The Application of Residue Theorem in The Calculations of Real Integrals

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Abstract. Complex analysis focuses on the functions of complex variables. The research on complex analysis began in 19th century by mainly Cauchy, Riemann, and Weierstrass. In 1825, Cauchy established Cauchy's integral theorem, indicating that the value of a complex function at a point within a closed contour C is entirely dependent on values of the functions at points on the contour C. Cauchy's integral theorem is a significant theorem in complex analysis, which was then further developed into Cauchy's residue theorem. In this paper, the authors introduce the fundamentals of complex analysis. In Introduction, the authors discuss the history of complex analysis and residue theorem and summarizes the content of published articles and books. In Method, the authors explain the definitions and fundamental properties of complex functions. In Result, the authors use examples of different forms of real integrals to demonstrate the application of residue theorem in the calculation of real integrals.

Keywords: Complex analysis; Residue theorem; Integral.

1. Introduction

Complex analysis focuses on the functions of complex variables. Complex numbers first appear in the general solutions for algebraic equations. A complex number z can be represented in Cartesian form as $z = a + bi$, where a, b are both real numbers and $i = \sqrt{-1}$. In Cartesian form, $Re(z) = a$, which is the real part of z, and $Im(z) = b$, which is the imaginary part of z [1]. With two independent variables a and b, complex analysis becomes very useful to solve problems that deal with two variables, which are independent to each other, at the same time.

In 18th century, mathematicians including Euler and Jean le Rond d'Alembert started to involve complex number and complex variables in their work with discoveries of important features of complex functions. The theorem currently called Cauchy-Riemann equations appeared in both d'Alembert's research on fluid dynamics and in Euler's calculations of real integrals using complex functions. However, mathematicians in 18th century consider complex variables as a tool for problem solving, rather than considering it as their research object.

The research on complex analysis started in 19th century by mainly Cauchy, Riemann, and Weierstrass. In 1825, Cauchy established Cauchy's Integral Theorem (CIT), indicating that the value of a complex function at a point within a closed contour C is entirely dependent on values of f at points on the contour C. CIT is a significant theorem in complex analysis, which was then further developed into Cauchy's Residue Theorem (CRT) [2].

Until 1846, Cauchy's main concern on his residue theorem was the calculation of real integrals and their values. According to H. J. Ettlinger, the application of residue theorem is mainly on simple integrals using residue theorems without any geometrical representations; Cauchy also avoids the usage of complex numbers by separating them into real parts and imaginary parts [3].

The change in Cauchy's view occurred in 1846, when he published two papers in which he extended the fundamental theorem that is independent of the contour and the residue theorem to the case of arbitrary closed curves. By about 1850, Cauchy had started to recognize the significance of his work as a fundamental result of complex functions.

In the article Geometric Residue Theorems by Harvey R. and Lawson H. B, the author mainly discusses the application of residue theorem based on Riemann's applications from the perspective of geometric singularities and topology [4]. O. J. Farrell suggests that a modified version of the usual procedure to calculate real integrals using residue theorem by using an example of a form of improper integral that is like $\int_{-\infty}^{\infty} \frac{e^{px}dx}{1+e^{x}}$ [5]. In the article Complex Analysis as Catalyst by Krantz. S. G., the author discusses the methods of applying knowledge from other fields of study to solve complex analysis problems including topology, hard analysis, partial differential equations, algebra, group theory, and geometry, by giving examples respectively from these fields of study [6]. In the book Methods of Mathematical Physics by Wu, C. and Gao C., the authors discuss about the applications of residue theorem in the field of physics [7]. A. P. Campbell and D. Daners link linear algebra with complex analysis to deduce the Jordan decomposition theorem and Cayley-Hamilton theorem by operating Laurent expansion [8]. Axler S. clarifies the elementary properties of harmonic functions based on a complex analysis perspective [9]. Henrici, P. demonstrates the application of Fast Fourier Transform method in complex analysis [10].

The theoretical basis of this paper is CRT.

CRT is a precise theorem that can be also applied to simplify the calculation of some real integrals, especially in the calculation of improper integrals. For improper integrals over infinite limits, it is nearly impossible to calculated by the methods in real integrals. Also, for some functions, calculating their integrals using real integral methods could be very complicated. In these conditions, it is necessary to apply residue theorem.

This paper divides the application of residue theorem in the calculation of real integrals to be two sections: the first section discusses the process of transforming real integrals to complex integrals, and the second section gives examples of the application of residue theorem in different types of real integral calculations.

The exist research on the application of residue theorem in the calculations of real integrals are relatively sophisticated for high school students to understand. Therefore, the purpose of this paper is to simplify the process of using residue theorem in real integrals for students to follow, and to give examples for students to better understand how residue theorem works in real integrals.

2. Method

2.1. Complex Numbers

Complex numbers in mathematics are created by extending real numbers. Any polynomial equation can have roots because of complex numbers. An element i of a complex number known as the "Imaginary Unit" is a square root of -1 and satisfies the equation i2 = -1. The formula a + bi can be used to express any complex number. Both a and b are real numbers for the complex numbers a + bi. The real part is referred to as a, and the imaginary part as b.

Either symbol C is used to represent the collection of complex numbers. A complex number is an algebraic extension of a real number that takes the form of the imaginary number i. It means that for addition, subtraction, and multiplication, complex numbers can be thought of as polynomials of variable i. In addition, the i2 = -1 rule is used. Furthermore, non-zero complex numbers can be used to divide complex numbers.

The horizontal axis and vertical axis serve as geometric representations of the real and imaginary halves, respectively, of a complex number. bringing two-dimensional space and the concept of one-dimensional space together.

However, complex numbers are permitted to use more complex algebraic structures. Other additional operations might not be possible in vector space. For instance, multiplying two complex numbers together will result in a new complex number. However, this should not be confused with the typical "product" in a vector.

2.2. Complex Conjugate

A complex conjugate in mathematics is a number whose real and imaginary components are the same as the complex number but have the opposite sign. The common notation for the complex conjugate is $a - bi$. If a and b are real numbers, it complies with the formula that reads "a+bi=a-bi." Typically, z* is used to indicate the complex conjugate's sign. The conjugate takes on the polar form $re^{-i\varphi}$. User may put it into words by utilizing Euler's formula.

2.3. Complex Analysis

A function whose independent and dependent variables are both complex is referred to as a function of a complex variable. The complex plane is broken down into the domain of definition and the domain of the value of a complex variable's function. The "argument" of the function is another name for the independent variable in the study of complex variables [11].

The independent and response variables can be split into real and fictitious components for functions of complex variables:

$$z = x + iy$$

$$\omega = f(z) = u(x, y) + iv(x, y)$$

Where x, y$\in$ R and $u(x, y), v(x, y)$ are real functions. It can be understood as a binary real function of the variables x and y.

①Holomorphic function

A holomorphic function is one that accepts values in the complex plane C, is differentiable at every point, and is defined on an open subset of the complex plane C.

A complex variable function is holomorphic if the real and imaginary sides of the function satisfy the Corsi-Riemann equation [12].

$$\frac{\vartheta u}{\vartheta x} = \frac{\vartheta v}{\vartheta y} \; and \; \frac{\vartheta u}{\vartheta y} = -\frac{\vartheta v}{\vartheta x}$$

②Cauchy integral theorem

According to Corsi's integral theorem, a holomorphic function's integral value is 0 if its closed integral path does not contain a singularity. The forward integral of the outer closed path is equal to the forward integral of the closed path over the inner ring containing the singularity if it contains a singularity.

③Cauchy integral formula

Assuming that the complex plane C has an open subset U, the complex differentiable equation f: U→C is on a closed disc D, which is a subset of U. Let C define D's perimeter. Then, one can infer that each point a that is located inside of D:

$$f(a) = \frac{1}{2\pi i} \oint_C \frac{f(z)}{z - a} dz$$

Where the integral is the integral along the counterclockwise direction.

④Morphological action

A meromorphic function is a function that is holomorphic in the region of D except for one or more isolated sets of points, which are referred to as the poles of the function in complex variable analysis.

2.4. Laurent Series Expansion

If z_0 is an isolated singularity of $f(z)$ and resolves within $0<|z-z_0|<R$, then the Laurent series expansion of $f(z)$ is

$$f(z) = \ldots + c_{-n}(z-z_0)^{-n} + \ldots + c_{-1}(z-z_0)^{-1} + c_0 + c_1(z-z_0)^1 + \ldots + c_n(z-z_0)^n + \ldots [13]$$

2.5. Analytic Function

A local convergent power series can provide an analytical function in mathematics. The two types of analytical functions—real and complex—exist. If and only if, for every x_0 in its domain, the Taylor series about the x_0 converges to the function in some neighbourhood, is a function considered analytical.

If one can define a function f for any $x_0 \in D$, that function is formally regarded as real analytic on an open set D in the real line.

$$f(x) = \sum_{n=0}^{\infty} a_n(x-x_0)^n = a_0 + a_1(x-x_0) + a_2(x-x_0)^2 + \ldots$$

It has real numbers for the coefficients $a_0, a_1, \ldots$ and converges to $f(x)$ for x in the vicinity of x_0. On the other hand, the Taylor series is an infinitely differentiable function at any point x_0 in the domain of a real analytic function.

$$T(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n$$

Converge to $f(x)$ for x in the pointwise neighbourhood of x_0. On a given set D, the set of all real analytical functions [14].

2.6. Residue

Assuming that the circular domain D may resolve z_0 as $f(z)$: $0<|z-z_0|<R$. C is any positive simple closed curve of D around z_0. The integral

$$\frac{1}{2\pi i}\oint_C f(z)dz$$

is the retention of $f(z)$ at z_0, Denoted $\mathrm{Res}[f(z), z_0]$. So

$$Res[f(z), z_0] = \frac{1}{2\pi i}\oint_C f(z)dz = c_{-1}$$

where c_{-1} is the coefficient of the negative power $(z-z_0)^{-1}$ in the Laurent series expansion of $f(z)$ [12].

2.7. Residue Theorem

The residue Theorem is an effective tool for complex analysis that may be used to calculate both the integral of a real function and the integral of an analytical function along the path of a closed curve. The Cauchy integral formula and the Cauchy integral theorem have been improved.

Let the function $f(z)$ be analytic everywhere in the region D except at a finite number of isolated singularities $z_1, z_1, \ldots, z_n$. C be a positive simple closed curve enclosing the singularities in D. Then [12]

$$\oint_C f(z)dz = 2\pi i \sum_{k=1}^{n} Res[f(z), z_k]$$

It is possible to determine the integrals of some difficult real functions by converting them into complex functions and then employing the residue theorem [15].

3. Result

3.1. Integral of the Form $\int_0^{2\pi} R(\cos\theta, \sin\theta)\, d\theta$

R (cos θ, sin θ) is a rational function with respect to cos θ, sin θ and is continuous on [0, 2π].

Let $z = e^{i\theta}$, with z = cos θ+ isin θ.

And let $z^{-1} = e^{-i\theta}$, therefore, z^{-1} = cos(-θ) + isin(-θ) = cos θ - isin θ.

Add the upper and lower equations to get $z + z^{-1} = 2\cos\theta$, which is $\cos\theta = \frac{z+z^{-1}}{2}$.

Subtract the upper and lower equations to get $z + z^{-1} = 2i\sin\theta$, which is $\sin\theta = \frac{z-z^{-1}}{2i}$.

At the same time, there are $d\theta = \frac{dz}{iz}$.

When θ goes through [0, 2π], z circles a circle in the positive direction of the circumference | z | = 1, hence

$$\int_0^{2\pi} R(\cos\theta, \sin\theta)\, d\theta = \int_{|z|=1} R\left(\frac{z+z^{-1}}{2}, \frac{z-z^{-1}}{2i}\right) \frac{dz}{iz}.$$

3.1.1 Example 1

One need calculates the integral $\int_0^{2\pi} (\frac{\cos\theta}{2+\sin\theta}) d\theta$.

Solutions

- Method 1, using the original mathematical analysis method.

$$\int_0^{2\pi} (\frac{\cos\theta}{2+\sin\theta}) d\theta = \int_0^{2\pi} (\frac{1}{2+\sin\theta}) d\sin\theta = 0.$$

- Method 2, converting it into a complex integral, and use the method of complex variable function to solve it.

Let $z = e^{i\theta}$, substitute $\cos\theta = \frac{z+z^{-1}}{2}$, $\sin\theta = \frac{z-z^{-1}}{2i}$, $d\theta = \frac{dz}{iz}$ into the original integral to get

$$\int_0^{2\pi} (\frac{\cos\theta}{2+\sin\theta}) d\theta = \int_{|z|=1} \left(\frac{\frac{z+z^{-1}}{2}}{2 + \frac{z-z^{-1}}{2i}} \right) \frac{1}{iz} dz$$

$$= \int_{|z|=1} \left(\frac{z^2+1}{z(z-(\sqrt{3}-2)i)(z+(\sqrt{3}+2)i)} \right) dz$$

Let $f(z) = \frac{z^2+1}{z(z-(\sqrt{3}-2)i)(z+(\sqrt{3}+2)i)}$, it can be seen that within |z| = 1, $f(z)$ has two first-order poles, which is z = 0, and another one is z = $(\sqrt{3}-2)i$, since $\operatorname*{Res}_{z=0} f(z) = -1$, $\operatorname*{Res}_{z=(\sqrt{3}-2)i} f(z) = 1$, as a result,

$$\int_0^{2\pi} (\frac{\cos\theta}{2+\sin\theta}) d\theta = \int_{|z|=1} f(z)\, dz = 2\pi i \left(\operatorname*{Res}_{z=0} f(z) + \operatorname*{Res}_{z=(\sqrt{3}-2)i} f(z) \right) = 0$$

3.1.2 Example 2

One need calculates the integral $I = \int_0^{2\pi} \frac{d\theta}{1-2p\cos\theta+p^2}$ ($|p| \neq 1$). Here is the solution.

- Let $\cos\theta = \frac{z+z^{-1}}{2}$, $d\theta = \frac{dz}{iz}$.

We substitute it into the original integral, that is calculating

$$\int_{|z|=1}\left(\frac{1}{1-2\mathrm{p}\times\frac{z+z^{-1}}{2}+p^{2}}\right)\frac{1}{iz}dz$$
$$=-\frac{1}{ip}\int_{|z|=1}\left(\frac{1}{\left(z-\frac{1}{p}\right)(z-p)}\right)dz$$

Let $f(z)=\frac{1}{\left(z-\frac{1}{p}\right)(z-p)}$, that can easily find that $f(z)$ has a first-order pole $z=\frac{1}{p}$ and a first-order pole z = p.

When p = 0, I = $\int_0^{2\pi} d\theta = 2\pi$.

When 0 < | p | < 1, it can be found that $\frac{1}{|p|} > 1$, so $f(z)$ has only one first-order pole, which is z = p within | z | = 1. Since $\operatorname{Res}_{z=p} f(z)=\frac{p}{p^2-1}$, so I = $-\frac{1}{ip}\times 2\pi i \operatorname{Res}_{z=p} f(z)=\frac{2\pi}{1-p^2}$.

When | p | > 1, it can also be found that $\frac{1}{|p|} < 1$, so $f(z)$ has only one first-order pole z = $\frac{1}{p}$ within | z | = 1. Since $\operatorname{Res}_{z=\frac{1}{p}} f(z)=\frac{p}{1-p^2}$, so I = $-\frac{1}{ip}\times 2\pi i \operatorname{Res}_{z=\frac{1}{p}} f(z)=\frac{2\pi}{p^2-1}$.

In summary, I = $\frac{2\pi}{|p^2-1|}$.

It can be found that if the original mathematical analysis method is used, it can be solved using the universal formula, but the latter is simpler than converting it to a complex integral using the residue theorem. Now use the method of complex variable function to calculate the integral [16].

3.2. Integral of the Form $\int_{-\infty}^{+\infty}\frac{P(x)}{Q(x)}dx$

To study this integral, can introduce the following lemma

Lemma 1 Let $f(z)$ be continuous along the arc S_R: z = $Re^{i\theta}$ ($\theta_1 \le \theta \le \theta_2$, R is sufficiently large), andc $\lim_{R\to+\infty} zf(z)$ = λ is consistent on S_R (that is, with θ1 ≤ θ in $\theta_1 \le \theta \le \theta_2$ is irrelevant), then

$$\lim_{R\to+\infty}\int_{S_R} f(z)dz = i(\theta_2-\theta_1)\lambda .$$

This lemma can be used to prove the following theorem. Theorem 2 Let $f(z)=\frac{P(z)}{Q(z)}$ be a rational fraction, where $P(z)=c_0z^m+c_1z^{m-1}+...+c_m$ ($c_0 \neq 0$) and Q(z)= $b_0z^n+b_1z^{n-1}$ +...+b_n ($b_0 \neq 0$) is a coprime polynomial and meets the conditions:

(1)$n-m \ge 2$;

(2) $Q(z) \neq 0$ can be shown on the real axis. So

$$\int_{-\infty}^{+\infty} f(x)\,dx = 2\pi i\sum_{\mathrm{Im}a_k>0}\operatorname{Res}_{z=a_k} f(z).$$

Theorem 2 shows that if the integrand satisfies the conditions of Theorem 2, the integral problem can be transformed into a residue problem.

3.2.1 Example 3

One need calculates the integral $\int_0^{+\infty}\frac{x^2}{x^4+1}\,dx$.

Solution

- The integrand is even, so

$$\int_0^{+\infty}\frac{x^2}{x^4+1}\,dx=\frac{1}{2}\int_{-\infty}^{+\infty}\frac{x^2}{x^4+1}\,dx.$$

Let $f(z) = \frac{x^2}{x^4+1}$, the function satisfies the conditions of Theorem 2 and has 4 first-order poles $a_k = e^{i\frac{\pi + 2k\pi}{4}}$ (k = 0, 1, 2, 3), where only a_0, a_1 are in the upper half plane. The residue is calculated as $\operatorname*{Res}_{z = a_0} f(z) = \frac{\sqrt{2}}{8} - \frac{\sqrt{2}}{8}i$, and $\operatorname*{Res}_{z = a_1} f(z) = -\frac{\sqrt{2}}{8} - \frac{\sqrt{2}}{8}i$.

So there is $\int_0^{+\infty} \frac{x^2}{x^4+1}\, dx = \frac{1}{2}\int_{-\infty}^{+\infty} \frac{x^2}{x^4+1}\, dx$

$$= \frac{1}{2} \times 2\pi i \sum_{Ima_k > 0} \operatorname*{Res}_{z = a_k} f(z)$$

$$= \pi i \,(\operatorname*{Res}_{z = a_0} f(z) + \operatorname*{Res}_{z = a_1} f(z)) = \frac{\sqrt{2}}{4}\pi \text{ [17].}$$

3.3. Integral of the Form $\int_{-\infty}^{+\infty} \frac{P(x)}{Q(x)} e^{Imx}\, dx$.

To study this integral, the authors also introduce the following lemma

Lemma 2 (Jordan's Lemma) Let the function g(z) be continuous along the semicircle Γ_R : z = R$e^{i\theta}$ ($0 \leq \theta \leq \pi$, R is sufficiently large), and $\lim_{R \to +\infty} g(z) = 0$ is consistent on Γ_R is established, then $\lim_{R \to +\infty} \int_{\Gamma_R} g(z)\, e^{Imx}\, dz = 0$ (m > 0).

The above lemma can help to prove the following theorem.

Theorem 3 Let g(z) = $\frac{P(z)}{Q(z)}$, where P(z) and Q(z) are coprime multiple term, and meets the conditions:

(1) The degree of P(z) is lower than that of Q(z).

(2) Q(z) ≠ 0 on the real axis.

(3) m > 0.

Then $\int_{-\infty}^{+\infty} g(x)\, e^{Imx} dx = 2\pi i \sum_{Ima_k > 0} \operatorname*{Res}_{z = a_k} [g(x) e^{Imx}]$

3.3.1 Example 4

One need calculates the integral $\int_{-\infty}^{+\infty} \frac{x\sin x}{x^2+1}\, dx$.

Solution

Let g (z) = $\frac{z}{z^2+1}$, g (z) satisfies the condition of Theorem 3, then let $f(z)$ = g(z) e^{iz}, that is $f(z) = \frac{ze^{iz}}{z^2+1}$, from Theorem 3, it can be known that

$$\int_{-\infty}^{+\infty} \frac{xe^{ix}}{x^2+1}\, dx = 2\pi i \sum_{Ima_k > 0} \operatorname*{Res}_{z = a_k} f(z) .$$

And $f(z)$ has two first-order poles, which is z = i and another one is z = $-i$, of which only z = i in the upper half plane, therefore, only need to calculate $\operatorname*{Res}_{z = i} f(z)$, it can be known easily that $\operatorname*{Res}_{z = i} f(z) = \frac{1}{2e}$.

so $\int_{-\infty}^{+\infty} \frac{xe^{Imx}}{x^2+1}\, dx = 2\pi i \times \operatorname*{Res}_{z = 1} f(z) = \frac{\pi i}{e}$.

notices that $e^{Imx} = \cos x + i\sin x$,

so $\int_{-\infty}^{+\infty} \frac{xe^{Imx}}{x^2+1}\, dx = \int_{-\infty}^{+\infty} \frac{x(\cos x + i\sin x)}{x^2+1}\, dx = \int_{-\infty}^{+\infty} \frac{x\cos x}{x^2+1} dx + i\int_{-\infty}^{+\infty} \frac{x\sin x}{x^2+1} dx = \frac{\pi i}{e}$,

so $\int_{-\infty}^{+\infty} \frac{x\sin x}{x^2+1}\, dx = \frac{\pi i}{e}$.

At the same time, it can be obtained that $\int_{-\infty}^{+\infty} \frac{x\cos x}{x^2+1} dx = 0$ [18].

4. Conclusion

The authors introduce the foundations of complex analysis in this study. The writers review the content of books and articles that have been published in the Introduction and talk about the history of complex analysis and the CRT. The definitions and fundamental characteristics of complex numbers and complex functions are covered by the authors in Method. The Result section illustrates how the residue theorem can be used to calculate real integrals with examples of various forms of real integrals. The outcome of this paper could be used to teach residue theorem in classrooms. This essay could be used by teachers to explain complex numbers and to further discuss the residue theorem. Students can comprehend the procedure for transforming a real integral to a complex integral and computing the complex integral using the CRT by studying the examples in Result.

This paper's primary drawback is that it does not account for all real integrals that can be computed via the residue theorem. This paper's focus is on specific integral forms, with little attention paid to other integral types. This paper could need more examples of various real integrals that can be solved using the residue theorem as well as a general explanation of how to convert real integrals into complex integrals.

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