Fourier Analysis and Its Application

Zinan Zhao
Department of Math, Xi'an Jiaotong University, Xi'an, China
* Corresponding Author Email: dzzzn2002@stu.xjtu.edu.cn

Abstract. In this article, mathematicianswill give a general summary of Fourier analysis and some of its applications. The Fourier transform will be organized in a developing order, the discrete Fourier transform, Fourier transform on the unit circle and Fourier transform on the real line. Some theorems about the convergence of Fourier series in different forms will be proved in detail. Finally, an estimation of the upper bound of Fourier transform will be discussed. Mathematicianscan prove that it is controlled by the 1-norm of the derivative of f.

Keywords: Fourier analysis; convergence of series; Upper bound estimation.

1. Introduction

Have you ever noticed that each phone model has a unique sound when you press the number buttons while on a call? This is because each button is composed of two different sinusoids that may be used to tell one button from another. When using your phone to explore a menu, the other person can find out what keys you press by applying a Fourier transform to the input and looking at the frequencies. Oscillations and vibrations regularly appear in physics-related situations. Typically, oscillatory motion may be described as a single sine or cosine function and is simple (e.g., weights on springs, pendulums, harmonic waves, etc.). In reaction to differences in the rate of wave propagation in the medium, a wave changes its speed as it travels across a heterogeneous media. So, by identifying a difference in phase between what is anticipated and what is measured, one may calculate the additional time delay, which in turn tells you how much the wave speed has changed in the medium. For laypeople, this is obviously a relatively simple explanation, yet it forms the basis for tomography. Thanks to Fourier methods, scientists have a set of powerful tools for expressing any periodic function as the sum of sines and cosines. In order to solve the heat equation in a basic physics issue, the Fourier series concept was originally proposed. Fourier analysis is a subject which has strong Physics background since the Fourier transform build a bridge from time space to frequency space [1-21]. It can also be a useful tool when mathematiciansneed to handle some partial differential equations such as the heat equation and wave equation. The wave equation in d dimension is $\frac{\partial^2 u}{\partial x^2}+\cdots+\frac{\partial^2 u}{\partial x_d^2}=\frac{1}{c^2}\frac{\partial^2 u}{\partial t^2}$. If rewrite $c=1$ and $\triangle=\frac{\partial^2}{\partial x_1^2}+\cdots+\frac{\partial^2}{\partial x_d^2}$, the wave equation becomes$\triangle u=\frac{\partial^2 u}{\partial t^2}$. Now, the equation will be solved based on the initial conditions, or the Cauchy problem,$u(x,0)=f(x)$ $\frac{\partial u}{\partial t}(x,0)=g(x)$ for $f,g\in S(\mathbb{R}^d)$. Physically speaking, this is to solve for the solution in the whole space knowing the position and velocity of all particles at a particular time $t=0$. Note the solution can be extended to $t<0$. To solve the system, take the Fourier transform of the equation and initial conditions with respect to the position coordinates. Then the problem becomes an ordinary differential equation with a time variable. Recall the property of differentiation rule in Fourier transform, $\mathcal{F}[\left(\frac{\partial}{\partial x}\right)^\alpha f(x)]=(2\pi i\xi)^\alpha\hat{f}(\xi)$. Then the equation becomes$-4\pi^2|\xi|^2\hat{u}(\xi,t)=\frac{\partial^2 u}{\partial t^2}(\xi,t)$. Fixing ξ, the solution with respect to time is$\hat{u}(\xi,t)=A(\xi)\,cos(2\pi|\xi|t)+B(\xi)\,sin(2\pi|\xi|t)$, where $A(\xi)$ and $B(\xi)$ are constants to be determined for each $\xi\in\mathbb{R}^{\mathrm{d}}$. Now take the Fourier transform of the initial conditions giving $\hat{u}(\xi,0)=\hat{f}(\xi)$ $\frac{\partial\hat{u}}{\partial t}(\xi,0)=\hat{g}(\xi)$. Then easily $A(\xi)=\hat{f}(\xi)$, $2\pi|\xi|B(\xi)=\hat{g}(\xi)$. Substituting the constants into the Fourier transform of the solution obtained above, $\hat{u}(\xi,t)=\hat{f}(\xi)\,cos(2\pi|\xi|t)+\hat{g}(\xi)\frac{sin(2\pi|\xi|t)}{2\pi|\xi|}$. Then taking the inverse Fourier with respect to ξ, the solution of the d dimensional wave equation can be obtained. A solution to the Cauchy problem of the d dimensional wave equation is $u(x,t)=\int_{\mathbb{R}^d}\left[\hat{f}(\xi)\cos(2\pi|\xi|t)+\hat{g}(\xi)\frac{\sin(2\pi|\xi|t)}{2\pi|\xi|}\right]e^{2\pi ix\xi}d\xi$. Note that since f and g are both Schwartz functions, u is at least twice differentiable. Now verify the equation using the solution given above. $\triangle u(x,t)=\int_{\mathbb{R}^d}\left[\hat{f}(\xi)\cos(2\pi|\xi|t)+\hat{g}(\xi)\frac{\sin(2\pi|\xi|t)}{2\pi|\xi|}\right](-4\pi^2|\xi|^2)e^{2\pi ix\xi}d\xi=\frac{\partial^2 u}{\partial t^2}(x,t)$. This shows u given above solves the equation. Then also, $u(x,0)=\int_{\mathbb{R}^d}\hat{f}(\xi)\,e^{2\pi ix\xi}d\xi=f(x)\frac{\partial u}{\partial t}(x,0)=g(x)$ verifies the initial conditions. Some conclusions in Fourier analysis can give us inspiration in function analysis. Take the inner product space as an example, mathematiciansknow the the Hilbert space is complete inner product space. Then separable Hilbert space is an isometric isomorphism of l^2 space. In conclusion, knowing how mathematiciansconstruct the Fourier analysis theory step by step is good for our further study and research.

2. Main Works

2.1. Basic properties of Fourier transform

2.1.1 Discrete Fourier Transform

Mathematiciansstart with the simplest form of the Fourier transform. Below is the definition of it.$\widehat{a_j}=\frac{1}{\sqrt{N}}\sum_{k=0}^{N-1}a_k\,\omega^{kj}$. Here $\{a_0,a_1,\dots,a_{N-1}\}$ is finite sequence, and ω is unit root of order-n. By making some discussion, mathematicianscan find some properties of DFT:

(1) (The inverse formula of DFT): $a_j=\frac{1}{\sqrt{N}}\sum_{k=0}^{N-1}\widehat{a_k}\,\omega^{-kj}$.

(2) (Keeping inner product): If the inner product between finite sequence is defined as below:

$\langle a,b\rangle=\sum_{j=0}^{N-1}a_j\,\overline{b_j}$, Then mathematicianscan prove that DFT keeps the inner product, i.e., $\langle a,b\rangle=\langle\hat{a},\hat{b}\rangle$.

(The convolution formula of DFT): If the convolution between finite sequence is defined as below:$(a*b)_j:=\sum_{k=0}^{N-1}a_{j-k}\,b_k$. Then mathematicianscan prove the convolution formula: $(\widehat{a*b})_j=\widehat{a_j}\widehat{b_j}$.

2.1.2 Fourier transform on the unit circle

Then mathematiciansconsider the case on the unit circle. Things will be much similar to the case of DFT, as long as the related conceptions are well defined.

Firstly, mathematicianshave to give a definition to Fourier transform on the unit circle:

For $f(\theta)(0\le\theta<2\pi)$, its Fourier coefficients are defined as below:$c_j=\frac{1}{2\pi}\int_0^{2\pi}f(\theta)e^{-ij\theta}d\theta, j\in Z$

It not hard to find that using Fourier transform mathematicianscan get a sequence. In other word, Fourier transform is a mapping from functions on the unit circle (Mathematiciansrequest it to be integrable) to l^2 space. Of course, mathematiciansmay hope that Fourier transform on the unit circle can have the similar properties of DFT, but before doing that we'd better check the convergence of Fourier series:$\sum_{-\infty}^{+\infty}c_j\,e^{ij\theta}$

Does it always converge to the original function f? Then how does it converge? Here mathematicianswill talk about this question. For better understanding, mathematiciansneed to supply some necessary conceptions.

Inner product (for vector space over complex numbers) $(\,,)$: $X\times X\to C$, satisfies:

(i)$(X,Y)=\overline{(Y,X)}$

(ii)$(\alpha X+\beta Y,Z)=\alpha(X,Z)+\beta(Y,Z)$

(iii)$(X,X)\ge 0,\ (X,X)=0\Leftrightarrow X=\theta$

※Remark: If a vector space equipped with an inner product is complete, it is called a Hilbert space.

Orthogonal

Two elements X, Y are orthogonal if $(X, Y)=0$. Mathematicianshave three important results:

(The Pythagorean theorem) If X, Y are orthogonal, then

$$\|X+Y\|^2 = \|X\|^2 + \|Y\|^2$$

(The Cauchy Schwartz inequality) $|(X, Y)| \leq \|X\|\|Y\|$

(The Triangle inequality) $\|X+Y\| \leq \|X\| + \|Y\|$

Useful inner product space:

R^n; C^n

$l^2(Z)$ over C : The set of all infinite sequences of complex numbers $(\ldots, a_{-n}, \ldots, a_{-1}, a_0, a_1, \ldots, a_n, \ldots)$, such that $\sum_{n\in Z}|a_n|^2 < \infty$;

※Remark: Mathematicianscan proof that all of these three inner product spaces is Hilbert space.

Pre-Hilbert space

If a vector space equipped with an inner product lacks the properties below, mathematicianscall it pre-Hilbert space:$(X, X) = 0 \Leftrightarrow X = \theta$. Here is an important example of pre-Hilebert space. Let X be the set of all complex-valued Riemann integrable functions on $[0,2\pi]$, equipped with the product:$(f, g) = \frac{1}{2\pi}\int_0^{2\pi} f(\theta)\overline{g(\theta)}\, d\theta$

Mathematiciansknow that $\int f = 0 \Leftrightarrow f = 0$, a.e. from real analysis, hence the property above cannot be satisfied.

Moreover, an example shows that is not a complete space: $f_n(\theta) = \begin{cases} 0 & \text{for } 0 \leq \theta \leq \frac{1}{n} \\ \ln(\frac{1}{\theta}) & \text{for } \frac{1}{n} < \theta \leq 2\pi \end{cases}$

Then let us talk about the convergence theorem of Fourier series:

Mathematiciansdefine three forms of sum as below: $(S_N f)(\theta) = \sum_{-N}^{N} c_j e^{ij\theta} = g_N * f, g_N(\sigma) = \frac{\sin(2N+1)\sigma}{\sin\sigma}$; $(F_N f)(\theta) = \frac{S_0 f + S_1 f + \ldots + S_N f}{N+1}(\theta) = F_N * f, F_N(\sigma) = \frac{1}{N+1}(\frac{\sin\frac{N+1}{2}\sigma}{\sin\frac{\sigma}{2}})^2$; $(P_r f)(\theta) = \sum_{-\infty}^{+\infty} r^{|j|} c_j e^{ij\theta} = P_r * f, P_r(\sigma) = \frac{1-r^2}{1-2r\cos\sigma + r^2}$, $(0 < r < 1)$;(Convolution on the unit circle: $(f * g)(\theta) := \frac{1}{2\pi}\int_0^{2\pi} f(t)g(\theta - t)dt$)

Then mathematicianshave the theorem about the convergence of these sums:

When f(x) is continuous function defined on the unit circle (f(0)=f(2π)), $F_N f$ uniformly converges to f, as $N \to +\infty$; $P_r f$ uniformly converges to f, as $r \to 1$; moreover, if the derivative off exists on the whole unit circle, then $S_N f$ uniformly converges to f, as $N \to +\infty$.

Proof: Here mathematiciansjust supply the proof about the convergence of $P_r f$, the rest is similar.

Firstly, mathematiciansobserve the properties of $P_r(\theta) = \frac{1-r^2}{1-2r\cos\theta + r^2}$

(i)$\frac{1}{2\pi}\int_{-\pi}^{+\pi} P_r(\theta)d\theta = 1$

(ii) $(P_r(\theta)$ is even function

(iii) For each θsettled not equal to 0, $\lim_{r\to 1} P_r(\theta) = 0$

(iv) $P_r(\theta)$ decreases on $[0, \pi]$

Since f(x) is continuous, $\forall\varepsilon > 0, \exists\delta > 0, s.t. |t - \theta| \leq \delta \Rightarrow |f(t) - f(\theta)| \leq \frac{\varepsilon}{2}$; moreover, $f(\theta)$ is a continuous function on unit circle which is compact, then $\exists\, M, s.t. |f| \leq M$.

Now mathematicianscan estimate the error to prove its uniform convergence: $\frac{1}{2\pi}\left|\int_0^{2\pi} P_r(\theta - t)(f(t) - f(\theta))dt\right| = \frac{1}{2\pi}\left|\int_{\theta-\pi}^{\theta+\pi} P_r(\theta - t)\big(f(t) - f(\theta)\big)dt\right| \leq \frac{1}{2\pi}\left|\int_{|t-\theta|\leq\delta} P_r(\theta - t)(f(t) - f(\theta))dt\right| + \frac{1}{2\pi}\left|\int_{|t-\theta|>\delta} P_r(\theta - t)(f(t) - f(\theta))dt\right| \leq \frac{1}{2\pi}\int_{|t-\theta|\leq\delta}|P_r(\theta - t)||(f(t) - f(\theta))|\, dt + \frac{1}{2\pi}\int_{|t-\theta|>\delta}|P_r(\theta - t)||f(t) - f(\theta)|dt$

$$\leq \frac{\varepsilon}{2}+\frac{1}{2\pi}\int_{|t-\theta|>\delta}|P_r(\theta-t)||f(t)-f(\theta)|dt \leq \frac{\varepsilon}{2}+\frac{M}{\pi}\int_{|t-\theta|>\delta}|P_r(\theta-t)|dt \leq \frac{\varepsilon}{2}+\frac{M}{\pi}\int_{|t-\theta|>\delta}P_r(\delta)$$

$\leq \frac{\varepsilon}{2}+\frac{\varepsilon}{2}=\varepsilon$(Take r close enough to 1, s.t. $P_r(\delta) \leq \frac{\pi}{4M}$). In this way mathematiciansfinish the proof of convergence. In fact, mathematicianscan talk about another form of convergence: Mean-square convergence of Fourier series.

Suppose f is integrable on the circle, then $\frac{1}{2\pi}\int_0^{2\pi}|f(\theta)-S_N(f)(\theta)|^2 d\theta \to 0, \;$ as $N \to \infty$

Proof:

Step1 (Finding orthogonal basis)

In the set X of all complex-valued Riemann integrable functions on $[0,2\pi]$, think about the family $\{e_n\}$ $n \in Z : e_n(\theta)=e^{in\theta}$. Mathematicianscan observe that the family $\{e_n\}$ $n \in Z$ is definitely an orthonormal basis, i.e.,$(e_n, e_m)=\begin{cases}1 & n=m\\ 0 & n\neq m\end{cases}$. Then mathematicianscan find that the Fourier coefficient can be written as an inner product: $(f, e_n)=\frac{1}{2\pi}\int_0^{2\pi} f(\theta)e^{-in\theta}d\theta=a_n$. As our definition, $S_N(f)=\sum_{|n|\leq N} a_n e_n$, it's easy to prove that $(f-S_N(f))\perp \mathrm{span}(e_n, |n|\leq N)$

Then applying the Pythagorean theorem, mathematicianshave$\|f\|^2=\|f-S_N(f)\|^2+\|S_N(f)\|^2=\|f-S_N(f)\|^2+\sum_{|n|\leq N}|a_n|^2(*)$

Step 2 (The best approximation lemma) If f is integrable on the circle with Fourier coefficients a_n, then

$\|f-S_N(f)\| \leq \left\|f-\sum_{|n|\leq N} c_n e_n\right\|$ for any complex numbers. Applying the Pythagorean theorem to $f-\sum_{|n|\leq N} c_n e_n = f-S_N(f)+\sum_{|n|\leq N} b_n e_n$, where $b_n=a_n-c_n$. Step 3 (Continuous function case). Assume that f is continuous, then using lemma 1 mathematicianscan find a trigonometric polynomial P of degree M, such that $|f(\theta)-P(\theta)|<\varepsilon_0 \Longrightarrow \|f-P\|<\varepsilon$, then using the best approximation lemma mathematicianscan get:$\|f-S_N(f)\| \leq \|f-P\|<\varepsilon$ whenever $N \geq M$. Step 4 (General case) If f is merely integrable, using lemma 2 mathematicianscan find a continuous function g on the circle satisfies those two properties, then:$\|f-g\|^2=\frac{1}{2\pi}\int_0^{2\pi}|f(\theta)-g(\theta)|^2 d\theta \leq \frac{B}{\pi}\int_0^{2\pi}|f(\theta)-g(\theta)|\, d\theta < C\varepsilon^2$. Then the case is back to the continuous function, hence the proof is finished. This theorem tells us that trigonometric polynomials are dense in $L^2(T)$.

The following talks about Riemann-Lebesgue lemma: If f is integrable on the circle, then $\hat{f}(n) \to 0$, as $|n| \to \infty$

The proof is given as follows: The terms of a converging series tend to 0. Remark (Bessel's inequality)

If $\{e_n\}$ is any othonormal family of functions on the circle, and $a_n=(f, e_n)$, then mathematicianshave:

$$\sum_{-\infty}^{\infty}|a_n|^2 \leq \|f\|^2$$

Equality holds precisely when the family is a basis.

After talking about the convergence of Fourier series, mathematicianscan find the similar properties like the DFT.

(Keeping inner product): If the inner product between functions on the unit circle is defined as $\langle f_1, f_2\rangle=\frac{1}{2\pi}\int_0^{2\pi} f_1(\theta)\overline{f_2(\theta)}\, d\theta$, then mathematicianscan prove that Fourier transform keeps the inner product, i.e.: $\langle f_1, f_2\rangle=\langle c^1, c^2\rangle$, where the definition of inner product between sequence is defined as before.

(Parseval identity) if $f(\theta)=\sum_{-\infty}^{+\infty} c_j\, e^{ij\theta}$, then $\frac{1}{2\pi}\int_0^{2\pi}|f(\theta)|^2 d\theta=\sum_{-\infty}^{+\infty}\left|c_j\right|^2$.

(Convolution formula) If the convolution between functions on the unit circle is defined as: $(f*g)(\theta):=\frac{1}{2\pi}\int_0^{2\pi} f(t)g(\theta-t)dt$, and the Fourier series of f, g converge to themself, then $c_j{}^{(f*g)}=c_j{}^{(f)}c_j{}^{(g)}$.

2.1.3 Fourier transform on the real line

$f \in L^1(R)$, define $\hat{f}(x) = \int_{-\infty}^{+\infty} f(y)e^{-2\pi ixy}\, dy, x \in R$, then $\hat{f}$ is called the Fourier transform of f.

To talk more about Fourier, transform on the real line, mathematiciansshall make some preparation. The definition of Schwartz space is $S_{(R)} := \{f: R \to C;\ f \in C^{\infty}(R);\ \forall k, l \in N, x^k f^{(l)}(x) \text{ is bounded}\}$, $S_{(R)}$ is called Schwartz space.

Properties:

(i) $S_{(R)}$ is linear space(ii) $f \in S_{(R)} \Rightarrow f^{(k)} \in S_{(R)}$(iii) $f \in S_{(R)}, P: R \to C$ is polynomial, then $Pf \in S_{(R)}$

(iv) $f \in S_{(R)} \Rightarrow f \in L^1{}_{(R)}$. Here is the idea how mathematiciansarrive that conclusion the Fourier transform is one to one mapping from $L^2(R)$ to itself. Firstly, mathematicianscan proof that it is a one-to-one mapping from Schwartz space onto itself, with the inverse form of Fourier transform:$f(x) = \int_{-\infty}^{+\infty} \hat{f}(y)e^{2\pi ixy}\, dy, x \in R$

The Schwartz space is not a complete inner product space, and its completion space is $L^2(R)$, using the Plancherel identity, mathematiciansknow the Fourier transform in Schwartz space keeps the inner product. Mathematicianscan extend this property to $L^2(R)$ and finally get a one-to-one mapping from $L^2(R)$ to itself.

2.2. Applications of Fourier transform

Mathematicianscan estimate the upper bound of Fourier transform after solving the following problem.

Let $f \in L^1(T)$ and let $\tilde{f}$ be the $2\pi -$ periodic extension of f to R, then:

$\hat{f}(n) = \frac{1}{4\pi}\int_{-\pi}^{+\pi}\left[\tilde{f}(x) - \tilde{f}(x + \frac{\pi}{n})\right] e^{inx}dx$ for all $n \in Z\backslash\{0\}$

Proof: $\hat{f}(n) = \int_{-\infty}^{+\infty} \tilde{f}(x)e^{-2\pi inx}dx = \int_{-\pi}^{+\pi} \tilde{f}(x)e^{-2\pi inx}dx + \sum_{k=-\infty}^{-1}\int_{(2k-1)\pi}^{(2k+1)\pi} \tilde{f}(x)e^{-2\pi inx}dx + \sum_{k=1}^{+\infty}\int_{(2k-1)\pi}^{(2k+1)\pi} \tilde{f}(x)e^{-2\pi inx}dx = \int_{-\pi}^{+\pi} \tilde{f}(x)e^{-2\pi inx}dx + \sum_{k=-\infty}^{-1}\int_{-\pi}^{\pi} \tilde{f}(x + 2k\pi)e^{-2\pi in(x+2k\pi)}dx + \sum_{k=1}^{+\infty}\int_{-\pi}^{\pi} \tilde{f}(x + 2k\pi)e^{-2\pi in(x+2k\pi)}dx = \int_{-\pi}^{+\pi} \tilde{f}(x)e^{-2\pi inx}dx + \sum_{k=-\infty}^{-1}\int_{-\pi}^{\pi} \tilde{f}(x)e^{-2\pi in(x+2k\pi)}dx + \sum_{k=1}^{+\infty}\int_{-\pi}^{\pi} \tilde{f}(x)e^{-2\pi in(x+2k\pi)}dx = \sum_{k=-\infty}^{+\infty}\int_{-\pi}^{\pi} \tilde{f}(x)e^{-4\pi^2 kni}e^{-2\pi inx}dx$. Estimating the error, mathematiciansarrive at our result: $\hat{f}(n) = \frac{1}{4\pi}\int_{-\pi}^{+\pi}\left[\tilde{f}(x) - \tilde{f}(x + \frac{\pi}{n})\right] e^{inx}dx$ for all $n \in Z\backslash\{0\}$. Then let's use this result to estimate the upper bound of Fourier transform. $\left|\tilde{f}(n)\right| = \frac{1}{4\pi}\left|\int_{-\pi}^{+\pi}\left[\tilde{f}(x) - \tilde{f}(x + \frac{\pi}{n})\right] e^{inx}dx\right| \le \frac{1}{4\pi}\int_{-\pi}^{\pi}\left|\tilde{f}(x) - \tilde{f}(x + \frac{\pi}{n})\right| dx = \frac{1}{4\pi}\int_{-\pi}^{\pi}|f'(x+\xi)|\frac{\pi}{|n|}dx \le \frac{1}{4\pi}\int_{-\pi}^{\pi}\|f'\|_{L^{\infty}(T)}\frac{\pi}{|n|}dx = \frac{\pi\|f'\|_{L^{\infty}(T)}}{2|n|}$

This is a good estimation of the upper bound of Fourier transform.

3. Summary

The most popular application of fourier analysis is as a univariate technique for either data simplification or modeling. It may also be applied as a multivariate data analysis approach. Fourier analysis and trend analysis are related in several ways. It examines the connections between data sets from a new angle. In the instance of Fourier analysis, the method clarifies the data set's time dimension variable. The most basic type of Fourier analysis makes the assumption that many occurrences have a periodic character, and that Fourier transforms may be used to eliminate the fluctuation in other variables caused by this periodicity. Fourier-transformed data can be subjected to more potent analysis by utilizing the residual (i.e., time-independent) variance from other variables. At the simplest level, Fourier analysis assumes that many events are periodic in nature, and that the variation in other variables due to this periodicity can be removed by using Fourier transforms. Fourier analysis allows one to evaluate the amplitudes, phases, and frequencies of data using the Fourier transform. More powerful analysis can be done on the Fourier transformed data using the remaining (i.e., time-independent) variation from other variables. Unfortunately, Fourier analysis may not be appropriate when there may be several overlapping cyclic, time-based periodicities, or if time cycle event analysis is required. Details regarding the calculations employed in this technique are beyond the scope of this chapter, but are available in reference texts on the subject. A wave changes its speed as it travels through a heterogeneous medium as a result of differences in the medium's rate of wave propagation. A shift in phase between what is expected and what is measured may thus be used to calculate the additional time delay, which in turn tells you how much the wave speed has changed in the medium. Naturally, this is a very simple explanation for laypeople, yet it forms the basis of tomography. Fourier methods have provided scientists with a set of useful tools for expressing any periodic function as the sum of sines and cosines. The Fourier series was initially used to solve the heat equation in a classical physics issue. In this paper, We've given a summary for the construction process of Fourier transform theory, from DFT to Fourier transform on the real line. And we've talked about the convergence of Fourier series both in uniform case and mean square case. The result allow some important properties to remain when the time space is extending. Finally, using the skills of Fourier transform, mathematiciansgive the upper bound of Fourier transform on each integer unequal to zero an estimation.

References

[1] Chang Gengzhe, Shi Jihuai. Mathematics Analysis. Higher Education Press. Beijing. Chapter 11. 2003.

[2] Stein, Elias M., and Rami Shakarchi. Fourier analysis: an introduction. Vol. 1. Princeton University Press, 2011.

[3] L. Carleson, On convergence and growth of partial sums of Fourier series, Acta Math. 116 (1966), 135–157.

[4] T. Cazenave, Semilinear Schr¨odinger equations. Courant Lecture Notes in Mathematics, 10. New York University, Courant Institute of Mathematical Sciences, New York; American Mathematical Society, Providence, RI, 2003. xiv+323 pp.

[5] E. Cand`es, J. Romberg, and T. Tao, Robust Uncertainty Principles: Exact Signal Reconstruction from Highly Incomplete Frequency Information. IEEE Transactions on Information Theory, Vol. 52, No. 2, February 2006.

[6] M. Christ and A. Kiselev, Maximal functions associated to filtrations. J. Funct. Anal. 179 (2001), no. 2, 409–425

[7] M. Christ and M. Weinstein, Dispersion of small amplitude solutions of the generalized Korteweg-de Vries equation. J. Funct. Anal. 100 (1991), no. 1, 87–109. [7] I. Daubechies, Ten Lectures on Wavelets. CBMS-NSF Regional Conference Series in Applied Mathematics, 61. Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 1992. xx+357 pp

[8] C. Fefferman, Pointwise convergence of Fourier series, Ann. of Math. 98 (1973), 551–571.

[9] G. Folland, A course in abstract harmonic analysis. Second edition. Textbooks in Mathematics. CRC Press, Boca Raton, FL, 2016.

[10] S. Foucart and H. Rauhut, A Mathematical Introduction to Compressive Sensing. Applied and Numerical Harmonic Analysis. Birkh¨auser/Springer, New York, 2013. [11] D. Griffiths, Introduction Quantum Mechanics. Pearson 2014.

[11] M. Lacey, Carleson's theorem: proof, complements, variations. Publ. Mat. 48 (2004), no. 2, 251–307.

[12] M. Lacey and C. Thiele, A proof of boundedness of the Carleson operator. Math. Res. Lett. 7 (2000), 361–370.

[13] Wheeden and Zygmund, Measure and Integral. An introduction to real analysis. Pure and Applied Mathematics, Vol. 43. Marcel Dekker, Inc., New York-Basel, 1977.

[14] Katznelson, An Introduction to Harmonic Analysis. Second corrected edition. Dover Publications, Inc., New York, 1976.

[15] M. Keel and T. Tao, Endpoint Strichartz estimates. Amer. J. Math. 120 (1998), no. 5, 955–980.

[16] R. Killip and M. Visan, Nonlinear Schr¨odinger equations at critical regularity. Evolution equations, 325–437, Clay Math. Proc., 17, Amer. Math. Soc., Providence, RI, 2013.

[17] E. Lieb and M. Loss, Analysis. Graduate Studies in Mathematics, 14. American Mathematical Society, Providence, RI, 1997.

[18] A. Martinez, An Introduction to Semiclassical and Microlocal Analysis. Universitext. Springer-Verlag,New York, 2002.

[19] H. Br´ezis and E. Lieb, A relation between pointwise convergence of functions and convergence of functionals. Proc. Amer. Math. Soc. 88 (1983), 486–490.

[20] Dym, H, and H.P. McKean. Fourier Series and Integrals. Academic Press: New York and London, 1972.