Complex Analysis and Residue Theorem

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Abstract. Mathematics is knitted into our lives by laying the fundamental base of many things. But people still think it is unnecessary to study math deeply due to its complexity. A lot of the great work from mathematicians is not noticed by other fields. Among all the essential mathematical methods and theorems, the Residue Theorem is one of the most significant ones. Therefore, this article will interpret the Residue Theorem step by step with its related theorems. Before jumping into the Residue Theorem, the article will introduce a series of mathematical methods, including the Triangle Inequality, the Euler's Formula, the Analytic function, the Taylor Series, and the Laurent Series. The applications and graphs of these mathematical methods will help to understand them further. Consequently, this article will demonstrate the most crucial part of this article — the Residue Theorem by its definition, formula, and example. Last but not least, this article will finish with how the Residue Theorem is used in our wind power generation system to express further how math correlates to our daily life. The result of this article shows that the Residue Theorem can be used to help explain other theorems and to extend new mathematical theories. It can be used not only in mathematics but also in a lot of fields, such as computer science, physics, and engineering. This article hopes a more extensive population can acknowledge the contribution of math to our world through the Residue Theorem.

Keywords: Complex Analysis, The Residue Theorem, Applications of the Residue Theorem.

1. Introduction

Mathematics is closely related to people's lives, and many mathematical algorithms are used in many studies. Complex analysis and Cauchy’s Residue Theorem are widely used, but most people do not deeply understand them. It is often used in many fields that are subtly related to mathematics, such as machine learning, statistics, and signal processing, such as core component analysis, Markov chain convergence analysis, community detection, concatenation, etc. All of them can show the diversity of use of the theorem.

This article will mainly focus on the definition and applications in other departments of the residue theorem. The history of the contribution to complex numbers and residues will be firstly introduced. In mathematical filed, the residue theorem can derive integral functions [1]. Besides, it can solve the Poles and Zeros of Meromorphic Functions [2]. Italian mathematicians Niccoló Fontana (1499-1557) and Gerolamo Cardano (1501-1576) were the first to consider crossing complex numbers by algebraic manipulation, such that a complex number can be represented as $a+\sqrt{-1}\sqrt{b}$ (a, b is both real numbers and b is positive) [3]. In real life circumstance, the residue theorem can also be used in Queueing Networks [4]. H. Poincaré made a great contribution to the elaboration of the concept of residuals, and in 1887 he introduced the concept of differential residual 1-form, which is understood as any rational differential 2-form in C2, with simple poles along a smooth complex curve [5]. The regularization penalty limits the real-valued function to smoothing in nonparametric estimation, with an example of a square norm of a derivative combination. The residue theorem is significant and can make most questions more obtainable [6]. This article will carry out a more extensive and widespread science popularization to help more people understand and be able to apply the Residue Theorem simply.

This article wants to make the concept of the residue theorem easier to comprehend by explain ing related theorems before unfold the applications of the theorem. Other related knowledge—the Cauchy Riemann Equation, the Laurent Series, and the Taylor Series— will be showed before this paper diving into Residue [7]. The Residue Theorem can be used in many fields to solve problems. As one of the most important theorems in complex analysis, the Residue Theorem can derive varieties of integral functions, solved the Poles and Zeros of Meromorphic Functions, and be used in Queueing Networks—specifically the wind power generation systems [8,9]. But to clarify, one crucial key term, “analytic function" will also be explained in the article [10]. The theorem is significant, and it can help make most questions to be more accessible. This article will illustrate how it not only make our daily life occurrences easier but also help physics knowledges to be more comprehensible. The definition of the Quantum theory of the electron will be explained first, then how the utilization of the Residue Theorem convened this theory will be shown [11,12].

The purpose of this article is carried out a more extensive and popular scientific explanation to help more people understand and be able to apply the Residue Theorem simply in other fields.

2. Related methods to Cauchy’s Residue Theorem

This part of the article will show some related methods, theorems, and examples of Cauchy's Residue Theorem to help further explain the application and expansion of Cauchy's Residue Theorem.

2.1. Triangle Inequality

Triangle Inequality has the definition that the length of one side of a triangle is less than or equal to the sum of the lengths of the other two sides. Both i and z are in a complex plane, where real numbers are run from left to right and imaginary numbers run up and down. Here is a theorem $|z_1 + z_2| \leq |z_1 + z_2|$ where Z_1 and Z_2 both stand for collinear [9].

2.2. Euler’s Formula

Euler’s formula connected the complex exponential planes with trigonometry. The formula is $e^{i\theta} = cos\theta + isin\theta$. Figure 1 provides a means of conversion between cartesian coordinates and polar coordinates [9]. It includes θ as an angle in radians and z as a radius vector.

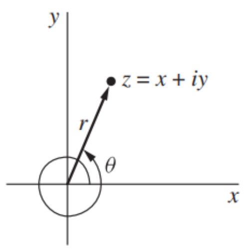

Figure 1. This is a graph of an equation in a complex plane to present z.

One can show that the polar form is $z = r(cos\theta + isin\theta)$ [9].

2.3. Analytic function

If a function is differentiable at each point of the region R, then it is an analytic function [8].

2.4. Cauchy Riemann Equation

Cauchy Riemann Equation is a system of differential equations concerned with differentiability—a function in one variable in calculus so that its derivative exists at each point in its entire domain—as well as continuity. It is a significant section of mechanical engineering. Here is given the theorem. It has a function like this $f(z) = u(x, y) + iv(x, y)$ and hypothesis that f’(z) at the point which is

$z_0 = x_0 + iy_0$. Then the first-order partial derivatives of u and v at (x_0, y_0) must be existed, and the Cauchy–Riemann equation shows like $u_x = v_y, u_y = -v_x$ need to be satisfied.

2.5. Taylor Series

Definition: assume a function f is analytic by a disk $|z - z_0| < R_0$, and it is center point at z_0 with radius R_0. Then f(z) has the power statement.

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^{n|} (|z - z_0| < R_0) \tag{1}$$

It can be expanded by the equation below.

$$f(z) = f(z_0) + \frac{f'(z_0)}{1!}(z - z_0) + \frac{f''(z_0)}{2!}(z - z_0)^2 + \cdots (|z - z_0| < R_0) \tag{2}$$

2.6. Laurent Series

Definition: According to Figure 2, analytic throughout an annular domain which of the function of $R_1 < |z - z_0| < R_2$, the center point at z_0, and let C bespeak any positively oriented simple closed contour around z_0 and lying in this domain above. Subsequently, at the domain of each point, f(z) has the series formation shown below:

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n + \sum_{n=1}^{\infty} \frac{b_n}{(z-z_0)^n} (R_1 < |z - z_0| < R_2) \tag{3}$$

Here a_{nd} is equal to the equation below:

$$a_n = \frac{1}{2\pi i} \int_C \frac{f(z)dz}{(z - z_0)^{n+1}} (n = 0,1,2, \dots) \tag{4}$$

, and b_n is equal to the equation below:

$$b_n = \frac{1}{2\pi i} \int_C \frac{f(z)dz}{(z-z_0)^{-n+1}} (n = 1,2, \dots) \tag{5}$$

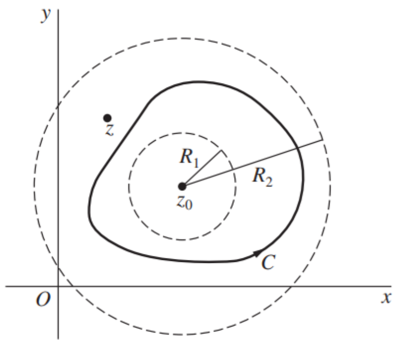

Figure 2. This graph includes a close counter C in the domain R_2.

2.7. Cauchy's Residue Theorem

Cauchy Residue Theorem is a powerful tool to evaluate line integrals of analytic functions over closed curves.

Definition: When z_0 is an isolated singular point of a function f which is a positive number R_2 such that f is analytic at each point z for which $0 < |z - z_0| < R_2$. Finally, f (z) has a Laurent series representation.

Let C be in the positive sense as a simple closed profile. As Figure 3 shows, if a function f is analytic inside and on C except for a finite number of singular points zk (k = 1, 2, ..., n) inside C to shows the equation below functions:

$$\int_C f(z)dz = 2\pi i\, Res(z = z_0)f(z). \tag{6}$$

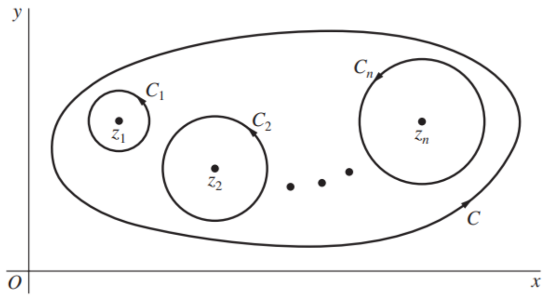

Figure. 3. This graph illustrates Cauchy Residue's Theorem.

Example: Define $f(z) = \frac{az^3+bz^2+cz+d}{z^4-1}$ with $a = 10, b = -2 - 2i, c = -2, d = 2i - 2$

Evaluate the integrals $\int_\gamma f(z)dz$, were

$$\gamma(t) = 1 + e^{it}, 0 \le t \le 2\pi$$
$$\gamma(t) = \frac{1-i}{2} + \sqrt{2}e^{it}, 0 \le t \le 2\pi$$
$$\gamma(t) = 2e^{it}, 0 \le t \le 2\pi$$

Here is the answer. Let

$$f(z) = \frac{10z^3 + (-2-2i)z^2 - 2z + 2i - 2}{z^4 - 1} = \frac{A}{z+1} + \frac{B}{z-1} + \frac{C}{z+i} + \frac{D}{z-i}$$

Then

$$f(z) = A(z+i)(z-i)(z-1) +$$
$$B(z+i)(z-i)(z+1) +$$
$$C(z+1)(z-1)(z-i) +$$
$$D(z+1)(z-1)(z+i)$$
$$= A(z^3 - z^2 + z - 1) +$$
$$B(z^3 + z^2 + z + 1) +$$
$$C(z^3 - iz^2 - z + i) +$$
$$D(z^3 + iz^2 - z - i)$$

$$\therefore \quad Res(f(z), -1) = A, Res(f(z), 1) = B, Res(f(z), -i) = C, Res(f, i) = D$$
$$A + B + C + D = 10, C - D = 2, B - A = -2, A + B - C - D = -2,$$
$$So, A = 3, B = 1, C = 4, D = 2$$

$$\therefore \quad \int_{\gamma 1} f(z)dz = 2\pi i(B) = 2\pi i \cdot 1 = 2\pi i$$
$$\int_{\gamma 2} f(z)dz = 2\pi i(B + C) = 2\pi i \cdot 3 = 6\pi i$$
$$\int_{\gamma 3} f(z)dz = 2\,\pi i(A + B + C + D) = 2\pi i \cdot 10 = 20\pi i$$

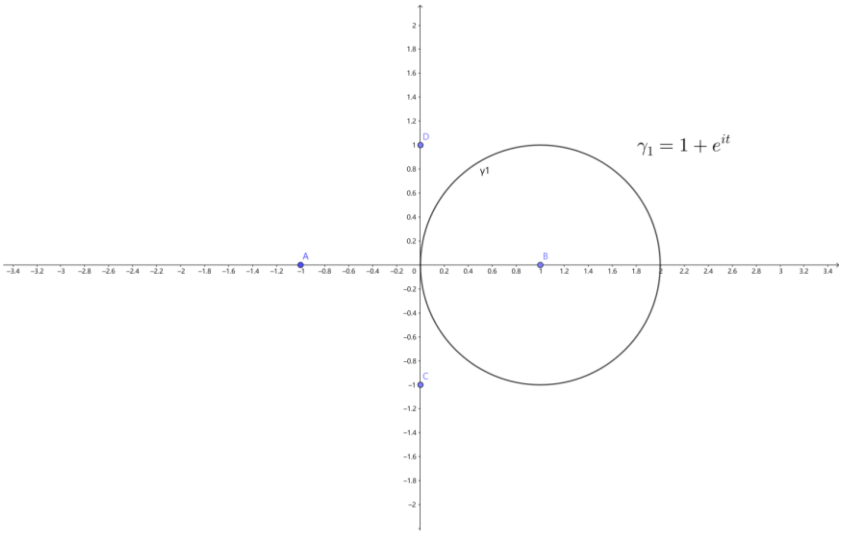

Figure 4. This graph includes the residue B.

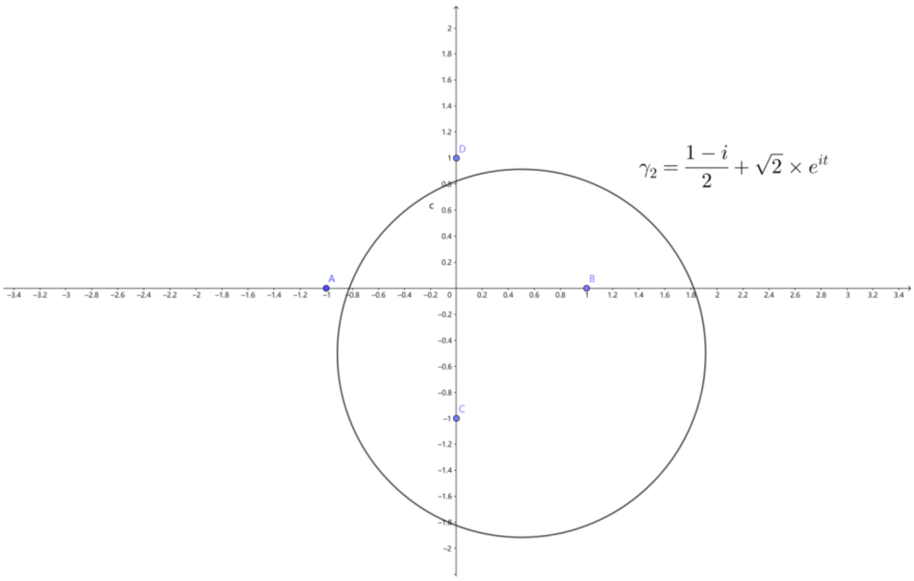

Figure 5. This graph includes the residues B and C.

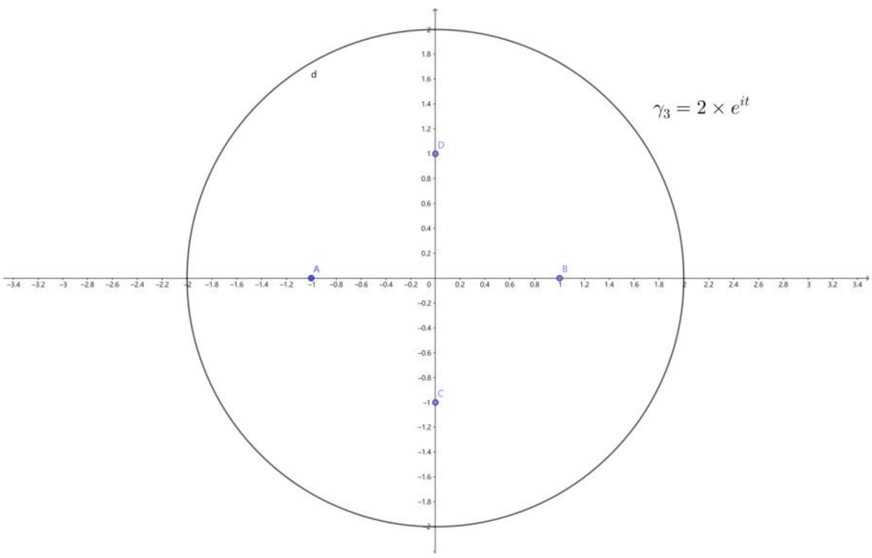

Figure 6. This graph includes the residuals A, B, C, and D.

2.8. Infinity residue

Definition: If an analytic function f is everywhere in the finite plane except for a finite number of singular points shows the integration to a positively oriented simple closed contour C, the equation below operates:

$$\int_C f(z)dz = 2\pi i \, Res(z=0)\left[\frac{1}{z^2} f\left(\frac{1}{z}\right)\right] \tag{7}$$

Suppose a function f is an infinite plane that is close to the contour C. Expressed in Figure 7, we may assume that R1 denotes a positive number is large enough that the point C has lied inside the circle |z| = R1. The function f is analytic around the domain $R_1 < |z| < \infty$.

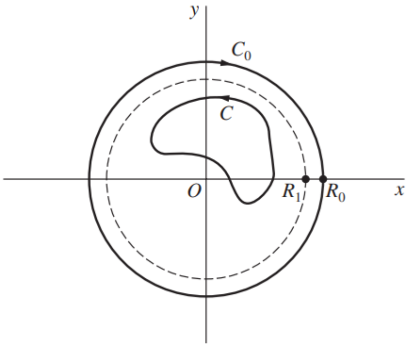

Figure 7. This graph shows the condition when finding the isolated singular point.

2.9. An Example of Cauchy's Residue Theorem

$$\int_C \frac{\boldsymbol{z^3(1-3z)}}{\boldsymbol{(1+z)(1+2z^4)}} \boldsymbol{dz}$$

The solution is shown below:

$$\int_C \frac{z^3(1-3z)}{(1+z)(1+2z^4)} dz$$
$$= 2\pi i \left(-\frac{3}{2}\right)$$
$$= -3\pi i$$

3. Application

Not only can the Residue Theorem be used in math, but it also can be used in real-life cases and engineering physics.

The Residue Theorem can be used in soft sliding mode for generation systems that wind power is controlled by [6,8]. Figure 8 shows the principle of the AI with the residue theorem, control with vector— a description wind generator system— and control of Residual for the controller of speed. Then, part b in Figure 8 shows speed in the control diagram of Residual approach and currents controller.

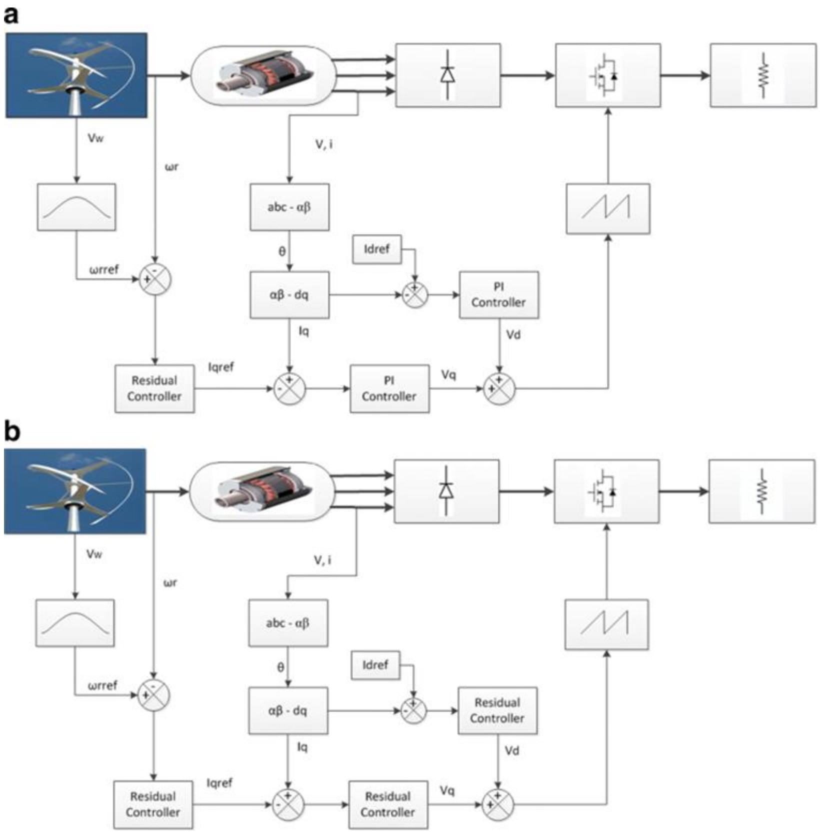

Figure 8. This graph shows the wind power generation system.

Then Figure 9 shows the residue theorem based on the system management controller. Let D be a simply connected domain, and let C be a simple closed positively oriented contour that lies in D. Except at the points z1, z2...., zn that lie inside C, then (appendix 1), if f(z) is analytic inside and on C.

Such a situation illustration is shown in Figure 9, and the residue theorem general form can be expressed in Eq. (2) of appendix. Then researchers use the Euler method to assure the concentration of the point to the reference point, so the system's firmness can be maintained.

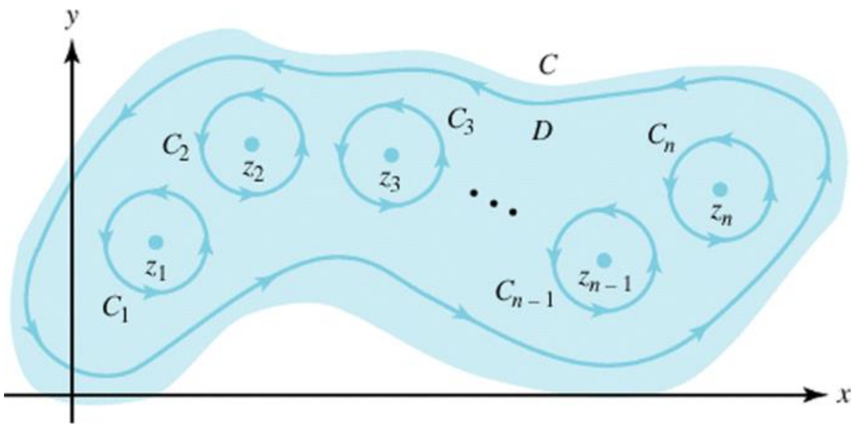

Figure 9. This graph shows the residue theorem based on SMC.

The Residue Theorem also can be used in Physics, such as Quantum mechanics. Considering a one-dimensional crystal-like semiconductor quantum wire (Nanowires) of length L, within which the electron having mass m is restricted to move freely in just one dimension and prevented from leaving the crystal by an infinitely large potential barrier at the ends of the crystal [11]. This situation is called electron in one-dimensional potential box and is represented as $V = 0$ for $0 < y<L$ and $V=\infty$ elsewhere.

Considering the uniform potential of an electron inside the crystal, the one-dimensional time-independent Schrodinger’s equation is given by what’s showing below [12]:

$$\ddot{\emptyset}(y) + \frac{2m}{h^2}\{E - V\}\emptyset(y) = 0 \equiv \frac{\partial}{\partial y} \quad (8)$$

When V=0, equation (12) becomes to the equation below:

$$\ddot{\emptyset}(y) + \frac{2m}{h^2}E\emptyset(y) = 0 \quad (9)$$

Here y belongs to [0, L] with boundary conditions, substituting the equation below:

$$\ddot{\emptyset}(y) + k^2\emptyset(y) = 0 \quad (10)$$

On taking the Laplace Transform of equation (10), then the equation below will be obtained:

$$L\left[\ddot{\emptyset}(y)\right] + k^2L[\emptyset(y)] = 0 \quad (11)$$

Applying Residue Theorem, then the equation below will be derived from the equation above:

$$\emptyset(y) = \frac{Ce^{-iky}}{-2ik} + \frac{Ce^{iky}}{2ik} \quad (12)$$

Last but not least, the application of the Cauchy's Residue Theorem math will be shown below. It is applied to the complex field, and the magnetic induction intensity B^w in the ampere loop theorem is a vector, so it can't directly to the residue theory used in electromagnetism of the ampere loop theorem be used to reconstruct a plural field [13]. For this purpose, one may consider an infinite intercepting wire around the magnetic field distribution of the space, as shown in Figure 10.

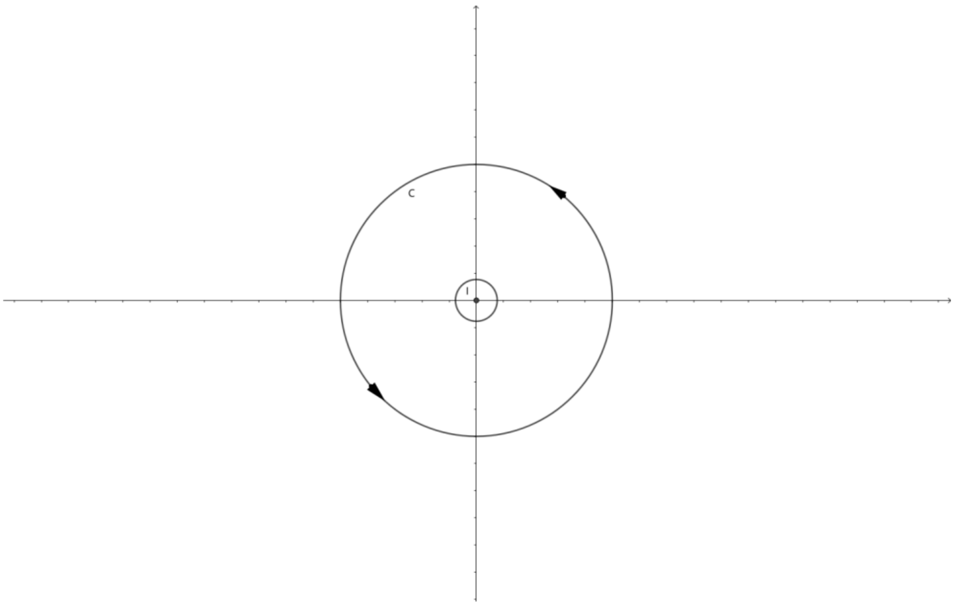

Figure 10. This graph shows an infinite intercepting wire around the magnetic field distribution.

Then from the inference of the graph, B^w can be written as this formula:

$$\tilde{B} = B_y + iB_x$$
$$= \frac{u_0I}{2\pi}\left(\frac{x - iy}{x^2 - y^2}\right)$$
$$= \frac{u_0I}{2\pi z} \quad (13)$$

When there are several n of current sources in loop C, we study Figure 11.

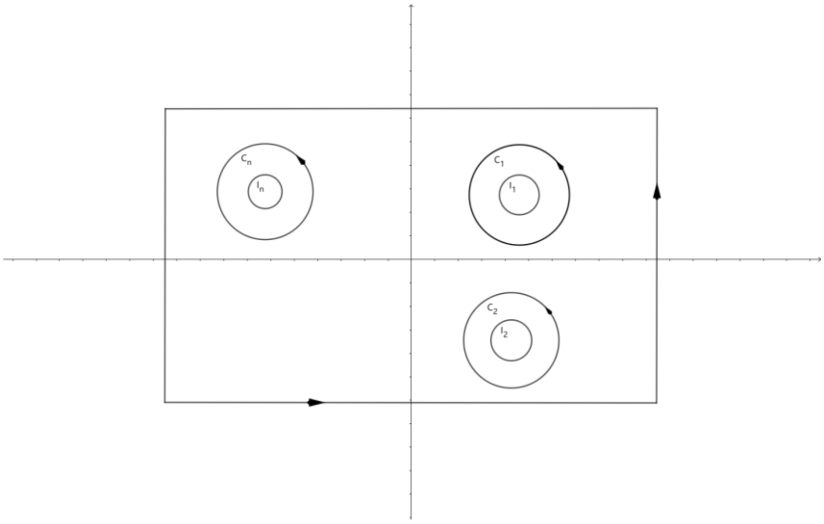

Figure 11. This graph shows n current sources in loop C.

Cauchy's Residue Theorem will be used in the loop. The formula is shown as follows:

$$\int_C \bar{B}^w \cdot dz = 2\pi i Res B(z)|_z$$
$$= 0$$
$$= i u_0 I \quad (14)$$

The process below will lead to the result:

$$\int_C B^w \cdot dI^w = \int_C \left(B_x dx + B_y dy\right)$$
$$\int_C \left(B_y + iB_x\right)(dx + idy) = \int_C \left(B_y dx - B_x dy\right) + i\int_C (B_x dx + B_y dy) \quad (15)$$

So, using both formulas (9) and (10), the final formula can be obtained from them,

$$\int_C \bar{B}^w \cdot dz = u_0 \sum_{k=1}^{n} I_k \quad (16)$$

So far, by using the complex function method to deduce the ampere loop theorem of electromagnetism, its method is simple, avoiding the complicated derivation of some materials. The derivation process above shows as long as choosing the appropriate number of complex numbers to represent the quantity of electricity and magnetism in electromagnetics, can use of the residue theorem is deduced in electromagnetics some valuable conclusions. The method above shows how to use the plural = B, B + iB and the Residue Theorem to get equations (9) and (10). Equation (11) is the Ampere Loop Theorem.

Above all, the three applications show the simple ways that Cauchy's Residue Theorem brings.

4. Conclusion

As this article showed, math theorems can be understood smoothly if they are learned piece by piece to build up knowledge with the assistance of virtual graphs and precise examples. Through the associated theorems and history of Cauchy's Residue Theorem, its evolution and importance can be shown directly. The basic conception contributed by the mathematicians build the bases for further extensions on theorems and applications. Like the triangle inequality which is originally a geometry theorem extended to complex analysis that have complex planes to carry on helping comprehending other theories. Different branches in one filed have strong connections and it also applying among in different fields of studying.

The residue theorem is not only crucial in mathematics, but other fields can also use it in many aspects. It is significantly important in the field of complex analysis. As this paper showed, the theorem can also be used on the wind power generation system by building up the process. At the same time, it can also be used to in physics to assist extending equations such as the quantum theory of the free election gas. There is countless example that can show how Complex Analysis serve other branches for people to explore and utilize.

Many people say that math is not a requirement to have a contented life, but they should recognize that abstract mathematical theorems are helping promote and convene our life. It is immersed into our life as the base of a lot of professional fields. If the abstractions are understood clearly, more convenience in both research and reality will both boost. Therefore, this article is intended to help more people in other fields to study and use mathematical Theorems like Cauchy’s Residue Theorems. This article hopes our life can gain more benefits from the mathematical world and learn more from the history to extend theorems consistently.

5. Appendix

(1) $$\begin{array}{@{}rcl@{}} \oint{f(z)}dz=2\pi\iota \sum\limits_{k=1}^{n} RES[f,z_{k}]. \End{array} $$.[8]

(2) $$\begin{array}{@ {} rcl@ {}} RES\, f(x)=\frac {1} {2\pi\iota}\int^{+\infty}_{-\infty} f(x) \, dx. \End{array} $$.

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