infty} r^{|n|}e^{in\theta}$, $0 < r < 1 = \sum_{n=1}^{\infty} r^n e^{-in\theta} + \sum_{n=0}^{\infty} r^n e^{in\theta} = \frac{re^{-i\theta}}{1-re^{-i\theta}} + \frac{1}{1-re^{i\theta}} = \frac{1-r^2}{1-2\,r\cos\theta+r^2}$. And the third type, Fejer Kernel: $F_N(\theta) = \frac{1}{N+1}\sum_{n=0}^{N} D_N = \frac{1}{N+1}\sum_{n=0}^{N}\frac{\sin\frac{(2n+1)}{2}\theta}{\sin\frac{\theta}{2}} = \frac{1}{N+1}\left(\frac{\sin^2\frac{\theta}{2}}{\sin^2\frac{\theta}{2}} + \cdots + \frac{\sin\frac{(2n+1)}{2}\theta \cdot \sin\frac{\theta}{2}}{\sin^2\frac{\theta}{2}}\right)$, $\sin\frac{(2n+1)}{2}\theta \cdot \sin\frac{\theta}{2} = \frac{1}{2}(\cos(nx) - \cos(n+1)x)$.$F_N(\theta) = \frac{1}{N+1}(\frac{\frac{1}{2}(1-\cos(N+1)x)}{\sin^2\frac{\theta}{2}}) = \frac{1}{N+1}\frac{\sin^2\frac{(N+1)\theta}{2}}{\sin^2\frac{\theta}{2}}$.

3.2. Convolution

Suppose there are two functions f and g which are 2π periodic integrable on the real line, then the convolution of two functions on the interval $[-\pi, \pi]$, denoted by $f * g$ can be calculated as following: $(f * g)(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(y)g(x-y)dy$. Since two functions are periodic, one can change variables inside and get $(f * g)(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x-y)g(y)dy$. Convolution in some sense can be seen as the weighted average of the value of function. It is also useful when one try to analyze the kernel. For example, for the partial sum $(S_N f)(x)$, it can be written as: $S_n f(x) = \sum_{n=-N}^{n=N} \hat{f}(n)e^{in\theta} = \sum_{n=-N}^{n=N}\frac{1}{2\pi}\int_0^{2\pi} f(t)e^{-int}dt\ \ e^{ijx} = \frac{1}{2\pi}\int_0^{2\pi}\left(\sum_{n=-N}^{N} e^{in(x-t)}\right) f(t)dt = \frac{1}{2\pi}\int_0^{2\pi}\frac{\sin\frac{(2n+1)}{2}(x-t)}{\sin\frac{x-t}{2}} f(t)dt = \frac{1}{2\pi}\int_{\pi}^{\pi} D_n(x-t)\, f(t)dt = (f * D_N)\ (x)$. D_N here is the nth Dirichlet kernel: $D_N(x) = \sum_{|n|\leq N} e^{inx}$

3.2.1 Properties of convolution

Below are several main properties of convolution

If f, g are two periodic integrable functions, then:

1. Linearity: $f * (g + h) = (f * g) + (f * h)$
2. Associativity: $(cf) * g = c(f * g) = f * (cg)$ for any $c \in \mathbb{C}$
3. Commutativity: $f * g = g * f$
4. Associativity: $(f * g) * h = f * (g * h)$
5. $f * g$ is continuous
6. $\widehat{f * g}(n) = f(n)g(n)$.

While the first 4 properties are on the algebraic level, the fifth one suggests that $f * g$ is continuous even if the two functions individually are only integrable. And the last one tells that the

Fourier transform can take convolution into multiplication. First 4 theorems are relatively easy to prove. To prove property 6, just do the integration: $\widehat{f * g}(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi}(f * g)(x)e^{-inx}\,dx = \frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{1}{2\pi}(\int_{-\pi}^{\pi}f(y)g(x-y)\,dy)e^{-inx}dx = \frac{1}{2\pi}\int_{-\pi}^{\pi}f(y)e^{-iny}(\frac{1}{2\pi}\int_{-\pi}^{\pi}g(x-y)\,e^{-in(x-y)}\,dx)dy = \frac{1}{2\pi}\int_{-\pi}^{\pi}f(y)e^{-iny}(\frac{1}{2\pi}\int_{-\pi}^{\pi}g(x)\,e^{-in(x)}\,dx)dy = \hat{f}(n)\hat{g}(n)$. In order to give proof for property 5, one need to first write: $(f * g)(x_1) - (f * g)(x_2) = \frac{1}{2\pi}\int_{-\pi}^{\pi}f(y)[g(x_1-y) - g(x_2-y)]\,dy$. Assume f and g continuous, then this implies that given $\varepsilon > 0$, there exists $\delta > 0$ such that for any $|s-t| < \delta$, it is true that $|g(s) - g(t)| < \delta$. Then for any $|x_1 - x_2| < \delta$, $|(x_1 - y) - (x_2 - y)| < \delta$ for any y.Thus, $|(f * g)(x_1) - (f * g)(x_2)| \le \frac{1}{2\pi}\int_{-\pi}^{\pi}|f(y)||[g(x_1-y) - g(x_2-y)]|\,dy \le \frac{\varepsilon}{2\pi}\int_{-\pi}^{\pi}|f(y)|dy \le \frac{\varepsilon}{2\pi}2\pi B$. It proves that $f * g$ is continuous.What if the functions are not continuous but just integrable? Then some Lemma need to be introduced: If function f is an integrable function on circle and it has a bound B. Then there exists a sequence $\{f_k\}_{k=1}^{\infty}$ of continuous functions such that $\sup(x \in [-\pi,\pi])\,|f_k(x)| \le B$ for k=1, 2, 3…And $\int_{-\pi}^{\pi}|f(x) - f_k(x)|\,dx \to 0$ as $k \to \infty$

Thus for functions f and g, one can have following sequences $f_{k,}\ g_g$. Then one have:$f * g - f_k * g_k = (f - f_k) * g + f_k * (g - g_k)$ .Based on the property of sequence, $|(f - f_k) * g(x)| \le \frac{1}{2\pi}\int_{-\pi}^{\pi}|f(x-y) - f_k(x-y)||g(y)|\,dy \le \frac{1}{2\pi}\sup|g(y)|\int_{-\pi}^{\pi}|f(y) - f_k(y)dy \to 0\ \ as\ k \to \infty$

Thus, one can conclude that $(f - f_k) * g$ goes to 0 uniformly as k goes to infinity, and $f_k * (g - g_k)$ is the same case. $f_k * g_k$ goes uniformly to $f * g$. Since $f_k * g_k$ is continuous, then so does $f * g$.

3.3. Good kernel family

When discussing about Fourier Series, the family of good kernels are very useful for recovering the original function. In general, Good kernels $\{K_n(x)\}_{n=1}^{\infty}$should satisfy these properties below:

1) $\frac{1}{2\pi}\int_{-\pi}^{\pi}K_n(x)dx = 1$, n>1
2) for all n>1, there exist M>0 such that $\int_{-\pi}^{\pi}|K_n(x)|\,dx \le M$
3) $\int_{\delta \le |x| \le \pi}|K_n(x)|dx \to 0$ as $n \to \infty$ for every $\delta > 0$,

For most of time, people will use good kernels $K_n(x) > 0$, thus $|K_n(x)| = K_n(x)$, which means the first property directly lead to the validity of the second one. One way to understand good kernels is the weight distribution on the circle. Property one means $K_n(x)$ assigns weight to the whole circle, and the property three implies that as n becomes larger, the mass will concentrate more around the origin.

Good kernels have insightful relationship with convolution, which leads to the following theorem: If $\{K_n(x)\}_{n=1}^{\infty}$ belongs to good kernels family, and f is integrable, then

$\lim_{n\to\infty}(f * K_n)(x) = f(x)$ whenever f is continuous at x. If function f is continuous everywhere, then the limit is uniform. This is why family $K_n(x)$ is also called the approximation to identity [1]. If one interprets the convolution as the weighted average, then $(f * K_n)(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi}f(x-y)K_n(y)dy$ is the weighted average of $f(x-y)$. According to the previous property, as n becomes large enough, K_n will tend to distribute all the mass at $y = 0$. Thus, $f(x)$ is assigned the full mass as $n \to \infty$.Below is the proof of the theorem: $(f * K_n)(x) - f(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi}f(x-y)K_n(y)dy - f(x)$. Since $\frac{1}{2\pi}\int_{-\pi}^{\pi}K_n(y)f(x)dy = f(x)\frac{1}{2\pi}\int_{-\pi}^{\pi}K_n(y)dy = f(x)$, $(f * K_n)(x) - f(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi}K_n(y)[f(x-y) - f(x)]dy$. $|(f * K_n)(x) - f(x)| = |\frac{1}{2\pi}\int_{-\pi}^{\pi}K_n(y)[f(x-y) - f(x)]dy|$. By assumption, f is continuous at point x, thus one can pick δ such that whenever $|y| < \delta$, $|f(x-y) - f(x)| < \varepsilon$. Therefore, the previous expression can be written as: $|(f *$

$K_n)(x) - f(x)| = \left|\frac{1}{2\pi}\int_{-\pi}^{\pi} K_n(y)[f(x-y) - f(x)]dy\right| \qquad = \frac{1}{2\pi}\int_{|y|<\delta} |K_n(y)||f(x-y) - f(x)|dy + \frac{1}{2\pi}\int_{\delta\le|y|\le\pi} |K_n(y)||f(x-y) - f(x)|dy \le \frac{\varepsilon}{2\pi}\int_{-\pi}^{\pi}|K_n(y)|\,dy + \frac{2B}{2\pi}\int_{\delta\le|y|\le\pi} |K_n(y)|dy$. The first part is due to the continuity of function f, which implied that $|f(x-y) - f(x)| < \varepsilon$. And the second part is because f is integrable, which means $|f(x-y) - f(x)|$ is bounded. Now according to property 2, $\frac{\varepsilon}{2\pi}\int_{-\pi}^{\pi}|K_n(y)|\,dy \le \frac{\varepsilon M}{2\pi}$. Also, according to property 3, $\frac{2B}{2\pi}\int_{\delta\le|y|\le\pi} |K_n(y)|dy \to 0$ as $n \to \infty$. Thus the theorem has been proved. If f is continuous everywhere, then δ can be chose independently, causing the limit to be uniformly converge to f. Remember the Dirichlet Kernel: $D_N(x) = \sum_{|n|\le N} e^{inx}$. Does this kernel belong to the good kernel family? If it does, then one can conclude that the Fourier Series converges to $f(x)$ as long as x is continuous. However, $\int_{-\pi}^{\pi} |D_N(x)|dx \ge c\log N$ as $N \to \infty$, which violates the second property. Dirichlet Kernel is not a good kernel. But it does satisfy the first property, that is: $\frac{1}{2\pi}\int_{-\pi}^{\pi} D_n(x)dx = 1$. These facts suggest that Dirichlet Kernel vibrates rapidly across the y-axis as N becomes large. But one can prove that the Fejer Kernel and the Poisson Kernel are both good kernels.

4. Convergence and Summability

4.1. Cesaro mean and Cesaro summation

Since sometimes Fourier Series does not converge, one need to use different methods to analyze the problem. This is when Cesaro mean and summation are introduced. Given a complex-valued series: $\sum_{k=0}^{\infty} c_k$, the nth partial sum, $s_n = \sum_{k=0}^{n} c_k$. The average of the first N partial sums are given by: $\sigma_N = \frac{s_0+s_1+\cdots+s_{N-1}}{N}$, where σ_N is the Nth Cesaro mean of sequence $\{s_k\}$, or Nth Cesaro sum of series $\sum_{k=0}^{\infty} c_k$. If σ_N converge to some σ as N goes to the infinity, then series $\sum_{k=0}^{\infty} c_k$ is Cesaro summable to σ. If a series converges to s, then it is also Cesaro summable to s. Although Dirichlet Kernel is not a gook kernel, its average does belong to family of good kernel. In fact, the nth Cesaro sum of Fourier series is: $\sigma_N(f)(x) = \frac{S_0(x)+\cdots+S_{N-1}(x)}{N} = \frac{(f*D_0)(x)+\cdots+(f*D_{N-1})(x)}{N} = f * (\frac{D_0(x)+\cdots+D_{N-1}(x)}{N}) = (f * F_N)(x)$ where $(F_N)(x) = \frac{D_0(x)+\cdots+D_{N-1}(x)}{N}$. One can prove that Fejer Kernel is a family of good kernel.

4.2. Abel means

Beside the Cesaro means, there is another average called Abel means. Consider

$A(r) = \sum_{k=0}^{\infty} c_k r^k$,where $\sum_{k=0}^{\infty} c_k$ is a series of complex number, $A(r)$ is the Abel means of the series and $0 \le r < 1$. If $\lim_{r\to 1} A(r) = s$, then the series $\sum_{k=0}^{\infty} c_k$ is Abel summable to s when $0 \le r < 1$. If one series is Cesaro summable, then it is Abel summable. However, Abel summable does not imply Cesaro summable. Now define the Abel means of function $f(\theta) \sim \sum_{n=-\infty}^{\infty} a_n e^{in\theta}$: $A_r(f)(\theta) = \sum_{n=-\infty}^{\infty} r^{|n|} a_n e^{in\theta}$, where $c_0 = a_0, c_n = a_n e^{in\theta} + a_{-n} e^{in\theta}, n > 0$. Notice that $A_r(f)(\theta) = \sum_{n=-\infty}^{\infty} r^{|n|} a_n e^{in\theta} \sum_{n=-\infty}^{\infty} r^{|n|} (\frac{1}{2\pi}\int_{-\pi}^{\pi} f(\varphi)e^{-in\varphi}d\varphi)\, e^{in\theta} = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\varphi)(\sum_{n=-\infty}^{\infty} r^{|n|} e^{in(\theta-\varphi)})d\varphi = (f * P_r)(\theta)$ where $P_r(\theta) = \sum_{n=-\infty}^{\infty} r^{|n|} e^{in\theta}$ is called the Poisson kernel. If f is integrable on the circle, then the Fourier series of f is Cesaro summable to f whenever f is continuous. If f is continuous, then the Fourier series is uniformly summable to f . If f is integrable on the circle and $\hat{f}(n) = 0$ for all n, then $f = 0$ at all points of continuity. This is because when $\hat{f}(n) = 0$, the Cesaro sum of Fourier series is also 0 for N. If a function is continuous, then it can be approximated by trigonometric polynomials, which means that for such function f and $\varepsilon > 0$, there exists trigonometric polynomial P such that: $|f(x) - P(x)| < \varepsilon$. This is because the Cesaro sum is trigonometric polynomials. Poisson Kernel is also a good kernel, thus another important theorem can be drawn:For an integrable function f on the circle, the Fourier series is Abel summable to f whenever f is continuous. If f is continuous on the circle, the Fourier series is uniformly Abel summable to f.

5. Summary

Spectral approaches provide benefits for numerical analysis. Cornelius Lanczos was the first to employ the Fourier series and Chebyshev polynomials to resolve ordinary differential equations. In the 1970s, Kreiss, Oliger, Orszag, and other researchers used spectral methods to solve the partial differential equations of fluid mechanics for the first time. The sums of a range of fundamental function choices, such as sines and cosines, are how general functions are expressed in Fourier analysis.In this article, the analysis on Fourier Series are based on several conditions. The paper gives a definition of Fourier coefficient and Fourier Series as well as good kernel families. The uniqueness of Fourier Series is proved with conditions; The convergence of Abel mean and Cesaro mean are also proved.

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