The Use of Fourier Transform in Heat Equation and Wave Equation

Chenxi He

The affiliated high school of SCNU, ShenZhen, China

* Corresponding Author Email: Chenjq@szu.edu.cn

Abstract. In the solve for the equation for heat transfer and wave movement, using Fourier Transform to simplify the differential equation and solve it. With the periodic property of the wave including standing wave and vibrating string and heat equation and its stable solution, it can provide many new properties of the equation. Then using the same method to solve the heat equation in space, it gives a similar equation like the plane heat transfer equation. The passage mainly discussed about using Fourier transform to solve standing waves equation heat equation in steady state and in the time dependent state. The general way of solving wave equation both standing wave and traveling wave) is finding differential equation for a half period, expending it into the R, using Fourier transform to solve the equation.

Keywords: Fourier transform; Standing wave and vibrating string Solving the heat equation in space

1. Introduction

The Fourier transform is defined as $f(x) = \int_{-\infty}^{\infty} \hat{f}(\lambda) e^{-i\lambda t} d\lambda$. In the formula f(x) is the function of physical space, and $\hat{f}(\lambda)$ is the function of the frequency space.

Consider a quantity can be write in the following way (the original function depends on time)$f(t) = \int_{-\infty}^{\infty} \hat{f}(\lambda) e^{-i\lambda t} d\lambda$The sum of the harmonic oscillators with frequency λ. In this way f(t) can simply be solved by using the $\hat{f}(\lambda)$ since it is easier to solve the function with λ because they share the same frequency [1-10].

There are mainly 3 types of Fourier transform:physical space's function is f(x). Its graph is a line. The frequency space $\hat{f}(\lambda)$. And the f(x) with the $\hat{f}(\lambda)$ have the relation of:$f(x) = \int_{-\infty}^{\infty} \hat{f}(\lambda) e^{-i\lambda t} d\lambda$. It is easy to find out that the frequency space' s grape is also a line (x is defined in all R,λ is defined in all R)when the physical space is the integer in the x axis. It can be solved by the formula $f(x) = \int_{0}^{2\Pi} \hat{f}(\lambda) e^{-i\lambda t} d\lambda$The grape of $\hat{f}(\lambda)$ is a circle (x is defined in Z, λ is defined in [0,2Π] when the physics space' graph is a circle it is easy to know that the $\hat{f}(\lambda)$ is supposed to be a series integer

$$f(\Theta) = \sum_{n=-\infty}^{\infty} \hat{f}(n) e^{in_\theta} \tag{1}$$

(Θ is defined in [0,2Π], n is defined in all Z)

Discrete Fourier transform is defined as

$$\widehat{a_j} = \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} a_k \omega^{kj} \tag{2}$$

In the formula ω is defined as $e^{\frac{2\pi i}{N}}$. For instance,if N is 2. Then ω=-1 $\widehat{a0} = (a0 + a1)/2$, $\widehat{a1}$ = (a0-a1)/ $\sqrt{2}$. The inversion formula of the DFT can be written as $a_j = \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} \widehat{a_k} \omega^{-kj}$. Proof is given as follows.

$\frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} = \frac{1}{\sqrt{N}} \sum_{e=0}^{N-1} a_e \omega^{ke}\, \omega^{-kj}$which is $= \frac{1}{N} \sum_{e=0}^{N-1} a_e \sum_{k=0}^{N-1} \omega^{k(e-j)}$ $= \frac{1}{N} \sum_{e=0}^{N-1} a_e \sum_{k=0}^{v-1} \omega^{km}$ $(m = e - j)$. Since $\omega =$ e to the $\frac{2\Pi i}{N}$. It is easy to know that: ω to the m = 0 is true any value that m is not 0. When the m is = 0 ω to the km is = 1 Then the original formula turns into: $=\frac{1}{N}\sum_{e=0}^{N-1} a_e$ (N when $e=j$. 0 when otherwise) $= a_j$. Assuming there are two set of complex number $a=\{a_0, a_2, \ldots a_{N-1}, b=\{b_1, b_2, \ldots. b_{N-1}\}$. Defined the operator <> as $\langle a, b\rangle = \sum_{j=0}^{N-1} a_j \overline{b_j}$. When mathematicianstake the DFT to the a and b and put the operator <> mathematicianscan find that $\langle a, b\rangle = \langle \hat{a}, \hat{b}\rangle$

Prove

$$\because \overline{\omega} = \omega^{-1}(\hat{a}, \hat{b}) = \sum_{j=0}^{N-1} \hat{a_j}\, \overline{\hat{b_j}} = \frac{1}{N}\sum_{j=0}^{N-1} a_k\, \omega^{kj} \sum_{e=0}^{N-1} \overline{b_e}\, \omega^{-ej} = \frac{1}{N}\sum_{k=0}^{N-1}\sum_{e=0}^{N-1} a_k \overline{b_e} \sum_{j=0}^{N-1} \omega^{j(k-e)} = \frac{1}{N}\sum_{k=0}^{N-1}\sum_{e=0}^{N-1} a_k \overline{b_e} \begin{pmatrix} \text{if} & k=e & N \\ \text{if} & k\neq e & 0 \end{pmatrix} = \frac{1}{N}\left(\sum_{k=0}^{N-1} a_k \overline{b_k}\right) N = \sum_{k=0}^{N-1} a_k \overline{b_k} = \langle a \;\; b\rangle \tag{3}$$

When mathematicianstake a second DFT to a series A. Mathematiciansfind that

$$\hat{\hat{a_j}} = a_{-j}$$

Prove:

Fourier coefficient of f(θ)

The Fourier coefficient is defined as

$$c_j = \frac{1}{2\pi}\int_0^{2\pi} f(\theta) e^{-ij\theta}\, d\theta \tag{4}$$

And for any periodic function. It can be written as

$$f(x) = \sum_{j=-\infty}^{\infty} c_j e^{\left(\frac{2\pi}{T}\right) jxi} \tag{5}$$

This is called Fourier series. However, this won't stop most of the properties of the DFT hold in the case of Fourier series. Consider the property of the operator<>. if mathematicians allow the series to become a function, which means that it makes the series a and b into two function f(θ) and g(θ). In order to make the operator result the same, its' definition will go through a slightly change: it is being integrate of a period and take the mean value (meaning that divided by one of its' periods)

It can be proved by the same way of proving the property of DFT on the operator<>. By using it, it is easy to get the property for Fourier coefficient

$< f, g >=< cf, cg >$(f and g means function, cf and cg means the coefficient of the function f and the function g)

When intergrading the f(θ) for one T and divided it by its T finding that it is the same as the infinite sum of the square of the absolute value of the Fourier coefficient. It has of great significance in the flied of quantum mechanic

If the sum of the $e^{-ij\theta}$ is not infinite, it has the upper bond and the lower bond of -N and N .and to find a function that equals to it. The function is written as $s_N f(\theta)$.This kind of approximation is very useful facing same hard function which is impossible or hard to solving for the infinite sum and this is a way to approximate to the original function since that sin and cos allows less than 1 and when the N goes higher the sin and cos N square are getting less and less. This kind of function has some special properties. $s_N f(\theta)$-f(θ) goes to 0 when N goes to infinity. f(θ) is square integrate limited. the integrate of the absolute value of the f(θ)- $\sum_{j=-N}^{N} c_j e^{\left(\frac{2\pi}{T}\right) jxi}$

goes to 0 when N goes to infinity. Also, in Fourier transform, define a notation that a(i) * b(i)= ∑a(x)b(i-x). It has $\widehat{f(x) * g}(x) = \hat{g}(\varepsilon)\hat{f}(\varepsilon)$ because that $F^{-1}\left(\hat{f}(x) * \hat{g}(x)\right) = F^{-1}\left[\int_{Rn} \hat{f}(\tau)\hat{g}(x-\tau)d\tau\right] = \left(\frac{1}{2\pi}\right)^n \int_{Rn}\int_{Rn} \hat{f}(\tau)\hat{g}(x-\tau)d\tau e^{ix} + dx = \left(\frac{1}{2\pi}\right)^n \int_{Rn} \hat{f}(\tau)\int_{Rn} \hat{g}(x-\tau)e^{i(x-\tau)t} dx e^{i\tau t} d\tau = \int_{Rn} \hat{f}(\tau)\hat{g}(t)e^{i\tau t} d\tau = (2\pi)^n g(t) f(t);$

Take fourier transform in both side

$$(2\pi)^n\widehat{fg} = \hat{f} * \hat{g} \Longrightarrow \widehat{fg} = (2\pi)^{-n}\hat{f} * \hat{g}f(x)\widehat{* g}(x) = \int_{Rn}\int_{Rn} f(\tau)g(x-\tau)d\tau e^{ix\varepsilon}dx =$$

$$\int_{Rn} f(\tau)\left[\int_{Rn} f_2(x-\tau)e^{-i(x-\tau)\varepsilon}dxe^{-i\varepsilon\tau}d\tau\right] = \hat{g}(\varepsilon)\int_{Rn} f(x-\tau)e^{i\varepsilon\tau}d\tau = \hat{g}(\varepsilon)\hat{f}(\varepsilon) \quad (6)$$

Also define $T_{x_0}f = f(x-x_0)$ Then $\widehat{T_{x_0}f(\varepsilon)} = e^{-ix_0\varepsilon}\hat{f}(\varepsilon)$ prove $T_{x_0}f = f(x-x_0)$,$\widehat{T_{x_0}f(\varepsilon)} = \int_{Rn} f(x-x_0)e^{-i\varepsilon t}dt$, Assume $t = x - x_0$. $\widehat{T_{x_0}f(\varepsilon)} = \int_{Rn} f(t)e^{-i\varepsilon(t+x_0)}dt = e^{-i\varepsilon x_0}\int_{Rn} f(t)e^{-i\varepsilon t}dt$ $= e^{-i\varepsilon x_0}\hat{f}(\varepsilon)$ .Define $S_\lambda f(x) = f(\lambda x)$ The $\widehat{S_\lambda f(\varepsilon)} = \lambda^{-n}\hat{f}(\varepsilon/\lambda)$.

Assume $t = x\lambda\widehat{S_\lambda}f(\varepsilon) = \int_{Rn} f(\lambda x)e^{-i\varepsilon x}dx = \int_{Rn} f(t)e^{-\frac{i\varepsilon t}{\lambda}}d\frac{t}{\lambda} = \lambda^{-n}\int f(t)e^{-i\frac{\varepsilon}{\lambda}t}dt = \lambda^{-n}\hat{f}(\frac{\varepsilon}{\lambda})$.
Assume F is defined in (a,b) and it has two continuous derivatives, It has

$$F(x,h) = F(x) + h'F'(x) + F''(x)\frac{h^2}{2} + h^2\varphi(1) \quad (7)$$

It can be deducted $\lim_{h\to 0} F(x+h) = a, a = F(x)$,So $\lim_{h\to 0} F'(x+h) = b + 2ch + o(h) = F'(x) = b$,Then $\lim_{h\to 0} F''(x+h) = F''(x) = 2c + o(1)$,So $c = \frac{F''(x)}{2}$ So $F(x+h) = F(x) + F'(x)h + \frac{F''(x)}{2}h^2 + o(h^2)$

2. Main body

2.1. The vibrating string

With one side fixed at its end point and the other end is able to vibrate freely.

First, the motion of the harmonic oscillator.

The motion of the harmonic oscillator is defined as the movement of the of oscillatory, which is foundation of all other oscillation

If there is an object with the mass m and it is bounded with a spring, which is a massless string which a side of it is bounded with a wall. The surface which the system is lied on is frictionless.

When the point which the object lies as the origin and take the line and the direction which the string's free moving end lies as the x axis. From hook's law. $F = -kx(t)$. And from the Newton's second law.

$F = ma$. Combining it with $ma = -kx(t)$.

This is a separatable differential equation. The answer to it can be given by

$$x(t) = a\cos ct + b\sin ct \quad (8)$$

In the result c =square root of k/m, a and b is given by the initial condition, both of them are constants. Since there are two numbers which are not determined. It will need two none equivalence equation to determine the unknown numbers

If the x (0) and the x' (0). the equation can be given by

$x(t) = x(0)\cos ct + x'(0)\sin ct/c$ In this formula, a = x (0), b=y' (0)/c

The formula $x(t) = a\cos ct + b\sin ct$ Can be written as $x(t) = A\cos(ct - \phi)$. In this formula A = $\sqrt{.a\,2 + b\,2}$, which is called the amplitude. C is the natural frequency the ϕ is phase. 2pi/c is the period of the motion

By using the Euler's formula

$$e^{it} = \cos t + i\sin t. \quad (9)$$

The formula $x(t) = A\cos(ct - \phi)$ can be transform in the complex number.

The standing waves and the traveling waves.

Standing waves are wavelike motions which is given by the equation $y = u(x.t)$ in which $u(x,t) = A(x)B(t)$

Another type is observed usually is the traveling wave. It is like pulling one side of the hormonic oscillation motion and start to make it moving at a constant speed assuming the speed is c

Mathematiciansgot the equation of the traveling wave is $u(x,t) = F(x - ct)$

The c is describing the speed of the motion of the wave. It is called the velocity of the wave

The function $u(x,t) = F(x - ct)$ is used to describe the one-dimensional wave. If wish to use a function for plane wave or higher dimension. The equation will turn into

For examples if it is a spherical wave $E(r) = \frac{Emax}{r} \exp[i(kr + b)]$

Even though it shares the form of one-dimension waves. But this is due to the spherical property and it has the property of three-dimension waves

2.2. Harmonics and superposition of tones

The motion of the vibrating string is a partial differential equation. A homogenous string which is in the x-y place, has the natural length L stretched along the x axis. $y = u(x,t)$ is the function to describe it. considering the string is divided into infinite small string (divided into N strings where N is approaching infinity).

Then the x of the nth string is given by $n = nL/N$.

If each oscillator only oscillates in the y axis, each point is oscillating by linking to its neighbor and the tension along with the string

Noting that

$$y(t) = u(x,t),\ x(n+1) - x(n) = \text{height},\ \text{height} = \text{total length/Total number} \tag{10}$$

with Newton's second law $\rho h y''_n$ (t) = F to the n point. Assuming all the force taken by the n particle is acted by the near two points and the force is proportional to height difference. The tension can be written as $\left(\frac{T}{h}\right)(y_{n+1} - y_n)$ T > 0 and is a constant equal to the coefficient of tension the string. For any neighbor points obey the above equation.

With newton's second law

Can transform it into $\rho h y_n''(t) = \frac{T}{h}\{y_{n+1}(t) + y_{n-1}(t) - 2y_n(t)\}$

Using the above notation, sees that

$y_{n+1}(t) + y_{n-1}(t) - 2y_n(t) = u(x_n + h, t) + u(x_n - h, t) - 2u(x_n, t)$ For any reasonable function F(x) which can be taken the second derivatives.

It holds $\lim_{h\to 0} \frac{F(x+h)+F(x-h)-2F(x)}{h^2} = F''(x)$

Also, as h→0, N→∞, so divide both side by h mathematiciansgot

$$\rho\frac{\partial^2 u}{\partial t} = T\frac{\partial^2 u}{\partial x^2}\text{ , assume } C = \sqrt{\frac{T}{\rho}} \tag{11}$$

$$\frac{1}{C^2}\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} \tag{12}$$

This relation is the one-dimensional wave equation or called the wave equation. The coefficient c > 0 is called the velocity of the wave.

In the above equation, if takes a and b properly, it will allow us to eliminate the coefficient. With some similarly transformation taking that $t = bT$ $U(X,T) = u(x,t)$,

Then $\frac{\partial U}{\partial X} = a\frac{\partial u}{\partial x}$,$\frac{\partial^2 U}{\partial X^2} = a^2\frac{\partial^2 u}{\partial x^2}$ Then$\frac{\partial^2 U}{\partial T^2} = \frac{\partial^2 U}{\partial X^2}$ / $0 \le x \le L \rightarrow 0 \le X \le \pi$

In the process of derivation, it has used many transformations of the constant, so the wave function is not sacrificed with the most common rules. But if return to the original equation by doing the inverse change of the variables

2.3. The wave equation's solution

Two kinds of methods are frequently used in solving the wave equation. They are

1 using traveling waves

2 using the superposition of standing waves

The first method is simple to use, however, it failed to directly give the insight into the problem to us insightfully. Meanwhile the second method is more applicable. It was used to believe that the second method can be only applied in the simple cases and the velocity and the position of the string are given as the initial condition However, with Fourier's ideas the problem can be worked out either way for all initial conditions.

The traveling waves.

In order to simplify the equation, assuming that c=1 and L= Π. The equation needed to be solved becomes

$$\frac{\partial^2 u}{\partial t} = \frac{\partial^2 u}{\partial x^2} \quad \text{on } 0 \leq x \leq \pi \tag{13}$$

The physical explanation is that if F is a function which can be differentiate for 2 times and the u (x, t) = F (x+ t) and the u (x, t) =F (x- t) at time t= 0.

It can be recognized as the graph of F translated to the right by the speed of

1. Then F(x-t) is a traveling wave with speed of 1 which travels to right

F(x+t) is in the opposite

From the above discussion. It represents that the wave equation is linear. It can the property that if a and b is the solution to the equation, the linear combinations of the a and b are the solutions to the equation as well.

Then using two waves which travel in the opposite direction, for instance F and G, are twice differential functions. Then

$$U(x,t)=F(x+t)+G(x-t) \tag{14}$$

Is supposed to be the solution to the wave equation, and all solution is in the form of this.

Supposing that the u is twice differentiable and considering

m = x + t   n= x- t

From the original purpose and those result above with the imposed restrictions 0 <= x <= Π

And the u(x,0) = f (x), with the fact that the end of the string is fixed u (0, t) =0 for all t.

In order to apply the Fourier transform. Expanding the function to the R by repeating it. Setting t = 0 shows that $\frac{\partial^2 v}{\partial m \, \partial n} = 0$

Intergrade the above equation

$$u(m,n)=F(m)+G(n) \tag{15}$$

F and G are not unique function to satisfy this equation. If another initial condition u is given, namely the initial velocity of the string which denoted by g(x)

∴u (x, t) =F(x+t) +G(x-t)

u(x,0)=f(x),.   u(x,t)=F(x+t)+G(x-t).   F(x)+G(x)=f(x). $\frac{\partial u}{\partial t}(x,0) = g(x)$

Where g (0) =g (π)=0 to satisfy the

boundary condition

extend g(x) to R

$$\begin{cases} F(x) + G(x) = f(x) & (16) \\ F'(x) - G\ `(x) = g(x) & (17) \end{cases}$$

Take the derivative of (17)

$$F'(x)+G'(x)=f'(x) \tag{18}$$

$$(18)+(19) \qquad F'(x)=f'(x)+g(x) \tag{19}$$

$$(19)-(18) \qquad 2G'(x)=f(x)-g(x) \tag{20}$$

Integrate (20) and (21)

$$F(x) = \frac{1}{2}\left[f(x) + \int_0^x g(y)\,dy\right] + C_1 \tag{21}$$

$$G(x) = \frac{1}{2}\left[f(x) - \int_0^x g(y)\,dy\right] + C_2 \tag{22}$$

C1 and C2 is constants,

Since f(x)=G(x)+F(x), ∴C1+C2=0

Therefore

$$u(x,t) = \frac{1}{2}[f(x,t) + f(x-t)] + \frac{1}{2}\int_{x-t}^{x+t} g(y)dy \tag{23}$$

This is the d' Alembert's formula.

Wisely pick f and g, to fit the boundary condition $u(0,t) = u(\pi,t) = 0$ for all t

The superposition of the standing waves

Some of the special solutions to the standing waves and the superposition of them, in the form of ϕ(x)ψ(t), can be used in other context like the heat equation. This is called the separation of variables. Due to the linear invariant property of the wave equation, the solutions to the wave equation can be linear combined together and result in another solution to the wave equation. To make a further assuming, it can be inferred superposition of the wave solution can get a new solution

Assuming the t and x are independent. The wanted form of it is$\varphi(x)\Psi''(t) = \varphi''(x)\Psi(t)$

So $\frac{\varphi''(x)}{\varphi(x)} = \frac{\Psi''(t)}{\Psi(t)}$ The left side include only t as variable. While the right is x only. Only possible Ψ and φ to fit the equation is that $\frac{\varphi''(x)}{\varphi(x)} = C = \frac{\Psi''(t)}{\Psi(t)}$ Can be only <0 since the

Basic differential form of the oscillation $x'' + \omega^2 = 0$

If the ω2 is ≤0, It won't be oscillating in the real Axis, so,the solution to the equation

$$\Psi(t) = A\cos\omega t + B\sin\omega t \qquad \varphi(x) = \widetilde{A}\cos mx + \widetilde{B}\sin mx \tag{24}$$

Consider the boundary condition that the string is fixed in the x=0 or x=π

Then $\varphi(0) = \varphi(\pi) = 0$, which suggest that $\widetilde{A} = 0, \widetilde{B} \neq 0$

C is constant, also, the period of theφ(x) is supposed to be $\frac{2\pi}{n}$ (n is constant)

If -1<ω<-1, It can never fit the boundary condition, and for allω<-1, since change of the sign of the

Ω won't affect the equation's form, it can be concluded that for all ω exist, It must larger

Then 0, $(A\omega\cos\omega t + B\omega\sin\omega t)\sin\omega x$

$u(x,t) = m$

If x∈ [0, π] uω(x,t) touch the axis only twice, It's called the Fist harmonic (for all t). if touch n time, it called (n-1) th harmonic the points which equal to 0 for all t are called notes. while those have the higher A are called anti-nodes. Recalled the former, wave equation is linear invariant, So the linear combination like all + Bv is the solution to it as well concluded that

$$u(x,t) = \sum_{\omega=1}^{\infty}(A\omega\cos\omega t + B\omega\sin\omega t)\sin\omega x \tag{25}$$

This is known as founier series. Require the initial condition for t=0. Assume f(x)=u(x,0)

$$f(x) = \sum_{\omega=1}^{\infty} A\omega\sin\omega x \tag{26}$$

If the (27) is true. time both side by sinnx and integrate [0, π]

$$\int_0^\pi f(x)\sin nx\,dx = \int_0^\pi\left(\sum_{\omega=1}^{\infty} A\omega\sin\omega x\right)\sin nx\,dx = \sum_{\omega=1}^{\infty} A\omega\int_0^\pi \sin\omega x\sin nx\,dx = An\frac{\pi}{2} \tag{27}$$

$$\text{So } \int_0^\pi \sin\omega x\sin nx\,dx = \begin{cases} 0 & \omega \neq n \\ \frac{\pi}{2} & \omega = n \end{cases} \tag{28}$$

If f is extended to [-π, π] and it is an odd function, the former answer still holds, if an even function is needed. Assuming $g(x) = \sum_{\omega=0}^{\infty} A'\omega\cos\omega x \qquad x \in [-\pi,\pi]$

Then $F(x) = \sum_{\omega=1}^{\infty} A\omega \sin \omega x + \sum_{\omega=0}^{\infty} A'\omega \cos \omega x \qquad x \in [-\pi, \pi]$
Applying Euler's identify

$$e^{ix} = \cos x + i \sin x \tag{29}$$

$$F(x) = \sum_{\omega=-\infty}^{\infty} a\omega e^{i\omega x} \tag{30}$$

According to (26)

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} e^{i\omega x} e^{-int} dx = \begin{cases} 0 & \omega \neq n \\ 1 & \omega = n \end{cases} \tag{31}$$

Then, $a_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} F(x) e^{-inx}$

2.4. The plucked string

Applying the former conduction into the plucked string.

To simply the question, choosing the unit in order to let the string fit condition that is taken $[0, \pi]$ and makes C=1 $C = \sqrt{T/P}$. Assuming the string is oscillated to height hat point $\rho(0 < \rho < \pi)$.

Those are the initial condition

So $$f(x) = \begin{cases} \frac{xh}{\rho} & 0 \leq x \leq \rho \\ \frac{h(\pi - x)}{\pi - \rho} & \rho \leq x \leq \pi \end{cases} \tag{32}$$

Finding the initial velocity g(x) identically equal to zero
Finding the Fourier coefficient of

$$f(x) = \sum_{\omega=1}^{\infty} A\omega \sin \omega x \qquad A_\omega = \frac{2h}{\omega^2}\frac{\sin \omega\rho}{\rho(\pi-\rho)} \tag{33}$$

$$u(x,t) = \sum_{\omega=1}^{\infty} A\omega \cos \omega t \sin \omega x \tag{34}$$

The solution can be rewritten as

$$u(x,t) = \frac{f(x+t)+f(x-t)}{2} \tag{35}$$

As the deduction before, First extend the f to [-π, π] by make it odd.
Then extend f to R with a period of 2π, So

$$\cos v \sin u = \frac{1}{2}[\sin(u+v) + \sin(u-v)] \tag{36}$$

2.5. Heat equation

Assuming there is an infinite large as well as super thin, which means the plate lies in place R^2

Distribute the system by giving it an initial heat distribution and assume it t=0. Assume the heat at point (x, y) at t is given by $u(x, y, t)$

Separating each square is infinite small

Then the energy is S is

$$H(t) = \sigma \iint_S u(x, y, t) dxdy \tag{37}$$

"S" is the area of the square $\sigma > 0$, σ is called the special heat of the material.

The flow of H into S' is

$$\frac{\partial H}{\partial t} = \sigma \iint_S \frac{\partial u}{\partial t} dxdy \tag{38}$$

$$\approx \sigma S \frac{\partial u}{\partial t}(x_0, y_0, t) \tag{39}$$

Since that mathematiciansconsider the change of T in a infinite small small is insignificant compare to "u" itself in (x,y,t)

So $\iint_S \partial xdy \approx S$

The flow of heat in one of the flour sides, let say the right side

$$-Rh\frac{\partial u}{\partial x_0}\left(x_0+\frac{h}{2}, y_0, t\right) \tag{40}$$

R>0, R is called conductivity of the material. So, the total tour sides heal flow of S is

$$Rh[\frac{\partial u}{\partial x}\left(x_0+\frac{h}{2}, y_0, t\right)-\frac{\partial u}{\partial x}\left(x_0-\frac{h}{2}, y_0, t\right)+\frac{\partial u}{\partial y}\left(x_0, y_0+\frac{h}{2}, t\right)-\frac{\partial u}{\partial y}\left(x_0, y_0-\frac{h}{2}, t\right)] \tag{41}$$

Using the mean value theorem and letting h→0

$$\frac{\sigma}{h}\frac{\partial u}{\partial t}=\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2} \tag{42}$$

This is the time dependent heat equation.

2.6. Steady state solution

According to common sense, if a system with no exchange of material or energy with the outer world, it will eventually to become uniformly this same T.

This is one of the descriptions of the second law of the void dynamics, In an isolated system. It will eventually stop all the transformation of the energy. This state of system is known as thermal equilibrium.

In the above disc heat equation, it reaches the steady state, which mathematically $\frac{\partial u}{\partial t}=0$

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0 \tag{43}$$

Using the sign" Δ" to represent that $\frac{\partial^2 u(x,y,...)}{\partial x^2}+\frac{\partial^2 u(x,y,...)}{\partial y^2}+\ldots\ldots$ the" Δ" is known as Laplace operator or Laplacian, (a) can be written in Laplacian $\Delta u=0$

The solutions to this equation are known as avionic function.

The demoing of the disc (unit) in the plane.

$$D=\{(x,y)\in R^2: x^2+y^2<1\} \tag{44}$$

Using polar coordinate to describe the Down Dan boundary condition (circle) C of the problem.

$$D=\{(r,\theta): 0\le r<1\} \quad C=\{(r,\theta): r=1\} \tag{45}$$

This is called pirichret problem.

Using the polar coordinates to describe the (a), name $f(\theta)=u(1,\theta)$, It can be written as

$$\Delta u=\frac{\partial^2 u}{\partial r^2}+\frac{1}{r}\frac{\partial u}{\partial r}+\frac{1}{r^2}\frac{\partial u}{\partial \theta^2} \tag{46}$$

Since $\Delta u=0$ according to previewers discussion

$$r^2\frac{\partial^2 u}{\partial r^2}+r\frac{\partial u}{\partial r}=\frac{\partial^2 u}{\partial \theta^2} \tag{47}$$

Assuming $u(r,\theta)$ is in the form of $F(r)g(\theta)$. It can be conducted that $\frac{r^2F''(r)+rF'(r)}{F(r)}=\frac{-G''(\theta)}{G(\theta)}$, then

$$\begin{cases} G''(\theta)+CG(\theta)=0 \\ r^2F''(r)+rF'(r)-CF(r)=0 \end{cases}$$

$G(\theta)$ is taking in as person,

So $\lambda\ge 0$, So like the heat equation $C=\omega^2$, where ω is an interior $G(0)=\tilde{A}\cos\omega\theta+\tilde{B}\sin\omega\theta$

Using Euler's identity , $e^{i\theta}=\cos\theta+i\sin\theta$,$G(\theta)=Ae^{i\omega\theta}+Be^{-i\omega\theta}$

If $C = \omega^2, \omega \neq 0$, It can be soon solution $F(r) = r^m, F(r) = r^{-m}$,If $\omega = 0, F(r) = 1, F(r) = \log r$

Since $u(r,\theta)$ is bounded when r→∞, so, only $F(r) = r^{-m}(c > 0)$ and

$F(r) = 1, C = 0$ so $u_\omega(r,\theta) = r^{(\omega)}e^{i\omega\theta}$ $\omega \in Z$

(a)Is linear equilibrium so

$$u(r,\theta) = \sum_{\omega\infty}^{\infty} \alpha\omega r^{(\omega)}e^{i\omega\theta} \tag{48}$$

Then $f(\theta) = u(1,\theta) = \sum_{\omega=-\varphi}^{\infty} \alpha\omega e^{i\omega\theta}$

3. Conclusion

The passage mainly discussed about using Fourier transform to solve standing waves equation heat equation in steady state and in the time dependent state. The general way of solving wave equation both standing wave and traveling wave) is finding differential equation for a half period, expending it into the R, using Fourier transform to solve the equation. For heat equation, the main step is to using the conservation of heat in an infinite square and then get the differential equation. Within the heat equation problem, it has been assumed that no radiation or heat lose through air or other media. It will be discussed later about the system with heat exchange. There are still some theorems the heat equation solution hold $\frac{\sigma}{h}\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}$. First, the straight lines in the second dimension will not be a solution to the differential equation. Second, any circle of the plane can’t be the solution to it.If take consideration of the heat transfer in the three dimensional space, if taking the same way the simplify the situation, assume there is an infinite large space in the 3 dimensional world, it is a prefect conduct with no loss of energy in the transform of heat, there are not any other body for it to transfer its energy, there is no radiation. Now giving it an initial distribution, consider this time as the time notated by t equals 0. Assuming each point in the given time t, consider the heat of the given point (x,y,z) at the time which is 0,then the energy in the volume v is $H(t) = \sigma \iiint_v u(x,y,z,t)\, dxdydz$. In the formula xina is the same as the before discussion, then the heat that flow into V is given by $\frac{\partial H}{\partial t} = \sigma \iiint_v \frac{\partial u}{\partial t} dxdydz$, which is about the same value as $\sigma h^3 \frac{\partial u}{\partial t}(x_0, y_0, z_0)$ .Since the partial derivative of H toward t is small since there is one a small initial distribution, and also when t is small, the difference caused by the change of derivative becomes less insignificant. Apply the Newton’s low of cooling, it is given that the heat through one surface of the cube, let’s say the right surface, the heat flow is given by$Rh\frac{\partial u}{\partial t}\left(x_0 + \frac{h}{2}, y_0, z_0, t\right)$. The R has the same meaning as before, the total heat which flow the V is given by $Rh\left[\frac{\partial u}{\partial t}\left(x_0 + \frac{h}{2}, y_0, z_0, t\right) - \frac{\partial u}{\partial t}\left(x_0 - \frac{h}{2}, y_0, z_0, t\right) + \frac{\partial u}{\partial t}\left(x_0, y_0 + \frac{h}{2}, z_0, t\right) - \frac{\partial u}{\partial t}\left(x_0, y_0 - \frac{h}{2}, z_0, t\right) + \frac{\partial u}{\partial t}\left(x_0, y_0, z_0 + \frac{h}{2}, t\right) - \frac{\partial u}{\partial t}\left(x_0, y_0, z_0 - \frac{h}{2}, t\right)\right]$If h goes to 0 and using the mean value theorem.$\frac{\sigma}{R}\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}$ .This is the 3D heat transfer equation (time dependent)

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