Application of Fast Fourier Transform

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Abstract. Fourier analysis is most frequently used as a univariate approach for either modeling or simplifying data. It may also be used as a method for multivariate data analysis. There are various connections between Fourier analysis and trend analysis. It takes a fresh look at how data sets are related. In the case of Fourier analysis, the technique clarifies the time dimension variable in the data set. The most fundamental kind of Fourier analysis works under the idea that many events have a periodic nature and that fluctuations in other variables brought on by this periodicity may be eliminated using Fourier transforms. By using the residual (i.e., time-independent) variance from other variables, Fourier-transformed data may be subjected to more powerful analysis.Based on differential matrices and semidiscrete Fourier transforms, this paper summarizes the key problems in Fourier analysis, FFT. Secondly, this paper points out the application of F FT in the field of modern science and technology and the main progress of current FFT research, and on this basis, the research prospects of FFT law are prospected.

Keywords: Fourier analysis; differential matrices; FFT.

1. Introduction

When you push the number buttons on a phone while on a call, have you ever noticed that each model makes a distinctive sound? This is due to the fact that each button is made up of two distinct sinusoids, which may be used to distinguish one button from another. By applying a Fourier transform to the input and analyzing the frequencies, someone may determine what keys you press when using your phone to navigate a menu. Physics-related circumstances frequently involve oscillations and vibrations. Typically, oscillatory motion is straightforward and may be represented by a single sine or cosine function (e.g., weights on springs, pendulums, harmonic waves, etc.). Nowadays, Fast Flourier Transformation is a popular function to extract important information and distinguish various signals. The essay will from differentiation Matrices and unbounded grids to illustrate FFT. FFT's first application is to transform time-domain signals to frequency domain signals in researching people's emotions. By using FFT, the researchers can find the features of volunteer's emotions. FFT's second application is to show the frequency weights among many signals. Apparently, FFT can not only be used in Psychology and Informatics, but it can also be applied in other areas [1-17].

2. Main Works

2.1. Differentiation Matrices

2.1.1 Discrete differential processes

For a given set of grid point $\{x_j\}$ and the corresponding function value $\{u(x_j)\}$, the derivative of u can be approximated by some finite difference formula. This method is through finite differences to utilize spectroscopic methods.

Specifically, set a uniform grid $\{x_1,...,x_N\}$,with $x_{j+1}-x_j=h$ for each j and a set of corresponding data values $\{u,...,u_N\}$. Then, suppose ω_j represents the derivative of u at x_j ,so the standard second-order finite difference is approximately

$$\omega_j = \frac{u_{j+1} - u_{j-1}}{2h} \tag{1}$$

The above equation can be simply derived by considering Taylor's expansion $u(x_{j+1})$ and $u(x_{j-1})$. For convenience, suppose the problem is periodic, and take $u_0 = u_N$ and $u_1 = u_{N+1}$, Discrete differential processes can therefore be represented as matrix vector multiplication.

$$\begin{pmatrix} \omega_1 \\ \\ \cdots \\ \\ \omega_N \end{pmatrix} = h^{-1} \begin{pmatrix} 0 & \frac{1}{2} & \cdots & & -\frac{1}{2} \\ -\frac{1}{2} & 0 & \cdots & & \\ & & \cdots & & \\ & & \cdots & 0 & \frac{1}{2} \\ \frac{1}{2} & & \cdots & -\frac{1}{2} & 0 \end{pmatrix} \begin{pmatrix} u_1 \\ \\ \cdots \\ \\ u_N \end{pmatrix} \tag{2}$$

This matrix is Toeplitz and has constant entries along the diagonal line, i.e, a_j depends only on $i - j$, it also "wraps" the matrix diagonally. Another way to derive this is through local interpolation and differentiation processes:

$$\begin{pmatrix} \omega_1 \\ \\ \cdots \\ \\ \omega_N \end{pmatrix} = h^{-1} \begin{pmatrix} & & \cdots & & \cdots & \frac{1}{12} & -\frac{2}{3} \\ & & \cdots & -\frac{1}{12} & \cdots & & \frac{1}{12} \\ & & \cdots & -\frac{2}{3} & \cdots & & \\ & & \cdots & 0 & \cdots & & \\ & & \cdots & -\frac{2}{3} & \cdots & & \\ -\frac{1}{12} & & \cdots & \frac{1}{12} & \cdots & & \\ \frac{2}{3} & -\frac{1}{12} & \cdots & & \cdots & & \end{pmatrix} \begin{pmatrix} u_1 \\ \\ \cdots \\ \\ u_N \end{pmatrix} \tag{3}$$

This is a pentagonal rather than a triagonal circulating matrix. The matrices of (2) and (3) are examples of differentiation matrices.

The first Matlab program, program 1 illustrates the behavior of (3). The program compares the finite difference approximation ω_j with the exact derivative, $u(x) = e^{\sin(x)}$, for various values of N. Having studied the finite differences of the second and fourth orders above, it is clear that considering the sixth, eighth, and higher order schemes will result in a circulation matrix with increased bandwidth. The idea behind spectral method is to take this process to the limit at least in principle and work with a differentiation formula of infinite order and infinite bandwidth, a dense matrix .

Given an infinitely isometric mesh, the following infinite matrix is obtained:

$$D = h^{-1} \begin{pmatrix} \cdots \\ \frac{1}{3} \\ -\frac{1}{2} \\ 1 \\ 0 \\ -1 \\ \frac{1}{2} \\ -\frac{1}{3} \\ \cdots \end{pmatrix} \tag{4}$$

This is a skew-symmetric (DT=-D) doubly infinite Toeplitz matrix, also known as a Laurent operator [Hal74, Wid65]. All its entries are nonzero except those on the main diagonal. But in practice one does not work with an infinite matrix.

For finite N , taking N even for simplicity , here is the N*N dense matrix

$$D_N = \begin{pmatrix} & .... & \\ \cdots & \frac{1}{2}\cot\frac{3h}{2} & \cdots \\ \cdots & -\frac{1}{2}\cot\frac{2h}{2} & \cdots \\ \cdots & \frac{1}{2}\cot\frac{1h}{2} & \cdots \\ & 0 & \\ \cdots & \frac{1}{2}\cot\frac{1h}{2} & \cdots \\ \cdots & -\frac{1}{2}\cot\frac{2h}{2} & \cdots \\ \cdots & \frac{1}{2}\cot\frac{3h}{2} & \cdots \\ & \cdots & \end{pmatrix} \tag{5}$$

Transforming the cotangent function, it can be seen that this matrix is indeed circular like Toprits. Program 2 was the same as program 1, but when (3) was replaced by (5), the error in output 2, which had some different outputs, would quickly decrease until it reached such high precision that the rounding error on the computer would prevent any further improvement.

In addition, it is important to note how different it is from the confluence rate of finite difference and finite element methods. When Nincreases, the error in the finite difference or finite element scheme usually decreases like $O\left(N^{-m}\right)$, and for some constants m, it depends on the smoothness of the approximate order settlement. For the spectral method, convergence at an $O\left(N^{-m}\right)$ rate per m is achievable if the solution is infinitely differentable; If the solution is analyzable, faster convergence is achieved at the $O\left(c^{N}\right)$ rate (0<c<1). The basic principle of spectral collocation is to interpolate the data globally given the discrete data on the grid, and then evaluate the derivative of the interpolated values on the grid. For periodic probabilities, triangle intersections are usually used on equally spaced points, and for non-periodic problems,mathematician usually use polynomial interpolation on points with uneven spacing.

2.2. The Semidiscrete Fourier Transform

The first spectral method is given by the double unlimited matrix. Because the method is suitable for discrete unbounded domains, it is not a widely applicable method. This method introduces the mathematical ideas required for the following practical scenarios.

The infinite grid is denoted by hZ , with grid points $x_j = jh$ for $j \in Z$.

Thought derivation using semi-discrete Fourier transforms and interpolation of finite sinc functions (4). The Fourier transform of a function $u(x)$ $, x \in R$ is the function

$\hat{u}(k)$ defined by

$$\hat{u}(k) = \int_{-\infty}^{\infty} e^{-ikx} u(x)dx, \ \ k \in R \tag{6}$$

$\hat{u}(k)$ can be expressed as the amplitude density at the wave enumeration k, so the process of decomposing the function into its constituent waves is called Fourier analysis. Similarly, u can be obtained by inverse Fourier transform.

$$u(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ikx}\hat{u}(k)dk \quad , x \in R \tag{7}$$

Equation (6) is Fourier synthesis. where the variable x is a physical variable,k corresponding to a Fourier variable or wave number.

It is important to note that, here, consider the Fourier transform of the R range exceeding hZ rather than the exact analogue of R, and its inverse existence in this case. The key point is that since the spatial domain is discrete, the wave number k will no longer cover all the ranges of R. Conversely, the appropriate wave number domain is a bounded interval of length $2\pi/h$, and a suitable choice is $[-\pi/h, \pi/h]$. The key is that k is bounded because r is discrete.

The root cause of these connections is aliasing in the Fourier transform.

If $k_1 \neq k_2$, then the two complex exponential $f(x)=\exp(ik_1x)$ and $g(x)=\exp(ik_1x)$ are not equal in the field of real numbers. Ifmathematicianlimit f and g to hZ , Then can get $f_j=\exp(ik_2x)$ and $g_j=\exp(ik_2x)$, If k_1-k_2 is an integer multiple of $2\pi/h$, Then can get $f_j=g_j$ for each j. This is true for any complex exponential. There are infinitely many complex exponents that match the grid Hz-alias of k, so it is sufficient to measure the number of paths of the grid at intervals of $2\pi/h$, and for reasons of symmetry, it can choose the $[-\pi/h, \pi/h]$ interval.

For a function v defined on hZ , the semi-discrete Fourier transform is defined $\hat{v}(k)=h\sum_{j=-\infty}^{\infty} e^{ikx_j} v_j$, $k\in[-\pi/h, \pi/h]$. The inverse semi discrete Fourier transformation is

$$v_j=\frac{1}{2\pi}\int_{\frac{\pi}{h}}^{\frac{\pi}{h}} e^{ikx_j}\hat{v}(k)dk, j\in Z \tag{8}$$

The (8) approximates $u(x)=\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ikx}\hat{u}(k)dk, x\in R$ (6) by truncating R to$[-\frac{\pi}{h},\frac{\pi}{h}]$. As$h\to 0$, the formulas converges.

For a function v defined on hZ with valuev_j atx_j , the semidiscrete Fourier transformation is defined by $\hat{v}(k)=h\sum_{-\infty}^{\infty} e^{-ikx_j} v_j, k\in[-\frac{\pi}{h},\frac{\pi}{h}]$ (2.3), (2.4) is in the same situation. It is the “space”variable that is discrete and the “Flourier”variable that is a bounded interval.

Evaluate the same formula for $x\in R$ rather than just $x_j\in hZ$ After determine $\hat{v}$, define interpolant which is provided,p by $p(x)=\frac{1}{2\pi}\int_{-\frac{\pi}{h}}^{\frac{\pi}{h}} e^{ikx}\hat{v}(k)dk, x\in R$, $p(x_j)=v_j$ for each j. By construction, the Fourier transformation $\hat{p}$, is $\hat{p}(k)=\begin{cases}\hat{v}(k), k\in[-\frac{\pi}{h},\frac{\pi}{h}]\\ 0, otherwise\end{cases}$. Thus, $\hat{p}$ has compact support in$[-\frac{\pi}{h},\frac{\pi}{h}]$, p is the band-limited interpolant of v. In this way, $\hat{p}$ has compact support and the support is contained in the particular interval$[-\frac{\pi}{h},\frac{\pi}{h}]$.

Give first two descriptions of spectral differentiation of a function v defined on hZ.

1) Given v, determine its band-limited interpolant p.
2) Set$\omega_j=p'(x_j)$

3) If u is a differentiable function with Fourier transform of $u^{'}$ is $ik\hat{u}(k)$ $\overset{\wedge}{u^{'}}(k) = ik\hat{u}(k)$ .Meanwhile, have an equivalent procedure for spectral differentiation:

i. Given v, compute its semidiscrete Fourier transform
ii. $\hat{v}$
iii. Define $\overset{\wedge}{\omega}(k) = ik\hat{u}(k)$
iv. Compute ω from $\overset{\wedge}{\omega}$
v. Derive the coefficients of the matrix.

Use the Fourier transform to get back and get a understanding of the band-limited interpolant p(x). Let δ be the Kronecker delta function, $\delta_j = \begin{cases} 1, j = 0 \\ 0, j \neq 0 \end{cases}$. Therefore, the semidiscrete Fourier transform of δ will be a constant :$\overset{\wedge}{\delta}(k) = h$ for all $k \in [-\pi/h, \pi/h]$. Also, the band-limited intrpolant of δ can be written as:

$p(x) = \frac{h}{2\pi}\int_{-\pi/h}^{\pi/h} e^{ikx}\mathrm{d}k = \frac{sin(\pi x/h)}{\pi x/h}$, which is also called sinc function. Because every general function can be written as $v_j = \sum_{m=-\infty}^{\infty} v_m \delta_{j-m}$, the band-limited interpolant of v can be written as $p(x) = \sum_{m=-\infty}^{\infty} v_m S_h(x - x_m)$. Hence, the derivative is $w_j = p'(x_j) = \sum_{m=-\infty}^{\infty} v_m S'_h(x_j - x_m)$. Using the result of the matrix D of (1.4), mathematiciancan get the derivative $S'_h(x_j) = \begin{cases} 0 & j = 0, \\ \frac{(-1)^j}{jh} & j \neq 0. \end{cases}$ It is important that the smoother the discrete function is, the more accuracy of interpolant will have. Sinc function doesn't provide a good accuracy on the oscillations near the discontinuity. What's more, the order of accuracy can be improved by 1 when each extra derivative is possessed by u. For example, the function $v(x) = \begin{cases} 1 & |x| \leqslant 3 \\ 0 & |x| \geqslant 4 \end{cases}$ cannot be interpolated well, especially the discontinuous grids. Therefore, it is important to get higher order spectral derivatives, which can improve the order of accuracy. For example, the higher order derivatives

$S''_h(x_j) = \begin{cases} -\frac{\pi^2}{3h^2} & j = 0, \\ 2\frac{(-1)^{j+1}}{j^2h^2} & j \neq 0 \end{cases}$ can tell entries of each column of D^2.

2.3. Flourier Analysis----a method to finish signal processing approach.

2.3.1 The first application of the method

The first application of the method is to go from time-domain signals to frequency domain signals. Alsomathematiciancan transform the frequency domain signals to time domain signals. The most popular method to finish Flourier Analysis is the Fast Flourier Transform(FFT) $X_k = \sum_{i=0}^{N-1} x_i(n) e^{\frac{-j2\pi ik}{N}}$, for $k = 0,1,2...N-1$ where X(k) denotes the discrete Flourier coefficient, $x_i(n)$ is the signal which will be input in the time domain.

For spectral analysis, it is convenience to take magnitude-squared of the FFT to obtain an estimate of the power spectral density (PSD).

In the Emotion recognition using Flourier transform and genetic programming, some researchers use Electroencephalogram (EEG) to record the volunteers ‘emotions when they watch emotional movie clips. By using Fast Flourier Transform(FFT), the researchers extracted the important features of four class of emotions(sadness, fear, calm, happy). Then they use the periodogram to compute.

2.3.2 The second application of the method

The second method is to show the frequency weights between the other frequency texts of signals. The method will use FD in FFT, which is very low-cost technique.

Then, in next step, for capturing overall shape of the frequency form. The features are found: spectral centroid, spectral variance, spectral skew and spectral kurtosis.

Through their feature selecting process, the researchers use the data to find the result from the wheel wear situation.

2.3.3 Application in some field

Sometimes, the initial spectrum is impossible to be analyzed. In this situation, the initial spectrum can be solved by FFT to get frequency spectra. Hence,mathematiciancan get information about the ratio of the value from different time. In chatter detection, this method is always utilized to get a frequency spectrum and the energy ratio, which can reflect the chatter level. The initial spectrum always has much noise information, which interrupts our judgements. In the area of moisture measurements in civil engineering. This method is always used to help get the peak value of different spectra and indicate qualitative changes in the water content

Also, another benefit of FFT is that it can obviously decrease the time computer use to calculate. It requires the calculation of the FFT only once at a computational cost of O(NlogN). Therefore, relatively more efficient FFT-based simulations are feasible on large grids. In the field of polycrystalline materials analysis, based on FFT, many useful methods have been discovered such as Galerkin formulation. It can help deal the weak form of equilibrium and use the same back-and-forth between Cartesian and Fourier space as FFT.In this way, this method can get rid of the dependence on alinear reference medium and the Green’s function of the classical schemes.

FFT performs well in the crack detection. FFT is efficient in solving the image of material of building. Wavelet transform and FFT can decrease the amount of noisy images and than reflect the discrete part of the image, By this way, the crack, which is revealed as the discrete part, can be detected obviously. This can be utilized in the detection of crack in building material.

3. Summary

Fourier analysis is most frequently used as a univariate approach for either modeling or simplifying data. It may also be used as a method for multivariate data analysis. There are various connections between Fourier analysis and trend analysis. It takes a fresh look at how data sets are related. In the case of Fourier analysis, the technique clarifies the time dimension variable in the data set. The most fundamental kind of Fourier analysis works under the idea that many events have a periodic nature and that fluctuations in other variables brought on by this periodicity may be eliminated using Fourier transforms. By using the residual (i.e., time-independent) variance from other variables, Fourier-transformed data may be subjected to more powerful analysis.Flourier analysis can convert signals from their original domain to a representation in the frequency domain. Now FFT can finish the transformations. Fast Flourier Transformation is used in science, engineering and mathematics. Though there was an interval between the time point which it was derived and the one that it was applied in different areas, it has been regarded as one of the top 10 algorithms of the last century.

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4. Appendix

Appendix

Program1

```
% p1.m - convergence of fourth-order finite differences
% For various N, set up grid in [-pi,pi] and function u(x):
    Nvec = 2.^(3:12);
    clf, subplot('position',[.1 .4 .8 .5])
    for N = Nvec
        h = 2*pi/N; x = -pi + (1:N)'*h;
        u = exp(sin(x)); uprime = cos(x).*u;
        % Construct sparse 4th-order differentiation matrix:
        e = ones(N,1);
        D = sparse(1:N,[2:N 1],2*e/3,N,N)...
      - sparse(1:N,[3:N 1 2],e/12,N,N);
        D = (D-D')/h;
        % Plot max(abs(D*u-uprime)):
        error = norm(D*u-uprime,inf);
        loglog(N,error,'.','markersize',15), hold on
end
grid on, xlabel N, ylabel error
title('Convergence of 4th-order finite differences')
semilogy(Nvec,Nvec.^(-4),'--')
text(105,5e-8,'N^{-4}','fontsize',18)
```

Program 2

```
% p2.m - convergence of periodic spectral method (compare p1.m)
% For various N (even), set up grid as before:
    clf, subplot('position',[.1 .4 .8 .5])
    for N = 2:2:100;
        h = 2*pi/N;
        x = -pi + (1:N)'*h;
        u = exp(sin(x)); uprime = cos(x).*u;
        % Construct spectral differentiation matrix:
        column = [0 .5*(-1).^(1:N-1).*cot((1:N-1)*h/2)];
        D = toeplitz(column,column([1 N:-1:2]));
        % Plot max(abs(D*u-uprime)):
        error = norm(D*u-uprime,inf);
        loglog(N,error,'.','markersize',15), hold on
end
grid on, xlabel N, ylabel error
title('Convergence of spectral differentiation')
```