The World of Fourier Series

Guanting Chen

Math Department, columbia university, NY, 10027, United States

* Corresponding Author Email: gc3027@columbia.edu

Abstract. The idea of Fourier analysis is to express general functions as sums of other choices of basic functions such as sines and cosines. One can interpret it as decomposing a vector into a set of basis vectors in the context of linear algebra. This paper will discuss differential equations and specifically focusing on approximating solutions using Fourier analysis. In modern times, Fourier analysis is not only used in the field of signal processing. As an important subfield of mathematical analysis, Fourier analysis has been broadly applied to various engineering fields, such as image processing, vibration analysis, acoustics, etc.

Keywords: Fourier series, good kernel, Dirichlet kernel.

1. Introduction

Numerical analysis can benefit from using spectral approaches. The first person to use Fourier series and Chebyshev polynomials to solve ordinary differential equations was Cornelius Lanczos. Additionally, spectral techniques are introduced to solve the partial differential equations of fluid mechanics in the 1970s by Kreiss and Oliger, Orszag, and others. As time went on, the use of spectral methods equaled that of their well-known equivalents, the finite difference and finite element approaches. Spectral techniques, however, are less well known [1-20]. The three basic sub-approaches used in spectral techniques are "Galerkin," "tau," and "collocation" (or "pseudospectral") [1-20].

2. Main works

Basic idea of Fourier analysis is that expressing a function $f(x)$ as the sum of exponentials $f(x) = \int_{-\infty}^{\infty} \widehat{f(\lambda)}\, e^{i\lambda x} d\lambda$. Where $x \in R$ in physical space and $\lambda \in R$ in frequency space. For an $\theta \in R$ on the circle and $n \in R$ as integers, the formula becomes:$f(\theta) = \int_{-\infty}^{\infty} \widehat{f(n)}\, e^{in\theta} d\theta$, w is defined as $w = e^{\frac{i2\pi}{N}}$, $w^N = e^{i2\pi} = \cos(2\pi) + isin(2\pi) = 1$. It states that $f(\theta)$ continues the circle, then $\lim_{r\to 1} Prf(\theta) = f(\theta)$. Let's denote out function by a0, a1, a2, and all the way to an-1. The define the DFT of this finite sequence by the following formula: $\widehat{a_j} = \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} a_k w^{kj}/$ where $0 \ll j \ll N-1$. Take an example, when $N = 2$, $\widehat{a_0} = \frac{1}{\sqrt{2}}(a_0 w^0 + a_1 w^1)\widehat{a_1} = \frac{1}{\sqrt{2}}(a_0 w^0 + a_1 w^1)$. Inversion formula for General N is $a_j = \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} a_k w^{kj}$. The proof of this formula is based on combining w series. Firstly, substitute a_j as: $\widehat{a_j} = \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} a_k w^{-kj}$. The left-hand side becomes: $\frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} \frac{1}{\sqrt{N}} \sum_{l=0}^{N-1} a_k w^{lk} a_k w^{-kj} = \frac{1}{N} \sum_{l=0}^{N-1} a_l \sum_{k=0}^{N-1} w^M$, where M=l-j. Since the determinant is zero, for a fixed integer M, the value of $\sum_{k=0}^{N-1} w^M$ becomes zero.$\sum_{k=0}^{N-1} w^M = (w^M)^0 + (w^M)^1 + \cdots + (w^M)^{N-1} = \frac{1-(w^M)^N}{1-w^M} = 0$. Thus, the right-hand side becomes:$\frac{1}{N} \sum_{l=0}^{N-1} a_l * \binom{N\ when\ l=j}{0\ otherwise} = a_j$. Hence, proved. Here are some main properties of Fourier transform. Denote a set of sequences a= {a0, a1, … aN-1}, $a_j \in N$ as a vector. Let b= {b0, b1, …, bN-1}, suppose that both a and b come to place and then compute the DFT of each of them:$\widehat{a_j} = (\widehat{a_0}, \widehat{a_1}, \ldots \widehat{a_{N-1}})\widehat{b_j} = (\widehat{b_0}, \widehat{b_1}, \ldots \widehat{b_{N-1}})$. Now compute $(\widehat{a_j}, \widehat{b_j}) : < a, b \geq \sum_{j=0}^{N-1} a_j * \overline{b_j} < \widehat{a_j}, \widehat{b_j} > = \sum_{j=0}^{N-1} \widehat{a_j} * \widehat{b_j} =$

$\frac{1}{N}\sum_{j=0}^{N-1} 1 \sum_{k=0}^{N-1} a_k w^{kj} \sum_{l=0}^{N-1} \overline{b_l}\, w^{-lj}$. Rearrange terms and get:$< \widehat{a_j},\ \widehat{b_j} \geq \frac{1}{N}\sum\sum a_k \overline{b_l} * \begin{pmatrix} k=l & N \\ k\neq l & 0 \end{pmatrix}$. Only $k = l$ left non-zero. Thus, have proven then moral convolution, with $< \widehat{a_j},\ \widehat{b_j} >= \sum_{k=0}^{N-1} a_k * \overline{b_{J=k}} =< a, b >$. Define the convolution of two sequences a and b as: $(a * b)j = \sum_{k=0}^{N-1} a_{j-k} b_k$. Another property of DFT is that if DFT has been done twice:$\widehat{\widehat{a_j}} = \frac{1}{\sqrt{N}}\sum_{k=0}^{N-1} \widehat{a_k} w^{-kj}$ On the circle, double DFT reflects the position of the point aj according to the horizontal diameter. Thus, computing the DFT four times doesn't change the position. Express it in equation $(DFT)^4 = I$ Fourier transform of circle to the frequency space is:$C_j = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(\theta)\, e^{-ij\theta} d\theta$ Where j is an integer. Inner product is defined as$< f_1(\theta), f_2(\theta) > = \frac{1}{2\pi}\int_0^{2\pi} f1(\theta) * \overline{f_2(\theta)} d\theta$ Using the equation $f(\theta) = \int_{-\infty}^{\infty} C_j\, e^{ij\theta} d\theta$ to deduce that: $< f_1(\theta), f_2(\theta) > = \frac{1}{2\pi}\int_0^{2\pi} (\sum_{j=-\infty}^{\infty} Cj\, e^{ij\theta})(\sum_{k=-\infty}^{\infty} \overline{C_k}\, e^{-ik\theta}) d\theta = \frac{1}{2\pi}\sum_{j,k} C_j^1 C_k^2 \int_0^{2\pi} e^{i(j-k)\theta} d\theta$ Integral part could be simplified: $\frac{1}{2\pi}\int_0^{2\pi} e^{in\theta} d\theta = \begin{pmatrix} n=0 & 1 \\ n\neq 0 & \left[\frac{e^{in\theta}}{in}\right]_0^{2\pi} = 0 \end{pmatrix}$. Using this important trick, easily write $< f_1(\theta), f_2(\theta) \geq \sum_{k=-\infty}^{\infty} C_j{}^1 \overline{C_j}^2 =< C^1, C^2 >$. The Fourier filter is a type of filtering function that is based on the manipulation of specific frequency components of a signal. The first filter is $S_N f = \sum_{n=-N}^{N} C_n e^{in\theta}$, where N is non-negative integers. This filter sets up a boundary and simply cuts the remaining parts that are located outside the boundary. The second filter is $F_N f = \frac{1}{N+1}\sum_{n=-N}^{N} S_N f$. This filter has an expression:Filter $= \begin{pmatrix} \frac{N+1-|j|}{N+1} & |j|\leq N \\ 0 & |j|\geq N \end{pmatrix} F_N f(\theta) = \frac{S0f+S1f+\cdots+SNf}{N+1}$. In frequency space, the graph lies on two equations $y_1 = 1 - \frac{1}{N+1}x$ and $y_2 = 1 + \frac{1}{N+1}x$ in the first two axis. Randomly select a number R between 0 and 1:$P_r f(\theta) = \sum_{n=-\infty}^{\infty} r^{|j|} C_j e^{ij\theta}$. It reaches maximum at N=0 and decrease gradually as $|N|$ increases. The first theorem states that if $f'(\theta)$ exists, then $|S_n f(\theta) - f(\theta)| \to 0$ as N approaches infinity. Define $D_N(\theta) = \sum_{j=-N}^{N} e^{ij\theta}$, immediately obtain that $\frac{1}{2\pi}\int_0^{2\pi} D_N(\theta) d\theta = 1$

Hence, $\frac{1}{2\pi}\int_0^{2\pi} D_N(\theta-\tau) f(\theta) d\tau = f(\theta)\frac{1}{2\pi}\int_0^{2\pi} D_N(\theta-\tau) d\tau = f(\theta)\, S_n f(\theta) - f(\theta) = \frac{1}{2\pi}\int_0^{2\pi} D_N(\theta-\tau)\{f(\tau) - f(\theta)\} d\tau$. By Riemann-Lebesgue theorem, this whole equation equals zero.

Convolution of $f(\theta)$ with $\delta_N(\xi) = \frac{\sin(2N+1)\frac{\xi}{2}}{sin\frac{\xi}{2}}$ is $S_n f(\theta)$. Thus $|S_n f(\theta) - f(\theta)| \to 0$ as $N \to \infty$. Theorem states that if $f(\theta)$ is continuous on the circle then $\lim_{r\to 1} P_r f(\theta) = f(\theta)\, P_r f(\theta) = \sum_{n=-\infty}^{\infty} r^{|j|} e^{ij\theta}$. Split the sum into $\sum_{n=-\infty}^{-1} r^{|j|} C_j e^{ij\theta}$ and $\sum_{n=0}^{\infty} r^{|j|} C_j e^{ij\theta}$. According to the exponential sum equation, $P_r(\theta) = \frac{re^{-i\theta}}{1-re^{-i\theta}} + \frac{1}{1-re^{i\theta}} = \frac{1-r^2}{1-2rcos\theta+r^2}$. when $\theta = 0$, $P_r(\theta) = \frac{1+r}{1-r}$. $P_r(\theta)$ decreases when θ moves further from the origin. The minimum of the function is when $\theta = \pi$ and $\theta = -\pi$. For a fixed value θ, $\lim_{r\to 1} P_r(\theta) = 0$. Similarly, $\frac{1}{2\pi}\int_0^{2\pi} P_r(\theta) d\theta = \frac{1}{2\pi}\int_0^{2\pi} \sum_{n=-\infty}^{\infty} r^{|n|} e^{in\theta}\, d\theta$. Since integral of periodic function has the following property, $\int_0^{2\pi} e^{in\theta} d\theta = \begin{pmatrix} 2\pi & n=0 \\ 0 & n\neq 0 \end{pmatrix}$. Hence, $\frac{1}{2\pi}\int_0^{2\pi} P_r(\theta) d\theta = 1$. If $f'(\theta)$ exists, want to find $|P_r f(\theta) - f(\theta)|$:$P_r f(\theta) = \frac{1}{2\pi}\int_0^{2\pi} P_r(\theta-\tau) f(\tau) d\tau$. Thus, $f(\theta) = \frac{1}{2\pi}\int_0^{2\pi} P_r(\tau) f(\theta)\} d\tau\, P_r f(\theta) - f(\theta) = \frac{1}{2\pi}\int_{-\pi}^{\pi} P_r(\tau)\{f(\theta-\tau) - f(\theta)\} d\tau$

The right-hand side is smaller than $\frac{1}{2\pi}\int_{-\pi}^{\pi} |P_r(\tau)|\{f(\theta-\tau) - f(\theta)\} d\tau$. Because an absolute sign increases the range and flips the negative parts. $\frac{1}{2\pi}\int_{-\pi}^{\pi} |P_r(\tau)|\{f(\theta-\tau) - f(\theta)\} d\tau =$

$\frac{1}{2\pi}\int_{-\delta}^{\delta}|P_r(\tau)|\{f(\theta-\tau)-f(\theta)\}d\tau+\frac{1}{2\pi}\int_{-\tau}^{\tau}|P_r(\tau)|\{f(\theta-\tau)-f(\theta)\}d\tau$ The definition of $f(\theta)$ being "continuous function" is that for all ε, there exists $\delta(\varepsilon)$ such that for any $|\tau|<\delta(\varepsilon)$, $|\{f(\theta-\tau)-f(\theta)\}|<\frac{\varepsilon}{2}$. Apply this restriction law, attain $\frac{1}{2\pi}\int_{-\delta}^{\delta}|P_r(\tau)|\{f(\theta-\tau)-f(\theta)\}d\tau<\frac{\varepsilon}{4\pi}\int_{-\delta}^{\delta}|P_r(\tau)|d\tau$

$|\delta(\varepsilon)|$ is less than π, could infer that $\frac{\varepsilon}{4\pi}\int_{-\delta}^{\delta}|P_r(\tau)|d\tau<\frac{\varepsilon}{4\pi}\int_{-\pi}^{\pi}|P_r(\tau)|d\tau$. Also, since function f is continuous, could also find a number M such that $|\{f(\theta-\tau)-f(\theta)\}|<M$. Therefore, the second equation $\frac{1}{2\pi}\int_{-\tau}^{\tau}|P_r(\tau)|\{f(\theta-\tau)-f(\theta)\}d\tau<\frac{M}{2\pi}\int_{-\tau}^{\tau}|P_r(\tau)|d\tau$.Given $\lim_{r\to 1}P_r(\tau)=0$, for any $\tau\neq 0$, there exists $\eta(\tau)$ such that $P_r(\tau)<\frac{\varepsilon}{2M}$ if $|r-1|<\eta$. Must have δ' such that $P_r(\tau)<\frac{\varepsilon}{2M}$ for any $P_r(\tau)<\frac{\varepsilon}{2}$, and thus combining those steps give $\frac{1}{2\pi}\int_{-\tau}^{\tau}|P_r(\tau)|\{f(\theta-\tau)-f(\theta)\}d\tau<\frac{M}{2\pi}\int_{-\tau}^{\tau}\left|\frac{\varepsilon}{2M}\right|d\tau<\frac{\varepsilon}{2}$. when $|r-1|<\delta'$. Overall, for all ε, there exists $\delta'(\varepsilon)$. If $|r-1|<\delta'$, then $|P_rf(\theta)-f(\theta)|<\varepsilon$. Get back to the same expression of being continuous at r approaches 1. Thus, $\lim_{r\to 1}|P_rf(\theta)-f(\theta)|=0$. Gibbs phenomenon refers to the case when f is discontinuous at certain points. Errors accumulates with N, so it results in an error comparing to the function f at the jump. Consider the function $f(\theta)=\begin{pmatrix}1 & 0\leq x<\pi\\ -1 & -\pi\leq x<0\end{pmatrix}$, the function is discontinuous at $\theta=0$$S_Nf(\theta)=\frac{1}{2\pi}\int_{-\pi}^{\pi}f(\tau)D_N(\theta-\tau)d\tau$. Again, split the range from the middle $S_Nf(\theta)=\frac{1}{2\pi}\left[-\int_{-\pi}^{0}D_N(\theta-\tau)d\tau+\int_{0}^{\pi}D_N(\theta-\tau)d\tau\right]$. .Next, changing the variable from $\theta-\tau$ to θ alters the range of integral $S_Nf(\theta)=\frac{1}{2\pi}\left[-\int_{\theta}^{\theta+\pi}D_N(\tau)d\tau+\int_{\theta-\pi}^{\theta}D_N(\tau)d\tau\right]=\frac{1}{2\pi}\left[\int_{-\theta}^{\theta}D_N(\tau)d\tau-\int_{-\theta+\pi}^{\theta+\pi}D_N(\tau)d\tau\right]$ .Rearranging terms and obtain $S_Nf(\theta)-\frac{1}{2\pi}\int_{-\theta}^{\theta}D_N(\tau)d\tau=\frac{1}{2\pi}\int_{-\theta+\pi}^{\theta+\pi}D_N(\tau)d\tau$. Has shown above that $D_N(\tau)=\sum_{j=-N}^{N}e^{ij\theta}=\frac{sin\frac{2N+1}{2}\tau}{sin\frac{\tau}{2}}$, and the function $\frac{1}{2\pi}\int_{-\theta+\pi}^{\theta+\pi}D_N(\tau)d\tau$ is approximate to one over N for a large N. After some trivial mathematical computation involving integration by substitution, finally obtain$S_Nf(\theta)=\frac{2}{\pi}\int_{0}^{\frac{2N+1}{2}\theta}\frac{sin\tau}{\tau}d\tau$. The lower boundary is fixed, the upper boundary is set to be the point when the first intersection between the function and x-axis. Hence, maximum of $S_Nf(\theta)$ is when $\frac{2N+1}{2}\theta=\pi$. $S_Nf(\theta)=\frac{2}{\pi}\int_{0}^{\pi}\frac{sin\tau}{\tau}d\tau=1.09$. Error is around 9 percent. Using the formula for the Fourier coefficients $S_Nf(x)=\sum_{j=-N}^{N}C_je^{ijx}$ Replacing $C_j=\sum_{j=-N}^{N}(\frac{1}{2\pi}\int_{0}^{2\pi}f(\tau)e^{-ijy}dy)e^{ijx}S_Nf(x)=\frac{1}{2\pi}\int_{0}^{2\pi}\sum_{j=-N}^{N}f(y)e^{-ij(x-y)}dy=\frac{1}{2\pi}\int_{-\pi}^{\pi}f(y)D_N(x-y)dy$, where $\{D_N\}$ is called the Dirichlet kernel and will be discussed it later. Such an expression is named as convolution of $f$and the function $\{D_N\}$. If the family of functions is indeed the "good kernel", then the convolution $S_Nf(x)$ will converge to $f(x)$ as $n\to\infty$, which means that $|S_nf(\theta)-f(\theta)|\to 0$ as $n\to\infty$. Fourier coefficients of $f(x)$ are defined as $a_n=\frac{1}{L}\int_{0}^{L}f(x)e^{-\frac{2\pi inx}{L}}dx$, where f is complex valued on $[0,L]$. Functions do not necessarily confine to real values. Functions in ascending order of catholicity is introduced below. First, those functions that are continuous at each single point on the segment is continuous is defined as Everywhere continuous functions. Second, if the functions have finite number of disjoints over the range $[0,L]$, it's called the piecewise continuous functions. The main concern with Fourier series is to determine whether the function is integrable or not. A real-valued function $f(x)$ defined on [0, L] is Riemann-integrable if $\forall\,\varepsilon>0$, there exists a list of numbers x_0, x_1, x_2,…, x_N that satisfy $0<x_0<x_1<x_2<\cdots<x_N<L$ such that the upper sum $\mho$ and lower sum $\mathcal{L}$ is closer than ε distance apart. Another word is that upper sum and lower sum matches when there are infinite subdivisions in range [0, L]. Generally speaking, integrable functions are not characterized by discontinuous segments. Discontinuities will count as "zero" in mathematical computation and usually they are not

considerable enough. For most cases, always presume that those functions are integrable, at least on the segment where integrals apply. If the function satisfies the condition $f(0) = f(L)$, determine that the function is on the unit circle. A better way to express a point on the unit circle is $e^{in\theta}$. For a real number θ, a function F on the circle is defined as: $f(\theta) = F\left(e^{in\theta}\right)$ Function f is periodic on R of period 2π, in another word, $f(\theta + 2\pi) = f(\theta)$ for all θ. Then, function F inherits all properties of f. For example, if f has a continuous derivative, then F is also differentiable. Since function f is periodic, then it makes no difference to shift the interval $[0, 2\pi]$, as long as the difference in range is 2π. When considering a function on the circle, simply replace the variable from x to θ. If f is an integrable function, then the nth fourier coefficient of f on the interval $[a,\ b]$ of length L is defined by the following: $\hat{f}(n) = \frac{1}{\mathrm{L}} \int_{\mathrm{a}}^{b} f(x) e^{-\frac{2\pi inx}{L}} d\theta$. Substituting $\hat{f}(n)$ by a_n. Fourier series of f is defined by $f(x) \sim \sum_{n=-\infty}^{\infty} a_n e^{\frac{2\pi inx}{L}} d\theta$. To illustrate, if the function f is given on interval $[0, 1]$, then $\hat{f}(n) = \int_0^1 f(x) e^{-2\pi inx}\, dx,\ f(x) \sim \sum_{n=-\infty}^{\infty} a_n e^{2\pi inx} dx$. If f is the full period the unit circle, then the nth Fourier coefficient is simplified to: $\hat{f}(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \theta e^{-in\theta}\, d\theta$. In a broader scope, every function g has the following expression is called trigonometric series: $g(x) \sim \sum_{n=-\infty}^{\infty} c_n e^{\frac{2\pi inx}{L}} dx$ Trigonometric polynomial is a trigonometric series that contains finite non-zero terms, which literally means that $c_n = 0$ for larger n. The degree of trigonometric polynomial is the maximum value of $|n|$ for $n \neq 0$. Finally, introduce the nth partial sum of the Fourier series, denote as $S_N f(x),\ S_N f(x) = \sum_{n=-N}^{N} \hat{f}(n) e^{\frac{2\pi inx}{L}}$

Based on the Fourier coefficient, the partial sum is symmetric over the origin. Need to prove that $S_N f(x)$ convergent to $f(x)$ as $N \to \infty$. Before solving that problem, it's necessary to understand the properties of Fourier analysis. Take an example, take $f(\theta) = \theta$. Calculate the Fourier coefficients by the formula $\hat{f}(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \theta e^{-in\theta}\, d\theta$. If $n \neq 0$, using integration by parts to compute $\hat{f}(n) = \frac{1}{2\pi}\left[-\frac{\theta}{in} e^{-in\theta}\right]_{-\pi}^{\pi} + \frac{1}{2\pi in} \int_{-\pi}^{\pi} e^{-in\theta}\, d\theta = \frac{(-1)^{n+1}}{in}$. If $n = 0$, find that $\hat{f}(0) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \theta\, d\theta = 0$. Therefore, the fourier series of f can be approximate to: $f(\theta) \sim \sum_{n \neq 0} \frac{(-1)^{n+1}}{in} e^{-in\theta} = 2 \sum_{n=1}^{\infty} (-1)^{n+1} \frac{sinn\theta}{n}$. The second equation is obtained by employing Euler's formula. The above Fourier series converges to $f(\theta)$. Next, define the series $f(\theta) = \frac{(\pi-\theta)^2}{4}$ for take $0 \leq \theta \leq 2\pi$. Same as the previous example, two integrations by parts ultimately eliminate the θ $f(\theta) \sim \frac{\pi^2}{12} + \sum_{n=1}^{\infty} \frac{cosn\theta}{n^2}$. Now it's time to introduce a new series. The Nth Dirichlet kernel is a trigonometric polynomial. In frequency space, a particular filter sets up a boundary N and simply cuts the remaining parts. Fourier coefficients possess the property that $a_n = 1$ if $|n| \leq N$ and $a_n = 0$ everywhere else. The Dirichlet Kernel defined as:$D_N(x) = \sum_{n=-N}^{N} e^{inx}$

w is defined as $w = e^{\mathrm{ix}}$. First separate DN (x) into two parts, which is$D_N(x) = \sum_{n=-N}^{-1} e^{inx} + \sum_{n=0}^{N} e^{inx}$. Thus, the function can be viewed as $D_N(x) = \sum_{n=-N}^{-1} \mathrm{w}^n + \sum_{n=0}^{N} \mathrm{w}^n$. The sum of geometric sequences gives the answer of each parts $\sum_{n=0}^{N} \mathrm{w}^n = \frac{1-\mathrm{w}^{\mathrm{N}+1}}{1-\mathrm{w}}.\sum_{n=-\mathrm{N}}^{-1} \mathrm{w}^n = \frac{\mathrm{w}^{-\mathrm{N}}-1}{1-\mathrm{w}}$. The sum is $\frac{1-\mathrm{w}^{\mathrm{N}+1}}{1-\mathrm{w}} + \frac{\mathrm{w}^{-\mathrm{N}}-1}{1-\mathrm{w}} = \frac{\mathrm{w}^{-\mathrm{N}}-\mathrm{w}^{\mathrm{N}+1}}{1-\mathrm{w}}\quad \frac{\mathrm{w}^{-\mathrm{N}}-\mathrm{w}^{\mathrm{N}+1}}{1-\mathrm{w}} = \frac{(\mathrm{w}^{-\mathrm{N}}-\mathrm{w}^{\mathrm{N}+1})/\mathrm{w}^{\frac{1}{2}}}{\frac{1-\mathrm{w}}{\mathrm{w}^{\frac{1}{2}}}} = \frac{\mathrm{w}^{-\mathrm{N}-\frac{1}{2}}-\mathrm{w}^{\mathrm{N}+\frac{1}{2}}}{\mathrm{w}^{-\frac{1}{2}}-\mathrm{w}^{\frac{1}{2}}}$. By employing the Euler's Formula $e^{iz} = cosz + isinz$, substitute $D_N(x) = \frac{sin\frac{2N+1}{2}x}{sin\frac{x}{2}}$. Hence, the closed form formula for the Dirichlet kernel is $D_N(x) = \frac{sin\frac{2N+1}{2}x}{sin\frac{x}{2}}$. The Poisson kernel is defined as: $P_r f(\theta) = \sum_{n=-\infty}^{\infty} r^{|\mathrm{n}|} e^{in\theta}$. Again, we split the sum into $P_r f(\theta) = \sum_{n=-\infty}^{-1} r^{|j|} C_j e^{ij\theta} + \sum_{n=0}^{\infty} r^{|j|} C_j e^{ij\theta}$.

Note that r is an fixed parameter ranging from 0 to 1, so the sum converges in θ $P_r(\theta) = \sum_{n=1}^{N} w^{-n} + \sum_{n=0}^{N} w^n$, where w is defined as $w = re^{i\theta}$. According to the exponential sum equation, $\sum_{n=0}^{N} w^n = \frac{1-w^{N+1}}{1-w}$. Fourier coefficient is "one to one" matched with the function. Every function has a unique expression of Fourier coefficient. If f and g share the identical Fourier coefficients, then function f and g are equal. Suppose that f is integrable along the circle. For all n $\in$ Z, replacing $\hat{f}(n) = 0$ with $f(n) = 0$. Then $f(\theta_0) = 0$ whenever f is continuous at the point θ_0.

Start our proof by stating that f is real-valued function. The function has meaning on the range $[\pi, -\pi]$, and set that $\theta_0 = 0$ and the function is positive at $f(0)$. Ideally, create a trigonometric polynomial $\{p_k\}$ with peak at $p_k(0)$ in order to generate contradiction with the theory that integrals are equal to zero. Choose a value $0 < \delta < \frac{\pi}{2}$ such that $f(\theta)$ is more than half of the peak whenever $|\theta| < \delta$. Let$p(\theta) = \epsilon + cos\theta, p_k(\theta) = p(\theta)^k$. Choose a positive ϵ value that is small enough to satisfy $p(\theta) < 1 - \frac{\epsilon}{2}$ whenever $\delta \leq |\theta| \leq \pi$. Symmetrically, choose another positive value ζ that is smaller than δ. This time, $p(\theta) > 1 + \frac{\epsilon}{2}$ for $|\theta| \leq \zeta$. Next, choose an upper bound C for all θ. Presumably $\hat{f}(n) = 0$. For every k and trigonometric polynomial $p_k(\theta)$, must satisfy $\int_{-\pi}^{\pi} f(\theta)p_k(\theta)d\theta = 0$. Here, the contradiction occurs:When $\delta \leq |\theta|$, $\int_{-\pi}^{\pi} f(\theta)p_k(\theta)d\theta$ is upper bounded; When $\zeta \leq |\theta| \leq \delta$, $\int_{-\pi}^{\pi} f(\theta)p_k(\theta)d\theta$ is upper bounded; When $|\theta| \leq \zeta$, $\int_{-\pi}^{\pi} f(\theta)p_k(\theta)d\theta \sim \left(1 + \frac{\epsilon}{2}\right)^k$;Thus, $\int_{-\pi}^{\pi} f(\theta)p_k(\theta)d\theta \to \infty$ if k approaches infinity. Write $f(\theta)$ in terms of real part $u(\theta)$ and imaginary part $v(\theta)$, where $u, v \in R$. Then, $u(\theta) = \frac{f(\theta)+\bar{f}(\theta)}{2}$; $v(\theta) = \frac{f(\theta)-\bar{f}(\theta)}{2i}$. In conclusion, Fourier coefficients are eliminated and $f = 0$ in continuous area. If a sequence of continuous functions converges uniformly, then its limit is also continuous. The concept of "smoothness" describes the speed of decay of Fourier coefficients. The faster the decay of Fourier coefficient, the smaller the gap between Fourier series and the function. A relatively smooth function can be approximate to the Fourier series. Here, introduces the big "O" notation. It is an asymptotic upper bound on the order of magnitude of a function described by another simpler function. $f(x) = O\big(g(x)\big)$. As $x \to a$ means that for some constant C, $f|x| \leq C|g(x)|$ as $x \to a$. Given a function f that is twice continuously differentiable function on the circle. Then Attempt to prove that the Fourier series of function f converges absolutely and uniformly to itself. Let's consider the case where $n \neq 0 \hat{f}(n) = \frac{1}{2\pi}\int_0^{2\pi} f(\theta)e^{-in\theta}\, d\theta = \frac{1}{2in\pi}\int_0^{2\pi} f'(\theta)e^{-in\theta}\, d\theta = -\frac{1}{2n^2\pi}\int_0^{2\pi} f''(\theta)e^{-in\theta}\, d\theta$. The remaining parts in integration by parts are eliminated because $f(\theta)$ and $f'(\theta)$ are 2π periodic. $\int_0^{2\pi} f'(\theta)e^{-in\theta}\, d\theta$. The function above is equal to $\hat{f}'(n)$. Before, this paper had made an approximation that if f is infinitely differentiable, then $f \sim \sum a_n e^{in\theta}\, f' \sim \sum a_n ine^{in\theta}\, f'' \sim \sum -a_n n^2 e^{in\theta}\, f''' \sim \sum -a_n in^3 e^{in\theta}$. In addition, the term $e^{-in\theta}$ doesn't boost the function, can prove that the function is bounded by a constant C, such that $\left|\int_0^{2\pi} f''(\theta)e^{-in\theta}\right| d\theta \leq \int_0^{2\pi} |f''(\theta)| d\theta \leq C$. Choose two functions f and g on R, they are both 2π-periodic integrable. Define their convolution $f * g$ by $f * g(x) = \frac{1}{2\pi}\int_0^{2\pi} f(y)g(x-y)\, dy$. The functions are 2π-periodic, which means that it can swipe variables to $f * g(x) = \frac{1}{2\pi}\int_0^{2\pi} f(x-y)g(y)\, dy$. Without altering the results. Convolution can be interpreted as calculating the average of two functions. Again, denote the partial sum of Fourier series $S_N f(x) = \sum_{n=-N}^{N} \hat{f}(n)e^{inx}$. Replacing $\hat{f}(n)$ by $\frac{1}{2\pi}\int_{-\pi}^{\pi} f(y)e^{-iny}dy$. Partial sum becomes $S_N f(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(y)\sum_{n=-N}^{N} e^{in(x-y)}dy = (f * D_N)(x)$

$D_N(x)$ is:$D_N(x) = \sum_{n=-N}^{N} e^{inx}$. Convolutions satisfy many properties. Given functions and their properties above, $f * g$ is continuous everywhere. It also satisfies the laws of addition and multiplication. More importantly, $\widehat{f*g}(n)=\hat{f}(n)\hat{g}(n)$ and this equation holds if it substitute the product of the two functions f and g by their convolution.In order to prove the continuity, write down $(f*g)(x_1)-(f*g)(x_2)=\frac{1}{2\pi}\int_{-\pi}^{\pi}f(y)[\mathrm{g}(x_1-\mathrm{y})-\mathrm{g}(x_2-\mathrm{y})]\,dy$ Under the condition that both functions are continuous and periodic. Continuity implies that for any given $\varepsilon>0$, there exists $\delta>0$ so that $|g(a)-g(b)|<\varepsilon$ whenever $|s-t|<\delta$ $(f*g)(x_1)-(f*g)(x_2)\leq\frac{1}{2\pi}\left|\int_{-\pi}^{\pi}f(y)[\mathrm{g}(x_1-\mathrm{y})-\mathrm{g}(x_2-\mathrm{y})]\,dy\right|\leq\frac{1}{2\pi}\int_{-\pi}^{\pi}|f(y)||[\mathrm{g}(x_1-\mathrm{y})-\mathrm{g}(x_2-\mathrm{y})]\,dy|$. Now it's time to simplify the long term in the bracket, $(f*g)(x_1)-(f*g)(x_2)\leq\frac{\varepsilon}{2\pi}\int_{-\pi}^{\pi}|f(y)|\,dy\leq\frac{\varepsilon}{2\pi}2\pi C,$ where $|f(y)|$ is upper bounded by $C.$ Using the lemma to prove the proposition property. There exists a sequence of continuous 2π periodic functions $\{f_k\}_{k=1}^{\infty}$, then $\sup_{x\epsilon[-\pi,\pi]}|f_k(x)|\leq B$. For all positive integers k $f*g-f_k*g_k=(f-f_k)*g-f_k*(g-g_k)$. Complete the proof of the proposition as follows: $|(f-f_k)*g(x)|\leq\frac{1}{2\pi}\left|\int_{-\pi}^{\pi}f(y)[\mathrm{g}(x_1-\mathrm{y})-\mathrm{g}(x_2-\mathrm{y})]\,dy\right|\leq\frac{1}{2\pi}\sup_{y}|\mathrm{g(y)}|\int_{-\pi}^{\pi}|f(\mathrm{y})-\mathrm{fk(y)}|\mathrm{dy}$. The later part, according to the lemma, tends to zero when k approaches to infinity. Thus, the $(f-f_k)*g(x)$ converges to zero uniformly on x. As an analogy, $f_k*(g-g_k)$ converges to zero uniformly. Hence, f_k*g_k converges to $f*g$. Since f_k*g_k converges uniformly to $f*g$, for each fixed integer n, will obtain $\widehat{f_k*g_k}(n)\to\widehat{f*g}(n)$ when k tends to infinity. In a broader sense, the Fourier series of f converges absolutely and uniformly to f if f satisfies a Holder condition of order α such that $\sup_{\theta}|f(\theta+t)-f(\theta)|\leq A|t|^{\alpha}$. For any selected t value with $\alpha>\frac{1}{2},$ Now, introduce a new notation C^K. If f is n times continuously differentiable, then $f\epsilon C^K$ Recall that the Dirichlet Kernel $D_N(x)$ is:$D_N(x)=\sum_{n=-N}^{N}e^{inx}$. It's not the good kernel. Although it satisfies the first property as the convolution will converge to $f(x)$ as $n\to\infty$, and then know that $\frac{1}{2\pi}\int_{-\pi}^{\pi}D_N(x)\,dx=1.$ However, the function $D_N(x)$ oscillates rapidly above and below the axis as N gets large, and thus violates the second property$\int_{-\pi}^{\pi}|D_N(x)|dx\geq c\log N$. For large N value. $\sum_{n=-\mathrm{N}}^{-1}\mathrm{w}^n=\frac{\mathrm{w}^{-\mathrm{N}}-1}{1-\mathrm{w}}$. Thus, $P_r(\theta)=\frac{re^{-i\theta}}{1-re^{-i\theta}}+\frac{1}{1-re^{i\theta}}=\frac{1-r^2}{1-2rcos\theta+r^2}$

3. Summary

This article provides a thorough explanation of the concept and associated features of Fourier series as well as a brief history of how Fourier series and its related theories were developed. The paper also teaches the concepts of covolution and excellent kernels in a suitable manner. And using these ideas in a methodical manner, the theory of the Fourier series' uniqueness and convergence is provided. In conclusion, the article provides a thorough explanation of the Fourier series, an introduction to related concepts, and examples of straightforward Fourier series applications.

References

[1] Trefethen, Lloyd N. Spectral methods in MATLAB. Society for industrial and applied mathematics, 2000.

[2] L. Carleson, On convergence and growth of partial sums of Fourier series, Acta Math. 116 (1966), 135–157.

[3] T. Cazenave, Semilinear Schr¨odinger equations. Courant Lecture Notes in Mathematics, 10. New York University, Courant Institute of Mathematical Sciences, New York; American Mathematical Society, Providence, RI, 2003. xiv+323 pp.

[4] E. Cand`es, J. Romberg, and T. Tao, Robust Uncertainty Principles: Exact Signal Reconstruction from Highly Incomplete Frequency Information. IEEE Transactions on Information Theory, Vol. 52, No. 2, February 2006.

[5] M. Christ and A. Kiselev, Maximal functions associated to filtrations. J. Funct. Anal. 179 (2001), no. 2, 409–425

[6] M. Christ and M. Weinstein, Dispersion of small amplitude solutions of the generalized Korteweg-de Vries equation. J. Funct. Anal. 100 (1991), no. 1, 87–109. [7] I. Daubechies, Ten Lectures on Wavelets. CBMS-NSF Regional Conference Series in Applied Mathematics, 61. Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 1992. xx+357 pp

[7] C. Fefferman, Pointwise convergence of Fourier series, Ann. of Math. 98 (1973), 551–571.

[8] G. Folland, A course in abstract harmonic analysis. Second edition. Textbooks in Mathematics. CRC Press, Boca Raton, FL, 2016.

[9] S. Foucart and H. Rauhut, A Mathematical Introduction to Compressive Sensing. Applied and Numerical Harmonic Analysis. Birkh¨auser/Springer, New York, 2013. [11] D. Griffiths, Introduction Quantum Mechanics. Pearson 2014.

[10] M. Lacey, Carleson’s theorem: proof, complements, variations. Publ. Mat. 48 (2004), no. 2, 251–307.

[11] M. Lacey and C. Thiele, A proof of boundedness of the Carleson operator. Math. Res. Lett. 7 (2000), 361–370.

[12] Wheeden and Zygmund, Measure and Integral. An introduction to real analysis. Pure and Applied Mathematics, Vol. 43. Marcel Dekker, Inc., New York-Basel, 1977. [15] Katznelson, An Introduction to Harmonic Analysis. Second corrected edition. Dover Publications, Inc., New York, 1976.

[13] M. Keel and T. Tao, Endpoint Strichartz estimates. Amer. J. Math. 120 (1998), no. 5, 955–980.

[14] R. Killip and M. Visan, Nonlinear Schr¨odinger equations at critical regularity. Evolution equations, 325–437, Clay Math. Proc., 17, Amer. Math. Soc., Providence, RI, 2013.

[15] E. Lieb and M. Loss, Analysis. Graduate Studies in Mathematics, 14. American Mathematical Society, Providence, RI, 1997.

[16] A. Martinez, An Introduction to Semiclassical and Microlocal Analysis. Universitext. Springer-Verlag, New York, 2002.

[17] Stein, Elias M., and Rami Shakarchi. Fourier analysis: an introduction. Vol. 1. Princeton University Press, 2011.