Application and Analysis of The Residue Theorem

Zihe Ren*

School of Beijing No. 12 Middle School International Department, Beijing, China

* Corresponding Author Email: 100427@yzpc.edu.cn

Abstract. This paper studies the derivation of the residue theorem and its application. The residue theorem is an important part of the theory of complex functions and plays a critical role in the development and application of complex functions. At the same time, the residue theorem advances the method of solving the value of definite integrals to a new stage. The issue of the Cauchy residue theorem has received considerable attention. Also, residue theorem lays the foundation for the development of integral theory, and it has been widely used in many scientific areas. The goal of this study is to investigate the properties and applications of the residue theorem. To this end, the strategies to deduce the residue theorem are summarized. Then this paper uses several examples that relate to the residue theorem to help people understand the residue theorem more deeply. The integrals under consideration include trigonometric integrals and integrals involving power functions.

Keywords: Residue theorem; Improper integral; Cauchy integral formula.

1. Introduction

The residue is the integral of a function at its isolated singularity. In view of the early development of the theory of complex analysis, this concept is of great significance to the classification of isolated singularities and the relationship between them [1,2]. The Cauchy residue theorem is a generalization of Cauchy integral theorem and Cauchy integral formula that are obtained in complex analysis [3]. Cauchy residue theorem has also been widely used in practical life. Residue theorem can put the integral path for smooth closed curve into a complex integral calculation in the sum of the residue of isolated singularity point. However, before that one must have a correct understanding of many concepts such as the isolated singularity [4]. Therefore, mastering the method of calculating the residue, especially the method of finding the residue at the pole, is the key point of applying the residue theorem. It is also the key point of solving the integral of this special real function in practice. For example, the residue theorem is an effective tool for calculating the path integral of analytic function along a closed curve and the integral of real function.

Throughout the paper, it will first introduce some basic concepts relating to the residue theorem. Afterwards, the residue theorem is applied to several improper integrals that are hardly calculated by conventional methods.

2. Residue Theorem

2.1. Cauchy Integral Formula

Cauchy integral formula plays an important role in analytical function. It is stated that $f(a) = \frac{1}{2\pi i}\oint \frac{f(z)}{z-a} dz$, or equivalently, $\lim_{z\to n} \frac{f(a)}{z-a} dz = f(a) \lim_{z\to n} \oint \frac{1}{z-a} dz = f(a) \cdot 2\pi i$.

The derivative of $\left(\frac{f(\gamma)}{((\gamma-z)^m}\right)$ is equal to $m\frac{f(\gamma)}{(\gamma-z)^{m+1}}$. According to the n-th squared variance formula [5,6], $f^n(a) = \frac{n!}{2\pi i}\oint \frac{f(z)}{z-a^{n=1}} dz$. Using the limit, $\lim_{z\to a} \frac{\oint \frac{f(\gamma)}{(\gamma-z)^n} d\gamma - \oint \frac{f(\gamma)}{(\gamma-z)^n} d\gamma}{z-a}$ is equal to the derivative at a. Using the n-th power formula, it is found that the above expression is $\lim_{z\to a}\oint f(\gamma) \cdot (\frac{1}{\gamma-z} - \frac{1}{\gamma-a}) \cdot (\frac{1}{(\gamma-z)^{n-1}} + \frac{1}{(\gamma-z)^{n-2}} + \cdots + (\frac{1}{(\gamma-z)^{n-1}})$. After a further simplification, it is obtained that the final result as $\oint f(\gamma) \cdot n \cdot \frac{1}{(\gamma-z)^{n+1}} d\gamma$=n$\cdot \oint \frac{f(\gamma)}{(\gamma-z)^{n+1}} d\gamma$ [7].

For the Taylor expansion [8],

$$f(z)=\frac{1}{2\pi i}\oint_C\frac{f(\zeta)}{\zeta-z}d\zeta=\sum_{k=0}^{\infty}f^{(k)}(b)\cdot(z-b)^k \quad (1)$$

While for the Laurent expansion [9],

$$f(z)=\frac{1}{2\pi i}\oint\sum_{k=-\infty}^{\infty}\frac{f(\zeta)(z-b)^k}{(\zeta-b)^{k+1}}d\zeta=\sum_{k=-\infty}^{\infty}C_k\cdot(z-b)^k \quad (2)$$

2.2. Residue Theorem

The Reisude theorem is a useful method to calculate the value of definite integral. It is stated that the integral has an intimate relation to the residue of the integrand at singular points, i.e.,

$$\oint_l f(z)\,dz=2\pi i\sum_{k=1}^{n}\mathrm{Res}(b_k) \quad (3)$$

Where $\mathrm{Res}\,f(b_k)=C_{-1}(b_k)$ [10]. This formula can be proofed easily. By complex Cauchy's theorem $\oint_1 t(z)\,dz=\sum_{k=1}^{n}\oint_l f(z)\,dz$ and by Laurent expansion $f(z)=\sum_{m=-\infty}^{\infty}\frac{1}{2\pi i}\oint_l\frac{f(\zeta)}{(\zeta-b_k)^{n+1}}d\zeta\cdot(z-b_k)^m$, it is found that

$$\oint_l f(z)\,dz=\sum_{k=1}^{n}\oint\sum_{m=-\infty}^{\infty}\frac{1}{2\pi i}\oint_l\frac{f(\zeta)}{(\zeta-b_k)^{n+1}}d\zeta\cdot(z-b_k)^m\,dz=2\pi i\sum_{k=1}^{n}\mathrm{Res}\,f(b_k) \quad (4)$$

3. Applications and Examples

3.1. $R(\sin\theta,\cos\theta)$-type Function

Let $R(u,v)$ be a rational function in two variables u and v. Generally, for the integral of the form $I=\int_0^{2\pi}R(\sin\theta,\cos\theta)d\theta$, By using the Euler formula $e^{i\theta}=\cos\theta+i\sin\theta$ and from which one can get that $\sin\theta=\frac{1}{2i}\left(z-\frac{1}{z}\right),\cos\theta=\frac{1}{2}\left(z+\frac{1}{z}\right)$ with $z=e^{i\theta}$. In addition, $dz=ie^{i\theta}\,d\theta=iz\,d\theta$, or $d\theta=\frac{dz}{iz}$. The example under consideration is

$$I=\int_0^{2\pi}\frac{1}{\sin\theta+4\cos\theta+9}d\theta. \quad (5)$$

With $z=e^{i\theta}$ in mind, the integration turns out to be $\rho(z)=\frac{2}{(1+2i)^2z^2+6iz+2i-1}$. It has two simple poles which are $z_{0,1}=-2-i,\frac{-2-i}{5}$. However, only one of the two lies inside the unite circle. Therefore, $\mathrm{Re}\,z(p(z),z_1)=\frac{5}{-5i+15i}=\frac{1}{2i}$ and the integral $I=\frac{\pi}{4}$.

3.2. $p(x)/q(x)$-type Function

In this subsection, the paper considers two distinct examples. The first example is

$$\mathrm{I}=\int_0^{\infty}\frac{dx}{1+x^3}, \quad (6)$$

in which the integrand is not even such that one cannot extend the lower limit to $-\infty$ directly. To conquer this problem, a third of circle should be considered, with the rays on the positive real axis $[0,\infty)$ and $[0,\infty)\cdot e^{2\pi i/3}$. The integral over the first path is the integral I, while the integral over the second path is $\int\frac{dz}{z^3+3}=\int_{\infty}^{0}\frac{d(xe^{2i\pi/3})}{(xe^{2i\pi/3})^3+1}=-e^{2i\pi/3}I$. This implies that $2\pi i\mathrm{Res}\left(\frac{1}{1+z^3};e^{\pi i/3}\right)=I\left(1-e^{2\pi i/3}\right)$. In addition, since $\mathrm{Res}\left(\frac{1}{1+z^3};e^{\pi i/3}\right)=\frac{1}{(1+z^3)'}\Big|_{z=e^{\pi i/3}}=\frac{1}{3e^{2\pi i/3}}$, one has

$$I=\frac{2\pi i}{3}\frac{e^{-2\pi i/3}}{1-e^{2\pi i/3}}=\frac{2\pi}{3\sqrt{3}}. \quad (7)$$

The other example is

$$I = \int_0^{\infty} \frac{\sqrt{x}}{x^2+1} dx, \tag{8}$$

Where the order of the denominator is only 1.5 order than that of the numerator. Suppose that C is a keyhole-like contour with a small circle r and big circle R. Let A and B be two paths along the real interval (r, R), using the positive x-axis as the branch cut. The real integral is equivalent to the form of $I = \lim_{r\to 0}\lim_{R\to\infty} \oint \frac{z^{1/2}}{z^2+1} dz$. The integrand has two simple poles at $z = i$ and $z = -i$. The residues at the two points are $\mathrm{Res}\left(\frac{z^{\frac{1}{2}}}{z^2+1}, i\right) = \frac{\sqrt{2}}{4} - i\frac{\sqrt{2}}{4}$ and $\mathrm{Res}\left(\frac{z^{\frac{1}{2}}}{z^2+1}, -i\right) = -\frac{\sqrt{2}}{4} - i\frac{\sqrt{2}}{4}$. According to the residue theorem, one has

$$\oint \frac{z^{\frac{1}{2}}}{z^2+1} dz = 2\pi i\left(\frac{\sqrt{2}}{4} - i\frac{\sqrt{2}}{4} - \frac{\sqrt{2}}{4} - i\frac{\sqrt{2}}{4}\right) = \sqrt{2}\pi. \tag{9}$$

On the other hand, the integrand along the A direction (i.e., r to R) is $\frac{z^{\frac{1}{2}}}{z^2+1} = \frac{\sqrt{x}}{x^2+1}$ while it is $\frac{z^{\frac{1}{2}}}{z^2+1} = \frac{\sqrt{x}e^{i\pi}}{x^2+1}$ when the path is along the B direction (i.e., R to r). Therefore,

$$I = \int_0^{\infty} \frac{\sqrt{x}}{x^2+1} dx = \frac{1}{2}\oint \frac{z^{\frac{1}{2}}}{z^2+1} dz = \frac{\sqrt{2}}{2}\pi. \tag{10}$$

3.3. Mixed-type Function

The residue theorem can also be used to calculate the integrals which are the mixture of the sine/cosine functions and polynomial functions. One of the examples is

$$I = \int_0^{\infty} \frac{\cos(x)}{x^2+3} dx, \tag{11}$$

Which has the cosine function in the numerator and the polynomial functions in the denominator. To execute this problem, by taking into account $t = \sqrt{3}x$, one find that $\int_0^{\infty} \frac{\cos(x)}{x^2+3} dx = \frac{1}{\sqrt{3}}\int_0^{\infty} \frac{\cos(\sqrt{3}t)}{t^2+1} dt = \frac{\pi}{2\sqrt{3}} e^{(-\sqrt{3})}$. In addition, assume that $R > 1$ and define the following two paths: $\gamma_R^1(t) = t + i0$ if $-R \le t \le R$, and $\gamma_R^2(t) = Re^{it}$ if $0 \le t \le \pi$.

In what follows the paper will consider the contour integration $\frac{1}{\sqrt{3}}\oint \frac{e^{(\sqrt{3}t)}}{z^2+1} dz$, which obeys the identity $\frac{1}{\sqrt{3}}\int_0^{\infty} \frac{e^{(\sqrt{3}t)}}{z^2+1} \mathrm{dz} = \sum_{\upsilon=1,2} \frac{1}{\sqrt{3}} \oint_{\gamma_R^{\upsilon}} \frac{e^{(\sqrt{3}t)}}{z^2+1} dz$. It is important to note that for the first part,

$$\lim_{R\to\infty} \int_{\gamma_R^1} \frac{e^{(\sqrt{3}z)}}{z^2+3} dz = \frac{1}{\sqrt{3}}\int_0^{\infty} \frac{e^{(\sqrt{3}z)}}{z^2+1} dz = \frac{\pi}{2\sqrt{3}} e^{(-\sqrt{3})} \tag{12}$$

While for the second part it is found that

$$\lim_{R\to\infty} \oint_{\gamma_R^2} \frac{e^{(\sqrt{3}z)}}{z^2+1} dz \le \mathrm{length}(\gamma_R^2)\} \cdot \sup_{\gamma_R^2} \frac{e^{\sqrt{3}t}}{z^2+3} \le \pi R \cdot \frac{1}{R^2-3} \to 0 \text{ as } R. \tag{13}$$

Thus, $I = \frac{1}{\sqrt{3}}\int_0^{\infty} \frac{\cos(\sqrt{3}t)}{t^2+1} dt = \frac{\pi}{2\sqrt{3}} e^{(-\sqrt{3})}$.

Another similar integral is of the form

$$I = \int_0^{\infty} \frac{\mathrm{x}\sin(\mathrm{x})}{(\mathrm{x}^2+1)^2}, \tag{14}$$

Which is again an even function but has the sine function in the numerator. Before calculating this integral, it is easily to find that the integrand $\frac{x\sin(x)}{(x^2+1)^2}$ is an even function with respect to x, while the function $\frac{x\cos(x)}{(x^2+1)^2}$ is odd. Hence,

$$\int_0^\infty \frac{x\sin(x)}{(x^2+1)^2}dx = \frac{1}{2i}\int_0^\infty \frac{xe^{ix}}{(x^2+1)^2}dx = \frac{1}{2}\mathrm{Im}\left(\int_{-\infty}^{\infty} \frac{xe^{ix}}{(x^2+1)^2}dx\right). \tag{15}$$

According to the residue theorem, the contour is expected to be a large semicircle in the upper half plane where it has a pole of order 2 located at $z = i$ [10]. Noticing that

$$\frac{d}{dz}\left(\frac{ze^{iz}}{(z+i)^2}\right)|_{z=i} = \frac{(2i)^2(e^{-1}-e^{-1}) - 2ie^{-1}(2i)}{(2i)^4} = \frac{1}{4e},$$

It is concluded that the final result of the integral

$$I = \int_0^\infty \frac{x\sin(x)}{(x^2+1)^2}dx = 2\pi i \cdot \frac{1/4e}{2i} = \frac{\pi}{4e}. \tag{16}$$

4. Conclusion

The residue and the residue theorem are widely used and can solve complex integral calculation problems in complex integral and real integral. It can be used to calculate some integrals, especially real integrals and improper products which are not easy to obtain directly. This paper considers several distinct integrals, such as integrals involving sine/cosine functions, integrals with multiple polynominal functions, and integrals which are a mixture of sine/cosine functions and polynominal functions. The study suggests that the residue theorem is indeed a versatile method to deal with a great number of improper integrals.

References

[1] Shen Y. Residue Theorem and Its Applications. Journal of Tonghua College, 2017, 267(38): 24-26.

[2] Zeng Q. Explore the application of the residue theorem in solving different types of integrals. Technology innovation and application, 2020, 23(11), 175-176.

[3] Cheng Y, Zhang L. Generalization of Cauchy Integral Theorem. Journal of Yuxi Normal University, 2017, 33(8): 19-22.

[4] Shen X. Research and extension of residual calculation method, journal of Jiujiang University (nature science), 2014, 107(4): 55-56.

[5] Lin Z. Experience in learning Cauchy integral theorem and residue theorem, Studies in College Mathematics, 2021, 24(1): 74-76.

[6] Liu K, Shao L. A Summary on Two Types of Real Integrals Using the Residue Theorem. Journal of Physics: Conference Series, 2021, 1903: 012017.

[7] Lang S. Complex analysis. New York: Springer, 2010.

[8] Shen Y, Li J, The residue theorem and its applications. Technical forum, 2014, 23(2): 34-34.

[9] Li Q, Lin S. A generalization of the residue theorem. Journal of Qiongzhou University, 2013, 20(5): 30-35.

[10] Yu S, Liu Z. Generalization and application of infinite first integration theorem for rational functions. Journal of South-Central University for Nationalities, 2005, 24(1): 85-87.