

# Complex Functions and Residue Theorem

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**Abstract.** The features and connections of the mappings between complex number fields are thoroughly and methodically summarized and inferred by the complex function. The functional relationship of the mapping between the fields of complex numbers is the subject of study. A complex function uses a complex number as both its independent and dependent variable. The remainder theorem generalizes both the Cauchy's integral theorem and the Cauchy's integral formula. Numerous complex calculations are resolved by the theory of Complex functions and Residue theorem, which has a wide range of applications. This article begins by defining ideas and formulas, demonstrating how formulas connect to concepts, and outlining the procedure and approaches used by mathematicians when working with formulas. The current aspect and the signal aspect are two of the most intimate relationships between Complex Functions and daily living.

**Keywords:** Complex functions; Residue Theorem; Cauchy-Riemann.

## 1. Introduction

If the function  $f$  is resolved in a concise closed profile  $C$ , apart from a limited number of singularities, these must all be isolated. The complex function is a comprehensive and systematic summary and induction of the characteristics and relationships of the mappings between complex number fields. The object of study is the functional relationship of the mapping between the fields of complex numbers. Therefore, in the study of complex functions, it is basically algebraic operations, and there is no case of approximation and so on after the number for the convenience of calculation.

One of the most ancient and traditional fields of mathematics is complex analysis, which dates back to the eighteenth century. In 1774, Euler considered two equations derived from the integrals of complex functions in one of his papers. Earlier than him, the French mathematician D'Alembert had this idea in his paper on fluid mechanics. Therefore, these two equations were later referred to as the "D'Alembert-Euler equation". D'Alembert-Euler has done many things to the society, for example, Euler's series laid the foundation of number theory as a separate branch of mathematics. A large part of Euler's work is related to the theory of divisibility of numbers. Euler's most important discovery in number theory was the quadratic antilaw. Euler also introduced the expansion of functions into infinite products and found the sum of elementary fractions, which played an important role in the later general theory of analytical functions. He developed a new and broader definition of the notion of sum of series. He also proposed two summation methods. These rich ideas had a profound impact on two themes in divergent series theory in the late 19th and early 20th centuries, namely asymptotic series theory and the concept of summability. By the nineteenth century, the above two equations had been studied in more detail when Cauchy and Riemann studied fluid mechanics, so these two equations were also called the "Cauchy-Riemann condition" [1]. For his contribution, in 1859, he studied the Riemann zeta function in his paper on the distribution of prime numbers less than a given value, and gave the integral representation of the function and the functional equations that satisfy. He pointed out that there is a deep relationship between the distribution of prime numbers and the Riemann zeta function. The full development of complex variable function theory was in the 19th century, and this new branch dominated nineteenth-century mathematics. In the 20th century, significant complex number mathematicians included Euler, Gauss, Riemann, Cauchy, Weierstrass, and others.

## 2. Complex Functions and Residue Theorem

### 2.1. Definition of Complex Functions

The range of complex functions ranges from plural to plural. In other words, it is a function, and its complex subset is defined as a complex subset of the domain as an upper domain. Complex function is a function takes complex number as independent variable and dependent variable. For example, assume that  $G$  is a group of points on the complex flat, there are one or more definite  $W$  correspond to any point that  $z$  in  $G$ . This is said to define complex function on  $G$ , so that  $W = f(z)$ . Many theorems and methods have been summarized in the study of complex functions. But each of these theorems and methods has its own very clear scope of application and method of use. This is to ensure that they are not misused and ultimately lead to biased or even complete errors in the results when used to solve the corresponding problems [2].

### 2.2. Definition of Residue Theorem

For the Residue Theorem, residue is introduced to use easier way to calculate the closed curve integral. The definition is that let  $z_0$  is an isolated singularity of the analytic function  $F(z)$ , and the coefficient  $C_-$  of the negative first power term in the Laurent expansion of  $F(z)$  at  $z_0$ . So this is called the residue of  $F(z)$  at  $z_0$ , as  $\text{Res}[f(z), z_0]$  [3].

If  $f$  is also an analytic on  $C$  and if  $C$  is positive, then the value of the integral of  $f$  around  $C$  is  $2\pi$  times  $f$  is a singularity inside  $C$  residuals at. In complex analysis, the remainder theorem is a powerful tool for calculating the integration of analytic functions along the path of a closed curve, and can also be used to calculate the integral of real functions.

### 2.3. Proof of Complex Functions

Here are solutions to verify the Euler's formula.

$dx$  is expressed to  $\lim x \rightarrow 0$ , that  $dy$  is expressed to  $\rightarrow 0$ , so  $y' = \frac{dy}{dx} = \frac{dy}{dx}$ .

$$\text{Therefore, } f(x)' = \frac{df(x)}{dx} \rightarrow df(x) = f'(x)dx. \quad (1)$$

The function  $\ln y$  take the derivative of  $y$ , so that

$$\frac{dy}{y} = \frac{1}{y} \cdot dy = d\ln y. \quad (2)$$

Let  $y = \cos x + i \cdot \sin x$ , take the derivative of  $x$ , find that:

$$\frac{dy}{dx} = -\sin x + i \cdot \cos x. \quad (3)$$

Look at the original type, let  $y$ , so that  $\frac{dy}{y} = i dx$  by transposing, so  $d\ln y = i dx$ . Therefore,

$$\ln y = ix + c \rightarrow y = e^{ix+c} \quad (c \text{ is any real number}) \quad (4)$$

Let  $x$  substitute into formula(1), so  $y = \cos 0 + i \cdot \sin 0 = 1$ . One may substitute into formula (4), solve that  $c = 0$ . So

$$e^{ix} = \cos x + i \sin x \quad (5),$$

And this is the Euler's formula's process of proof [4].

### 2.4. Proof of Residue Theorem

e.g.

$$\int_{-\infty}^{\infty} \frac{e^{itx}}{x^2 + 1} dx.$$

It follows a real straight line from  $-a$  to  $a$ , and then counterclockwise along a semicircle centered on 0 from  $a$  to  $-a$ . We take  $a$  as greater than 1 so that the imaginary unit  $i$  is enclosed inside the curve. The path integrals are:

$$\int_C f(z) dz = \int_C \frac{e^{itz}}{z^2 + 1} dz.$$

Since  $e^{itz}$  is an entire function, it only has singularities if  $z^2 + 1$  is zero. Since  $z^2 + 1 = (z + i)(z - i)$ , it has a singularity when  $z = i$  or  $z = -i$ . Due to  $f(z)$  is:

$$\frac{e^{itz}}{z^2 + 1} = \frac{e^{itz}}{2i} \left( \frac{1}{z - i} - \frac{1}{z + i} \right) = \frac{e^{itz}}{2i} \frac{1}{z - i} - \frac{e^{itz}}{2i(z + i)},$$

The residue of  $f(z)$  at  $z = i$  is:

$$\text{Res}_{z=i} f(z) = \frac{e^{-t}}{2i}.$$

According to the residue theorem, then can get that:

$$\int_C f(z) dz = 2\pi i \cdot \text{Res}_{z=i} f(z) = 2\pi i \frac{e^{-t}}{2i} = \pi e^{-t}.$$

Path  $C$  can be divided into

$$\int_{\text{straight}} + \int_{\text{arc}} = \pi e^{-t}.$$

$$\text{Therefore, } \int_{-a}^a = \pi e^{-t} - \int_{\text{arc}}.$$

If  $t > 0$ , then when the radius of the semicircle tends to infinity, the integral along the path of the semicircle tends to zero:

$$\int_{\text{arc}} \frac{e^{itz}}{z^2 + 1} dz \leq \int_{\text{arc}} \left| \frac{e^{itz}}{z^2 + 1} \right| |dz| = \int_{\text{arc}} \frac{|e^{itz}|}{|z^2 + 1|} |dz| = \int_{\text{arc}} \frac{1}{|z^2 + 1|} |dz| \leq \int_{\text{arc}} \frac{1}{a^2 - 1} |dz| = \frac{\pi a}{a^2 - 1} \rightarrow 0 \text{ as } a \rightarrow \infty. \text{ Therefore,}$$

if  $t > 0$ , then:

$$\int_{-\infty}^{\infty} \frac{e^{itz}}{z^2 + 1} dz = \pi e^{-t}.$$

Analogously, we have

$$\int_{-\infty}^{\infty} \frac{e^{itz}}{z^2 + 1} dz = \pi e^t,$$

So:

$$\int_{-\infty}^{\infty} \frac{e^{itz}}{z^2 + 1} dz = \pi e^{-|t|}.$$

(If  $t = 0$ , this integral can be quickly calculated using the elementary method, and its value is  $\pi$ .)

## 2.5. Application

Complex functions are the focus of the field of mathematical analysis known as complex functional analysis, which was formerly known as complex functional theory. Numerous domains of mathematics, such as algebraic geometry, number theory, analytic synthesis, and applied mathematics can benefit from complex analysis. Complex analysis pays special attention to the analytic function of a complex number since it is identical to its Taylor series (i.e. complex numbers' different functions) (i.e. the holomorphic function). Analytic number theory makes extensive use of complex analysis, particularly the theory of conformal mappings, which has several practical applications. Assuming that  $W = f(z)$  is a two-sided single-valued continuous function in region  $D$ , and  $Z_0$  is any point within  $D$ , if any two simple smooth curves  $C_1$  and  $C_2$  over  $Z_0$  are mapped by the function  $W = f(z)$  like two simple smooth curves  $L_1$ ,  $L_2$  and  $C_1$ ,  $C_2$  in  $Z_0$  are the same size and direction as  $L_1$ ,  $L_2$  in  $W_0$ , then  $W = f(z)$  is said to be the angle-preserving map of region  $D$ . It has the invariance of the rotation angle and the invariance of the retention angle and the invariance of the elongation. If a bidirectional single-valued map of region  $D$  to  $G$  maps the major linear portion of  $W = f(z) = u(x, y) + iv(x, y)$  in the neighborhood of any point in  $D$  can be represented as either the first type of orthogonal transformation or bit-like transformation or similar transformation [5]. With the help of complex dynamics and the images of fractals created by iterating holomorphic functions, it has recently gained a lot of popularity.

The study of conformal invariants in quantum field theory in string theory is a significant application of complex analysis [6]. Additionally, sections of physics, such as fluid mechanics, thermodynamics, and particularly quantum mechanics, have been broadened.

Complex Functions are also closely related to everyday life, and two of the most intimate aspects are: the current aspect and the signal aspect. The current in daily life is AC three-phase, and the phase is more complex and abstract if calculated through trigonometric functions, many engineering problems cannot be solved, but the introduction of imaginary numbers greatly simplifies the process of calculation, so that many engineering problems are solved. Phone calls and text messages are transmitted through electromagnetic waves. The complex functions are also greatly used in the signal. Signal analysis and other fields using plural, it can easily represent periodic signals.

The Cauchy's integral theorem and Cauchy's integral formula are both generalized in the remainder theorem. It happens while attempting to use elementary calculus to determine the eigenfunctions of the Cauchy distribution. People depict this integral as the limit of a route integral that travels from point  $a$  to point  $-a$  in a real straight line, then in the opposite direction around a semicircle centered at point  $a$ . Assume that  $n$  is bigger than 1, enclosing the hypothetical unit  $I$  inside the curve. Analytical methods are generalized to include complex variables in addition to real variables in complex analysis [7]. The possibility of universal solutions in algebraic equations is where complex numbers first appear.  $A$  is referred to as the complex number's real portion,  $b$  is its imaginary part, and  $I$  is its imaginary unit, the root number  $-1$ . A specific category of complex functions is called analytical functions. The mathematical world has acknowledged its central theorem, the Cauchy-Riemann equation system, as being inseparable for more than 200 years. Although the analytical function has a fairly full theory and has been used in many contexts, Wang Jianding discovered that there are few phenomena in nature that can satisfy the constraints of the "Cauchy-Riemann" equation system, which limits the use of analytical functions.

## 2.6. Cauchy-Riemann Condition

Cauchy-riemann equations are two partial equations that make functions to be holomorphic in the open set. Geometrically, such a matrix is always a composite of a rotation and a scaling, and thus keeping the angles constant. Thus, functions with nonzero derivatives satisfying the Cauchy-Riemann equation keep the Angle of the plane curve constant. That is, the Cauchy-Riemann equation is the condition for a function to be a conformal mapping.

The Cauchy–Riemann equations of  $u(x, y)$  on a pair of real-valued functions  $u(x, y)$  and  $v(x, y)$  include two equations:

$$(1) \quad \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$$

$$(2) \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

The Cauchy–Riemann equation is a necessary condition for a function to be differentiable at one point.

If the function  $f(z) = u(x, y) + iv(x, y)$  is defined in the region  $D$ , then it is sufficiently necessary to resolve in  $D$  as:

(1)  $u(x, y)$  and  $v(x, y)$  are differentiable everywhere in  $D$ ;

(2)  $u(x, y)$  and  $v(x, y)$  satisfy the first-order system of partial differential equations  $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ ,  $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$  everywhere in  $D$  [8].

## 2.7. History of Cauchy-Riemann Condition

The Cauchy–Riemann condition (i.e. the Cauchy–Riemann differential equation) in complex analysis is two partial differential equations that provide the filling conditions for the totality of differentiable functions in open sets, named after Cauchy and Riemann [9]. This system of equations first appeared in D'Alembert's writings. Later Euler linked this system of equations to analytic

functions. Cauchy then used these equations to construct his theory of functions. Riemann's thesis on this function theory was published in 1851.

## 2.8. Proof of Cauchy-Riemann Condition

Let the function be defined within the region D, and at one point within D derivable (or differentiable), and thus

$$f(z + \Delta z) - f(z) = f'(z) \Delta z + \rho(\Delta z)$$

$$\rho(\Delta z) \text{ satisfies } \lim_{\Delta z \rightarrow 0} \frac{\rho(\Delta z)}{\Delta z} = 0.$$

Thereinto

$$\Delta z = \Delta x + i\Delta y, f(z + \Delta z) - f(z) = \Delta u + i\Delta v, f'(z) = a + bi, \rho(\Delta z) = \rho_1(\Delta z) + i\rho_2(\Delta z).$$

Then

$$\Delta u = u(x + \Delta x, y + \Delta y) - u(x, y), \Delta v = v(x + \Delta x, y + \Delta y) - v(x, y).$$

Substitution

$$f(z + \Delta z) - f(z) = f'(z) \Delta z + \rho(\Delta z)$$

And compare the real part and the virtual part

$$\Delta u = a\Delta x - b\Delta y + \rho_1(\Delta z)$$

$$\Delta v = b\Delta x + a\Delta y + \rho_2(\Delta z) \quad (|\Delta z| = \sqrt{\Delta x^2 + \Delta y^2})$$

$$\text{because } \left| \frac{\rho_k(\Delta z)}{\Delta z} \right| \leq \left| \frac{\rho(\Delta z)}{\Delta z} \right| \rightarrow 0 \quad (\Delta z \rightarrow 0) \quad (k = 1, 2),$$

Thus and in point differentiable, and established

$$a = \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, -b = \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \quad [10]$$

## 2.9. Example of Complex Functions

Define  $f(z) = \frac{az^3 + bz^2 + cz + d}{z^4 - 1}$  with  $a=10, b=1+i, c=-4, d=1-i$ ,

evaluate the integrals  $\int_{\gamma} f(z) dz$

where (a)  $\gamma(t) = \frac{1-i}{2} + \sqrt{2}e^{it}, 0 \leq t \leq 2\pi$

(b)  $\gamma(t) = 2e^{it}, 0 \leq t \leq 2\pi$

(c)  $\gamma(t) = 1 + e^{it}, 0 \leq t \leq 2\pi$

(a) with two singular points :1 and -i

$$\int_{\gamma} f(z) dz = 2\pi i \text{Res}(f, 1) + 2\pi i \text{Res}(f, -i)$$

$$\text{Res}(f, 1) = \left( \frac{az^3 + bz^2 + cz + d}{z^4 - 1} \right) (z - 1) \Big|_{z=1}$$

$$= \frac{az^3 + bz^2 + cz + d}{(z^2 + 1)(z + 1)} \Big|_{z=1}$$

$$= \frac{a + b + c + d}{4}$$

$$= \frac{10 + 1 + i - 4 + 1 - i}{4}$$

$$= 2$$

$$\text{Res}(f, -i) = \left( \frac{az^3 + bz^2 + cz + d}{z^4 - 1} \right) (z + i) \Big|_{z=-i}$$

$$= \frac{az^3 + bz^2 + cz + d}{(z^2 - 1)(z - i)} \Big|_{z=-i}$$

$$= \frac{ai - b - ci + d}{(-2)(-2i)}$$

$$= \frac{10i - 1 - i + 4i + 1 - i}{4i}$$

$$= \frac{14i - 2i}{4i}$$

$$= 3$$

Therefore,  $\int_{\gamma} f(z) dz = 10\pi i$

(b) with four singular points :  $\pm 1$  and  $\pm i$

$$\int_{\gamma} f(z) dz = 2\pi i \text{Res}(f, 1) + 2\pi i \text{Res}(f, -1) + 2\pi i \text{Res}(f, i) + 2\pi i \text{Res}(f, -i)$$

$$\text{Res}(f, 1) = 2$$

$$\text{Res}(f, -i) = 3$$

(done the check before)

$$\text{Res}(f, -1) = \left( \frac{az^3 + bz^2 + cz + d}{z^4 - 1} \right) (z + 1) \Big|_{z=-1}$$

$$= \frac{az^3 + bz^2 + cz + d}{(z^2 + 1)(z - 1)} \Big|_{z=-1}$$

$$= \frac{-a + b - c + d}{-2}$$

$$= \frac{-10 + 1 + i + 4 + 1 - i}{-4}$$

$$= 1$$

$$\text{Res}(f, i) = \left( \frac{az^3 + bz^2 + cz + d}{z^4 - 1} \right) (z - i) \Big|_{z=i}$$

$$= \frac{az^3 + bz^2 + cz + d}{(z^2 - 1)(z + i)} \Big|_{z=i}$$

$$= \frac{-ai - b + ci + d}{(-2)(2i)}$$

$$= \frac{-16i}{-4i}$$

$$= 4$$

Therefore,  $\int_{\gamma} f(z) dz = 2\pi i(1 + 2 + 3 + 4) = 20\pi i$

(c)  $\int_{\gamma} f(z) dz = 2\pi i \text{Res}(z, 1)$

$$\text{Res}(f, 1) = \left( \frac{az^3 + bz^2 + cz + d}{z^4 - 1} \right) (z - 1) \Big|_{z=1}$$

$$= \frac{az^3 + bz^2 + cz + d}{(z^2 + 1)(z + 1)} \Big|_{z=1}$$

$$= \frac{a + b + c + d}{4}$$

$$= \frac{10 + 1 + i - 4 + 1 - i}{4}$$

$$= 2$$

Therefore,  $\int_{\gamma} f(z) dz = 4\pi i$ .

## 2.10. Conformal mapping

The image implemented by the analytical function that is not zero everywhere in the derivative is a conformal image, and the conformal image is also called the angle-preserving transformation. A complex function can provide a geometric description of its properties through conformal image theory. Conformal images have been widely used in fluid mechanics, aerodynamics, elastic theory, electrostatic fields, circuit theory [11].

## 3. Research implications

This section introduces the concepts and social impact. The theory of Complex functions and Residual theorem is widely used and many complex calculations can be used to solve. For example, when designing an aircraft, Zhukovsky of Russia used complex function theory to solve the structural problem of aircraft wings. He also made contributions to solve problems in fluid mechanics and aviation mechanics by using complex function theory [12].

These functions and theorems are not only widely used in other disciplines, but also in many branches of mathematics. It deeply involves differential equations, integral equations, probability theory and number theory, and has a great impact on their development.

#### 4. Research progress

Now talking about earlier about this article. Complex function is a function takes complex number as independent variable and dependent variable. For example, let  $G$  is a set of points in the complex plane, there are one or more definite  $W$  correspond to any point that  $z$  in  $G$ . This is said to define complex function on  $G$ , so that  $W = f(z)$ . Many theorems and methods have been summarized in the study of complex functions. But each of these theorems and methods has its own very clear scope of application and method of use. And also know that for the Residue Theorem, residue is introduced to use easier way to calculate the closed curve integral. The definition is that let  $z_0$  is an isolated singularity of the analytic function  $F(z)$ , and the coefficient  $C_{-1}$  of the negative first power term in the Laurent expansion of  $F(z)$  at  $z_0$ . So this is called the residue of  $F(z)$  at  $z_0$ , as  $\text{Res}[f(z), z_0]$ .

Before this section there are some demonstrations and conclusions about the Complex Function and Residue Theorem but also Cauchy-Riemann condition. Here are some conclusions for these three.

The first result is the complex Function:  $e^{ix} = \cos x + i \sin x$ , the second conclusion is the Residue Theorem: when  $\int_{-\infty}^{\infty} \frac{e^{itx}}{x^2 + 1} dx$ , so  $\int_{-\infty}^{\infty} \frac{e^{itz}}{z^2 + 1} dz = \pi e^{-|t|}$ . and the last one is the Cauchy-Riemann condition:  $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ ,  $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$ .

The main research object of complex variable functions is analytical functions, including single-valued functions, multivalued functions and geometric theory. In the long historical process, through the efforts of many scholars, the complex variable function theory has achieved great development, and some specialized research fields have been formed.

#### 5. Insufficient previous studies and the introduction of our concepts

There are some shortcomings in the previous studies: first, the form is too complex. To be specific, first of all, there are many formulas, which are easy to confuse the readers. Secondly, the content is complicated, difficult to understand, and boring, so the readers will not want to read any more. Second, there is a deviation in the center of the article [13]. This article should talk about the concept and proof of complex function and residue theorem, but I have been talking about mathematicians related to public signs, so it will deviate from the topic and make readers feel strange.

The topic of this time is about the concept of complex function and residue theorem, and the explanation and proof of some related formulas, followed by the introduction of some mathematicians. In this paper, there are explanations and proofs of formulas, such as Euler formula, Cauchy-Riemann condition and so on. This article starts with explaining formulas and concepts, proving formulas related to concepts and introducing the sequence and methods of mathematicians related to formulas. The purpose of this article is to let readers understand the concepts and connotations of complex functions and residue theorems and this passage also want to help the readers who don't understand the complex function and residue theorem.

#### 6. Conclusion

This passage mainly talking about the introduction of the complex functions and residue theorem, the proof of formula related to complex functions, residue theorem and some research progress. In the future will explore the complex functions and residue theorem deeper and also will talk about more about complex functions and residue theorem.

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