The Application of Complex Variables Function and Residue Theorem

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Abstract. The history of complex function can be dated back to about 18 century when Euler came up with two equations derived from integrals of complex function. Over the age, mathematicians have been trying to explore and discuss more in this mysterious field, and this new branch gradually became prevalent in the 19 century mathematics like when calculus ruled the 18 century mathematics. In particular, the concept of residue was invented and became an important part in complex function, being applied in some special types of real integral. Also it is a generalization of Cauchy integral theorem and Cauchy integral formula. Nowadays, the area of complex function is filled with crystallization of the wisdom of countless scholars and researchers, and the precious mathematical treasure can also meet the needs of other academic areas like physics and biology. Summarizing the various approaches and examples of application in fields of mathematics and others can be a valuable topic. In this paper, the history of application and some examples for references of complex functions, and introduce the main concepts and formulas about them will be briefly discussed, including the area of complex numbers, analytic function, and residue. There are some relationships between some of the applications to be dig out and considerable real-world problems, and they are supposed to the generalization of those subjects can provide certain enlightenment to people interested in or during the study in fields of complex function.

Keywords: Complex function; analytic function; residue.

1. Introduction

Complex variables function and residue theorem are important extensions when it comes to complex numbers. The theory of complex variables functions can include single-valued analytic theory, Riemann surface theory, and geometric function theory, which are all significant tools in investigating fields of complex number and others. For example, Biao (2022) [1] uses possible conic singularities to study the curve shortening flow (CSF) based on Riemann surface, and the geometric function theory is used in quaternionic slice domains (Graziano& Caterina, 2020) [2]. In particular, residue theorem makes computing the integral of complex function more convenient than using the line integral. The definite integral of a real function can be transformed into the integral of a complex function along the curve of a closed loop, and then the fundamental theorem of residues can be used to calculate the residue of the integrated function on the isolated singularity inside the curve of a closed loop. Some application of this theorem can be made like extending characteristics of residue fields to function fields of conic. Furthermore, the residue theorem can be applied in other academic fields, like employing the frequency averaged power fed into a linear second-order system during design optimization using residue theorem (Kunmo et al. 2014) [3, 4]. The paper provides concepts and formulas of primary theorems in the field of complex functions, the aim of which is to summarize present methods and process in solving mathematics problems or applying in real-world situation of complex functions and residue theorems. Thus, it would be meaningful to explore the meaningful applications of complex functions in particular conditions and many other fields can be a valuable topic.

The theory of complex functions emerged in the 18th century. In 1774 Euler, in one of his papers, considered two equations derived from integrals of complex functions. The French mathematician D 'Alembert had obtained them earlier in his treatise on fluid mechanics.

In the eighteenth century, the emerging field of complex functions dominated mathematics. The theory of complex functions was dubbed the mathematical delight of the century by mathematicians of the time, who recognized it as the most fecund branch of mathematics. It has been recognized as one of abstract science's most aesthetically pleasing hypotheses.

2. Method

2.1. Complex Function

The base of the complex function is the complex number. The concept of the complex number are defined as a component of a number system that adds a certain element i, which is called the imaginary unit and satisfying the equation $i^2 = -1$, to the real numbers. The real number x is defined as one of element in the complex number, which is considered as the real part of the complex number z, denoted $Re\, z = x$. While the real number y is defined as the element of the imaginary part of the complex number z, denoted $Im\, z = y$. So, the complete imaginary part of a complex number z is defined as $i \cdot y = iy$. Combining the real part and the imaginary part, the complex number z appears in the form of $z = x + iy$ in many different versions of textbooks that introduce the concept of complex functions. Additionally, complex numbers have another way to be expressed, that is $z = (x,\ y)$. The basic logic of this type of expression is put the complex number z into a rectangle coordinate system. In the rectangle coordinate system, the value of real part of the complex numbers will be considered as the value on the x-axis and the value of imaginary part of the complex numbers will be considered as the value on the y-axis. Therefore, the complex number z will become a coordinate value, which is $z = (x, y)$.

In addition, by taking the Euler formula in the complex numbers, which is $e^{i\theta} = (cos\theta + isin\theta)$, there will be another way to express a complex number. Switching the element of the real part and imaginary part of a complex number into the form of trigonometry expression will result a form of $z = r(cos\theta + isin\theta)$, r is a real number. Thus, another way to express a complex number will be $z = re^{i\theta}$.

Some of the same laws that apply to the basic four operations and functions, and even calculus, can be used in complex numbers as well, such as addition, subtraction, multiplication and division. For example, defining $z_1 = x_1 + iy_1$and $z_2 = x_2 + iy_2$ and combining these two complex numbers by addition can result:

$$z_1 + z_2 = x_1 + iy_1 + x_2 + iy_2 = x_1 + x_2 + i(y_1 + y_2) \tag{1}$$

Based on the properties above, some further algebraic properties could be discussed. For example, since the existence of multiplicative inverses, if product $z_1 z_2$ is zero, at least one of the factors of $z_1$or z_2 is zero, which is a simple concept among the area of pure mathematics

This means if $z_1 \cdot z_2 = 0$, either $z_1 = 0$ or $z_2 = 0$, or both of z_1 and z_2 are zero.

Furthermore, when it is defined as $z_1 = (x_1, y_1)$ and $z_2 = (x_2, y_2)$, the result by using subtraction with them will be:

$$z_1 - z_2 = (x_1, y_1) - (x_2,\ y_2) = (x_1 - x_2, y_1 - y_2) \tag{2}$$

Finally, binomial formula in the area of the complex numbers could be applied :

$$(z_1 + z_2)^n = \sum_{k=0}^{n} \binom{n}{k} {z_1}^k \cdot {z_2}^{n-k} \tag{3}$$

where $\binom{n}{k} = \frac{n!}{k!(n!-k!)}$, $k = 0, 1, 2, \dots, n$

Notably, complex numbers can also be applied to other fields like physics. For instance, by proposing a procedure for merging various strains by utilizing a complex number system, simple load states can be converted into an equation for a comparable strain matching the Macha-previously established criteria. Then a strained fatigue criterion is formed. (Lagoda et al. 2021) [4]. In addition, for obtaining qualitative electroanalytical methods in the means of electronic noses, researchers use electrochemical impedance spectroscopy (EIS) to get complex number data, which has higher predictive ability than real numbers. (Dayvison et al. 2021) [5].

After understanding the concept of the complex numbers, the complex functions could be introduced. In mathematics, a function from a set X to a set Y divide among each element of X exactly one element of Y. In the easiest concept of functions, X can be just a real number like $1, \frac{2}{3}, \pi$,etc. Therefore, in the concept of the complex functions, X is a set which includes both real numbers and imaginary numbers. (When the element of the imaginary part of a complex number z equals to zero, z would be a real number). Besides, the so-called set Y in the complex functions is also different to that in the basic functions. Each element of X could be assigned to several different elements of Y. Back to the definition of the complex functions, let S be a set of complex numbers that is corresponding to the set X in the basic functions. Elements in so-called set Y in the complex functions would be w, denoted by $f(z)$. So that $w = f(z)$, and the set S would be the domain of definition of f.

2.2. Analytic function

After the introduction for the complex functions, it would be analytic function, which is a complex function differentiable everywhere on a region K. Weierstrass called the sum of function of a power series converging on a disk an analytic function, and the analytic function of a region refers to the function that is displayed as the sum of power series on every small circle neighbourhood in the region. Different definitions of analytic functions were shown to be equivalent in the early 20th century. Based on definition from Weierstrass, an analytic function on a region can be regarded as an analytic expansion of a power series on a neighbourhood of any circle within it. In particular, let f be defined at every points z in several removed neighborhood of a point z_0, then $f(z)$ has a limit w_0 as + approaches z_0. If f has a constant value w_0, the image of z is always the centre of the neighborhood. Then a central theorem about uniqueness of limits is shown: when a limit of $f(z)$ exists at a point z_0, it is the sole one. The explorations of analytic function like this are numerous, but this theorem is only taken as an example as it used to be an important branch that is widely used in many theoretical and practical problems.

Deeper consideration about analytic function can be done. For example, Ref [6] obtained theorems of multidimensional space of analytic function. The order of imaginary section of non-trivial zeros of the Riemann zeta-function is contained in the used shifts of periodic zeta-functions. Moreover, associated with pascal distribution series, which is a discrete probability distribution in statistics often used to describe biome clustering and the distribution of infectious in medicine, researchers find necessary and sufficient conditions for the sequences in the classes Wδ (α, γ, β) of analytic function, see Frasin et al [7].

In particular, two partial differential equations that are named after Cauchy and Riemann are provided for the sufficient and necessary conditions for differentiable functions to be holomorphic in the open set, which is the Cauchy-Riemann Equation. This system of equations emerged firstly in the work from D 'Alembert. Then Euler linked this system of equations to analytic functions. After that Cauchy used them to organize his theory of functions. The paper from Riemann about this function showed up in 1851. The content of Cauchy-Riemann Equation is shown below:

$u(x, y)$ in a sequence of functions of real values $u(x, y)$ and $v(x, y)$ include two equations:

$$u_x = v_y \text{ (1)}, \; u_y = -v_x \tag{4}$$

In general, Cauchy-Riemann Equation suggests that if $f(z) = u(x,\ y) + iv(x,\ y)$ and $f'(z)$ exists at a point $z_0 = x_0 + iy_0$, then the derivatives of first order part of u and v must appear at $(x_0,\ y_0)$, and they must satisfy the Cauchy–Riemann equations $u_x = v_y,\ u_y = -v_x$, which are the same equation as the two equations that are mentioned above. Also, $f'(z_0)$ can be written as $f'(z_0)$=$u_x + iv_x$ at the point that are evaluated at these partial derivatives , which is point $(x_0,\ y_0)$.

In details, this Cauchy-Riemann Equation could be proved by starting with the assumption that $f'(z_0)$ exists:

$$z_0 = x_0 + i \cdot y_0 , (1) \quad \Delta z = \Delta x + i \cdot \Delta y \tag{5}$$

$$\Delta w = f(z_0 + \Delta z) - f(z_0) \tag{6}$$

Then, taking two formulas above into the formula below:

$$\Delta w = f(x_0 + i \cdot y_0 + \Delta x + i \cdot \Delta y) - f(x_0 + i \cdot y_0) \tag{7}$$

Also,

$$f(z) = u(x,y) + iv(x,y) \tag{8}$$

The change in value w can be expressed in the form of:

$$\Delta w = [u(x_0 + \Delta x, y_0 + \Delta y) + iv(x_0 + \Delta x, y_0 + \Delta y)] - [u(x_0, y_0) + iv(x_0, y_0)]. \tag{9}$$

Then, the equation of derivative function could be written:

$$\frac{\Delta w}{\Delta z} = \frac{u\,(x_0 + \Delta x, y_0 + \Delta y) - u\,(x_0, y_0)}{\Delta x + i \cdot \Delta y} + i \cdot \frac{v(x_0 + \Delta x, y_0 + \Delta y) - v(x_0, y_0)}{\Delta x + i \cdot \Delta y} \tag{10}$$

According to the nature of differentiation, this expression should remain valid as $(\Delta x, \Delta y)$ tends to (0, 0) in any different manners that are chosen to be proved it. In particular, when the Horizontal approach is chosen to evaluate the function, that is when $\Delta y = 0$. Also, $(\Delta x, 0)$ are required to tend to be $(0, 0)$. Taking these values in to the equation will result:

$$\frac{\Delta w}{\Delta z} = \lim_{\Delta x = 0} \frac{u\,(x_0 + \Delta x, y_0\,) - u\,(x_0, y_0)}{\Delta x} + i \cdot \lim_{\Delta x = 0} \frac{v(x_0 + \Delta x, y_0\,) - v(x_0, y_0)}{\Delta x} \tag{11}$$

That is:

$$f'(z_0) = u_x(x_0,\ y_0) + iv_x(x_0,\ y_0) \tag{12}$$

Then, the function could be evaluated again through the vertical approach. That is when $\Delta x = 0$ and (0, Δy) tends to be (0, 0). Taking these values in to the equation will result:

$$\frac{\Delta w}{\Delta z} = \lim_{\Delta y = 0} \frac{u\,(x_0\,, y_0 + \Delta y\,) - u\,(x_0, y_0)}{i\Delta y} + i \cdot \lim_{\Delta y = 0} \frac{v(x_0, y_0 + \Delta y\,) - v(x_0, y_0)}{i\Delta y} \tag{13}$$

Because $\frac{1}{i} = -i$, the equation can be also written as:

$$\frac{\Delta w}{\Delta z} = \lim_{\Delta y = 0} \frac{v\,(x_0\,, y_0 + \Delta y\,) - v\,(x_0, y_0)}{\Delta y} - i \cdot \lim_{\Delta y = 0} \frac{u(x_0, y_0 + \Delta y\,) - u(x_0, y_0)}{\Delta y} \tag{14}$$

Now, the equation comes to:

$$f'(z_0) = v_y(x_0,\ y_0) - iu_y(x_0,\ y_0) \tag{15}$$

The two final equations not only give $f'(z_0)$ by partial differentiating the component function u and v, but also show some indispensable condition to allow $f'(z_0)$ exists. By connecting these two expressions, which is that equating the parts of real values and the parts of imaginary value in these two expressions, these necessary conditions will appear. That is:

$$u_x(x_0,\ y_0) = v_y(x_0,\ y_0) \tag{16}$$

And

$$iv_x(x_0,\ y_0) = -iu_y(x_0,\ y_0) \tag{17}$$

$$v_x(x_0,\ y_0) = -u_y(x_0,\ y_0) \tag{18}$$

The final two equations are the Cauchy-Riemann equations, which is named in remembrance of the mathematician A. L. Cauchy (1789-1857), who discovered the two important conditions for a partial derivative in the complex functions to exist and use them. Also, named in honor of the German mathematician G. F. B. Riemann (1826-1866), the one that made them important in developing the theory of a complex variable's function.

In conclusion for Cauchy-Riemann equation, that is :If the function can be defined in region D, then the necessary and sufficient conditions for its analysis in D are:

(1) $u\ (x, y)$ and $v\ (x, y)$ are differentiable everywhere in region D

(2) $u\ (x, y)$ and $v\ (x, y)$ satisfy the first order partial differential equations above everywhere in region D.

The equation can be explored more deeply with differential function in complex plane. For example, Pashaei et al. (2020) [8] derive fractional Cauchy-Riemann equations with respect to multi-valued functions, which in case of yield classical Cauchy-Riemann equations. Then talk about two difficult conformable differential equations and their Riemann surface solutions.

Moreover, mathematicians are still coming up with new ideas based on analytic function. Take the semi-analytic function theory as an example, which was initially proposed by Professor Wang Jianding in 1983 [9], and the conjugate analytic function theory was first proposed and rigorously verified by him in 1988. Electric fields have seen the successful application of the two theories. fields of magnetism, dynamics of fluids, elasticity as well as other areas. Many professionals have accepted both hypotheses. Double analytical functions are created as a result of scholarly development and citation. A number of new disciplines of mathematics, including differential and integral equations, like double-half analytic functions and analytical semi-conjugate functions and related boundary value issues. In general, semi-analytic function serves to provide a method for the study of common complex functions which cannot be solved by analytic functions. The content of semi-analytic function is shown below:

If the function $f\ (z)\ =\ u\ (x, y)\ +\ iv\ (x, y)$ is defined in the neighborhood of a point z (x, y) and has a first order continuous partial derivative:

$$u_x\ = v_y,\ u_y\ = -v_x$$

then f (z) is the first kind of complete semi-analytic function.

(1)

If the function $f\ (z)\ =\ u\ (x, y)\ +\ iv\ (x, y)$ is defined in the neighborhood of a point $z\ (x, y)$ and has a first order continuous partial derivative:

$$u_x\ \neq v_y,\ u_y\ = -v_x$$

then $f\ (z)$ is the second kind of complete semi-analytic function.

2.3. Residue

Residue--The integral of an analytic function along any forward simple closed curve enclosing an isolated singularity in a ring domain divided by 2πi.

Residue theorem: the integral problem along a closed circuit can be resolved using the residue theorem by computing the residue at an isolated point. In addition, some of the anti-derivative functions of the integrand in mathematical analysis and real-world issues cannot be stated by elementary functions or cannot be expressed by complex elementary functions. Such integrals can be determined using the residue theorem.

Let us say that the contour is straightforward and closed. If a function f has a finite number of singular points but is otherwise analytic inside and outs

$$zk\ (k = 1, 2 ... n)\ \text{inside}\ C.\ \oint f(z)dz = 2\pi i \sum Res(f(z), ak)$$

3. Results

Through exploring and concluding various ways of applying different theories in the fields of complex function, the versatility and practicability of this function could be found. Numerous complex calculations are resolved by the theory of complex functions, which has a wide range of applications. For instance, physics has a wide variety of stable planar fields. For each point pair, the so-called field is a region of physical quantities. Complex functions are used to calculate these fields. Russian aviation designer Zhukovsky, for instance, applied theories to address the structural issues with airplane wings. He also contributed to the use of theory about complex functions to address issues in fluid mechanics and aviation mechanics. In addition, the complex function can be used in more basic academic tools. For example, The development of a chaotic particle optimization method for complicated functions aims to improve the efficiency of the Particle Swarm Optimization (PSO) algorithm in addressing continuous function optimization problems [10].

4. Conclusions

In conclusion, Residue theorem, which is also called Cauchy's Residue Theorem, can evaluate line integrals of analytics function over a closed curve in a very useful and easy way. What is more, the Residue Theorem is closely connect to the Laurent Expansion, Cauchy-Goursat Theorem, etc, which not only has made significant contribution in the pure mathematics area, but also has been applied to the study of engineering.

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