Solving Real Integrals Using the Residue Theorem

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Abstract. Previously, it is difficult or even impossible to tackle some real integrals that may not be calculating by classic methods in calculus. However, this paper would summarize methods using the Residue theorem to calculate some types of these “impossible integrals”. However, before one calculates the integral using the residue theorem, one will introduce two important conclusions, one is about when the function decays faster than $\frac{1}{z}$, which involve integral inequality and the method finding the upper bound. The second one would give reader a general conclusion of integration. At the same time, the readers have to understand the proof of the Cauchy’s Residue Theorem, and other theorems such as Cauchy’s theorem, Cauchy’s Integral formula. One should notice that the importance of branch cut and branch points, since the integrand one face could be multifunction, so one may try to turn them into single valued function. Also, the integral inequality plays a very important role in the first conclusion.

Keywords: The Residue Theorem Residue Integral inequality.

1. Introduction

The Residue theorem, as known as The Cauchy’s residue theorem. The ultimate goal of it is to find the area under the given curve which is the integral of that function. However, it is sometimes hard to solve the integral in Calculus. For this reason, people transform this function into the complex plane and using the Residue theorem since it gives us simple way to find integral instead of using the traditional way. It is vital to understand the core of the residue theorem. Given that a complex number is made of two variables which it is hard to find out the area under the curve. In fact, the integration in complex variable is along a contour. Sometimes it is not easy just to use the Residue theorem such as certain integration of rational functions, trigonometric integral, improper integral, logarithmic integral and sine and cosine integral. In this case, one would use other ‘tricks’ to compute the integral like triangle inequality and branch cut [1].

Reference [2] gives us some ideas about how to deal with real integrals which are hard to tackle. It uses the Residue Theorem as well in order to make calculation possible. There are several steps. First, one should find out the appropriate integrand. Next, one may try to define the simple closed contour which contain the branch cut. Finally, one mainly used the residue theorem with other methods to compute the integral. If there is a semicircle on the upper plane, one may assume $\left|f(z)\right|\cdot|z|<M$, which is the upper bound of the function, one may have $\lim_{R\to\infty}\int_{C_R} f(z)\,dz=0$. Subsequently, one may have different methods in two cases. One is when the function decays faster than $\frac{1}{z}$ and the other one is the function decays as fast as $\frac{1}{z}$. In fact, the core method is trying to connect the integral with triangle inequality and modulus of complex number. In the first case, one may try to use integral inequality and get when $R\to\infty$, $\lim_{R\to\infty}\int_{C_R} f(z)\,dz=0$. This is followed by using the Residue theorem [2].

When the function $f(z)$ decays as fast as $\frac{1}{z}$, one may assume $C_1+C_2+C_3$ is the rectangular path in the upper half-plane. $C_1:\gamma=x_1+it$, $C_2:\gamma=t+i(x_1+x_2)$, $C_3:\gamma=-x_2+it$, but this time one may have $e^{i\alpha z}$ in the integral. The domain is from 0 to $x_1 + x_2$. Obviously, for C_1 $Me^{-\alpha(x_1+x_2)}$ goes to zero as $x_1 + x_2$ go to ∞. There is a special case that the function is in the form of $\frac{1}{\left(1+x^n\right)^n}$. It seems that one cannot use our two theorems in [2]. However, when x^n becomes larger and larger, one may know that $x^n \approx x^n + 1$, which this form could be connected to our theorem. Notice that the method for getting the residues is different sometimes. Our first method is to calculate the Laurent series of $f(z)$ then find b_1. Next method is for finding out the simple poles and the formula is $\operatorname{Re} s(z_0) = \lim_{z \to z_0} (z - z_0) f(z)$. The third method is a little different compare with method 2. The method 3 is for poles who have order larger than 1 using $\frac{1}{(n-1)!} \lim_{z \to z_0} \frac{\partial^{n-1}}{dz^{n-1}} ((z - z_0) f(z))$. one may use method 3 but it could become complicated when the order is too big. It is also a problem happened in our example. In example 5.9 of [3], cosine makes it impossible to use the Theorem, so one may just replace it with e^{ix}, The section 9.5 is what we mainly discuss in our paper. It is noticeable that the integral of C_1 and C_2 is totally different from each other. They are on each side of the branch cut and their arguments are different, so one should calculate their integral respectively. Cauchy's principal value is explained in 9.6, one think it is important to know the function is whether convergent or not. One can turn sine and cosine in the form of natural logarithm. The reason is that they are periodic function which goes to infinity in both upper and lower plane which means our two theorems could not be satisfied. Generally speaking, one would replace the integrand by $\frac{e^{ix}}{x}$. However, it is not convergent and has pole of naught. For these reasons, if the function $f(\mathrm{z})$ is continuous on the real line except at x_1, the Cauchy's principal value could be represented in the sum of two integral. The upper bound and lower bound is -R and $x_1 - r_1$, or -R to $x_1 + r_1$, as r_1 goes to zero. This method is working as well, if there are multiple values of discontinuities. We can expand the theorem when the integral is converges, the Cauchy's principal value is also convergent. All the example we discuss above are closed curve instead a portion of circles. z_0 is a pole of $f(z)$. Since it is a semicircle, we could think to build connection between it and its Laurent expansion. The is that we can, since goes to zero [3].

For [4], one may mainly discuss improper integral in our paper, one does have ways to tackle this integral, but as the paper mentioned before, these methods are usually complicated than ways in Complex Variable. In the process, one may use the form of fractional. One may assume the radius of the contour is large enough so that all the residues are in it. $\int_0^\infty f(z)\,dz$ is a special case, the thing one may immediately do is to turn this into $\frac{1}{2}\int_{-\infty}^{\infty} f(z)\,dz$ if $f(z)$ is an even function. For sine and cosine integral, one may use Taylor Series for sine, cosine or natural logarithm [4].

Integral inequality plays a very important role in solving the integral. The theorem that are concluding in [2] is supported by this, since they involve finding the upper bound of the integrand. But before going to the proof of the upper bound, there is a lemma says that $\left|\int_a^b w(t)\,dt\right| \le \int_a^b \left|w(t)\,dt\right|$. It has a hidden property when the left side of the inequality is zero, since there is modulus on the right side, one can get the integral clearly. One may assume the value is a nonzero complex number $r_0 e^{i\theta_0}$, and try to write r_0 in the form of integration. Because r_0 is a real number, so it equals to real part of that integral. Moving on to the proof of the ML inequality. We suppose C denote the contour of length L, and if M is a nonnegative constant which is larger than modulus of $f(z)$, then $\left|\int_C f(z)dz\right| < ML$, and one can use the lemma and get $\left|\int_C f(z)dz\right| < M\int_a^b \left|z'(t)\right|dt$. one can easily think that the right integral represents the length of C. It is very important to use this to proof theorem of $f(z)$ decays as fast as or faster than $\frac{1}{z}$, and try to connect it with integral inequality [5].

Before readers have a better understanding of the Cauchy's Residue theorem, one may be required to understand other related conclusion, and [6] give us a really good explanation. Definition 1 assume that D is open, define partial derivatives are continuous on D. Then, it mainly introduces the Cauchy's-Riemann differential equations, which could be effectively know whether the given function is differentiable or not. People can easily find out the derivative of real part and imaginary part of the function respectively. Then, it discusses the concept of analytic, which would be little different from differential. Analytic or holomorphic could be used if $\frac{\partial f}{\partial \bar{z}} = 0$. Next, the Cauchy's Theorem and the Cauchy's integral theorem are represented. The Cauchy's Theorem tell us if The C is a closed simple contour and analytic inside C, the integrals of all function in it are all zero. If a is a pole order of 1, and the integral of the contour encircles it could be got by The Cauchy's Integral Formula. It is interesting that if one differentiates. The original Cauchy's integral n time, e gets generalized Cauchy's Integral Formulas [6].

There are some kinds of connection between the residue theorem and Laurent Series and the Residue Theorem, the paper would discuss the condition of the function around the points where the functions are not analytic. One may know in this case one may use the Cauchy's Integral Formula, which is for the contour around the poles. But one can learn something new in this paper, if the function could not be written down in Taylor series, one may change our strategies by using Laurent series. The main idea in the paper is to view one kind of Laurent series into the form of $\frac{1}{z-z_0}$. One may assume that R represent the radius. Next, the function could be separated in two parts. The next step is one want to make sure in which case the Laurent series is converges or diverges. The coefficient of the Laurent series may not usually get by original formula of integral. Moving on to the problem of singularities. There are isolated singularities, they are points in C where a function is no analytic. One also split the isolated singularities as well, there are three types, poles, essential singularities and removable singularities. If one may want to have a better understanding of the method, like simple curve, smooth curve, Jordan contour [7].

Because the Cauchy's Residue Theorem is very important in Complex Analysis. It is very common for it to be described in every textbook about Complex Analysis, and the residues play crucial roles in Complex Variable, and residues have strong connection with the calculation of the integral. One definitely will use the Residue Theorem to handle it. The readers should notice that the function should in the form of fractional like $\frac{az^3+bz^2+cz+d}{z^4-1}$, and the values of a, b, c and d are given. Firstly, one should find out the poles of the function (1, i , -1, -i) and one would use these to find out the residues at these points (A, B, C, D). Then one would use the partial fraction. Partial fraction[8] is very important, and involve reduction of fractions to a common denominator. One may try to split one fraction into many fractions which the denominators are the poles of the function, and try to combine them together in order to get the values of a, b, c, d in the form of $aA+iB+oC+fD$ where a,i,o,f are constant. Next, one can get equations group by these conditions. However, if the equations are too complicated, one may transform our equations group into the form of matrix, and using the inverse matrix. Finally, one can find out the values of A, B, C, D. In the following one may present a few theorems.

Theorem 1. Suppose C is a simple closed contour in Figure 1. If there is a holomorphic function f inside and on C except for several singularities $z_k(k=1,2...,n)$ inside C [5].

$$\int_C f(z)dz = 2\pi i \sum_{k=1}^{n} \operatorname*{Res}_{z=z_k} f(z)$$

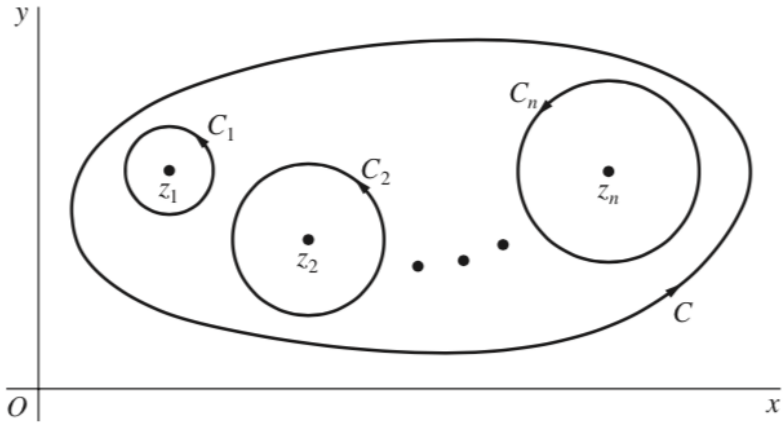

Figure 1. It shows several residues in the contour.

Theorem 2. If a complex valued function f(z) is analytic (continuous and differential) everywhere on and inside simple closed domain C, then

$\int_C f(z)dz = 0$

The reason for the value of naught is because readers always come back to the starting point in the complex variable. Furthermore, one could extend the theorem. Suppose the function is analytic everywhere inside C, except for some points inside C, people call this point singularities. And by the Laurent's Theorem, there exists a disc around these points, then one may get

$\int_z f(z)dz = \int_\gamma f(z)dz$ [5]

Definition 3. When z_0 is a singularity of function f, f is analytic at each z for $0<|z-z_o|<r$,where r is a positive number, the Laurent series is:

$$f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n + \frac{b_1}{z-z_0} + \frac{b_2}{(z-z_0)^2} + \ldots + \frac{b_n}{(z-z_0)^n} + \ldots$$

$b_1 = \frac{1}{2\pi i}\int_C f(z)dz$ Where b_1 is the residue at the point z_0 [5].

Definition 4. The set of all complex numbers $\left(\mathbb{C} = \{a+bi \mid (a,b) \in \mathbb{R}\times\mathbb{R}\}\right)$.

Definition 5. The modulus is the distance between the origin between the origin and the point $z=(a,b)$, which can be considered as the length of a vector.

Definition 6. The angle between the positive real ais and the vector created by complex number z is called the argument of a complex number, there are infinite number of arguments in the interval of 2π [9].

Definition 7. A point $z=z_0$ is named as singularity or singular points is the function is not defined at $z=z_0$.

Definition 8. Suppose $f(z)=g(z)\frac{g(z)}{(z-z_0)^n}$ where f is analytic in the punctured disc $\{z:0<|z-z_0|<R\}$ and where g is analytic n neighborhood of z_0. If n>1 then f is said to have a pole of order n at z_0 [9].

Definition 9. A many-valued relation which associates subsets of $\mathbb{C}$ with each point in some subset of $\mathbb{C}$ will be called a multifunction. A point $\alpha\in\mathbb{C}$ is a branch point of a multiple function g, defined on some open subset A of $\mathbb{C}$, if there exists at least one closed curve ξ enclosing α and lying in A, such that upon completion of a circuit of ξ for which any argument of $z-\alpha$ increase continuously by 2π , g does not return to its original chosen value. A branch cut of g is then a chosen line segment with initial point at a branch point α of g, such that there is one and only one branch cut in any neighborhood of α containing no other branch point of g. A branch is any function with the same domain as g and obtained from g by making it single-valued, which is continuous at all points of its domain except along any chosen branch cuts of g. In our thesis, readers make efforts to avoid the multi-valued function, since they may not be named as 'correct' function. The reason for the appearance of multiple values is the argument of the complex function. This is just the time people require branch cut. People usually consider the x-axis branch cut, and make the paths which go along the branch cut, and these paths are really closed to the cut [10].

Theorem 10. Suppose the function $f(z)$ decays faster than $\frac{1}{z}$, and is defined on the upper plane in which $a>1$ and $M>0$, one can say [3]

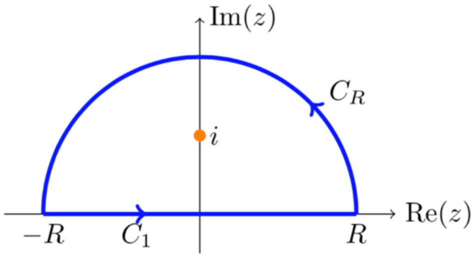

Figure 2. It explains the path of our integral.

$$f(z)<\frac{M}{|z|^a}$$

Then

$$\lim_{R\to\infty}\int_{C_R} f(z)dz=0$$

Clearly only one residue is presented in Figure 2.

Proof

$$\left|\int_{C_R} f(z)dz\right|\le\int_{C_R}|f(z)||dz|\le\int_{C_R}\frac{M}{|z|^a}|dz|=\int_0^\pi\frac{M}{\mathrm{R}^a}Rd\theta=\int_0^\pi\frac{M}{\mathrm{R}^{a-1}}d\theta$$

Since a>1 which means as $R\to\infty$, this goes to zero. Then, it is clear that $\lim_{R\to\infty}\left|\int_{C_R} f(z)dz\right|\le 0$, then one may get

$$\lim_{R\to\infty}\int_{C_R} f(z)dz=0$$

Example. See Example 1 using Theorem 10 in the main body below. This paper proceeds with a new theorem.

Theorem 11. One assumes that there is $u=f(z)=\ln(z)$ and $z=re^{i\theta}$, then one may have

$\ln z = \ln r + \mathrm{i}\left(\theta + 2\pi \mathrm{n}\right)$

Proof

$u = f\left(z\right) = \ln\left(z\right) = u + \mathrm{iv} = \ln\left(re^{i\theta}\right) \qquad z = re^{i\theta}$

$z = e^{u} = e^{u} \cdot e^{iv} = re^{i\theta}$

So: $e^{u} = r$ and $\ln r = u$

Next: $\because e^{iv} = e^{i\theta}$

$\sin v + \cos v = \sin\theta + \cos\theta$

$\therefore v = \theta + 2\pi n$

$\ln z = \ln r + \mathrm{i}\left(\theta + 2\pi \mathrm{n}\right)$

The methods described later on are special for three kinds showed below. Functions in fractional form while it decays faster than1/z. Functions in fractional form while its numerator is not 1 and $-1 < n < 0$.

2. Main body

(a)

Example 1

Compute $I = \int_{-\infty}^{\infty} \frac{1}{\left(1+x^{6}\right)^{6}} dx$

Solution: Let $f\left(z\right) = \frac{1}{\left(1+z^{6}\right)^{6}}$

When z becomes larger, the value of z^{4} is much larger then1, then

$$f\left(z\right) \approx \frac{1}{z^{36}}$$

Turns out, this function satisfies Theorem 10, and using The Residue Theorem, then one may have

$$\lim_{R\to\infty} \int_{C_R} f\left(z\right) dz = 0$$

and

$$\int_{C_1+C_R} f\left(z\right) dz = 2\pi i \cdot \text{the sum of the residues}$$

Obviously, one can figure out that

$$\int_{C_1+C_R} f\left(z\right) dz = \int_{C_R} f\left(z\right) dz + \int_{C_1} f\left(z\right) dz$$

When $R \to \infty$, equation becomes

$$\int_{C_1+C_R} f\left(z\right) dz = \lim_{R\to\infty} \int_{C_R+C_1} f\left(z\right) dz = \lim_{R\to\infty} \int_{C_1} f\left(z\right) dz = \lim_{R\to\infty} \int_{-R}^{R} f\left(z\right) dz = I$$

Next, one may find out the poles of the function, and calculate the residues. The function $f\left(z\right)$ has poles of order 6 at $\pm i$, but only $z = i$ is inside the contour.

$$\mathrm{Re}\, s\left(f\left(z\right), i\right) = \frac{1}{\left(6-1\right)!} \lim_{z\to i} \frac{\partial^{6-1}}{\partial x^{6-1}} \left(\left(z-i\right)^{6} \cdot \frac{1}{\left(1+z^{6}\right)^{6}} \right)$$

$$= \mathrm{Re}\, s\left(f\left(z\right), i\right) = \frac{1}{\left(5\right)!} \lim_{z\to i} \frac{\partial^{5}}{\partial x^{5}} \left(\left(z-i\right)^{6} \cdot \frac{1}{\left(1+z^{6}\right)^{6}} \right)$$

$$=\operatorname{Re}s\left(f\left(z\right),i\right)=\frac{1}{(5)!}\lim_{z\to i}\frac{\partial^5}{\partial x^5}\left(\left(z-i\right)^6\cdot\frac{1}{\left(\left(z^2+1\right)\left(z^4-z^2+1\right)\right)^6}\right)$$

$$=\operatorname{Re}s\left(f\left(z\right),i\right)=\frac{1}{(5)!}\lim_{z\to i}\frac{\partial^5}{\partial x^5}\left(\left(z-i\right)^6\cdot\frac{1}{\left(\left(z-i\right)\left(z+i\right)\left(z^4-z^2+1\right)\right)^6}\right)$$

$$=\operatorname{Re}s\left(f\left(z\right),i\right)=\frac{1}{(5)!}\lim_{z\to i}\frac{\partial^5}{\partial x^5}\left(\frac{1}{\left(z^6-z^4+z^2+iz^4-iz^2+i\right)^6}\right)=A$$

$$I=2\pi i\cdot A$$

(b)

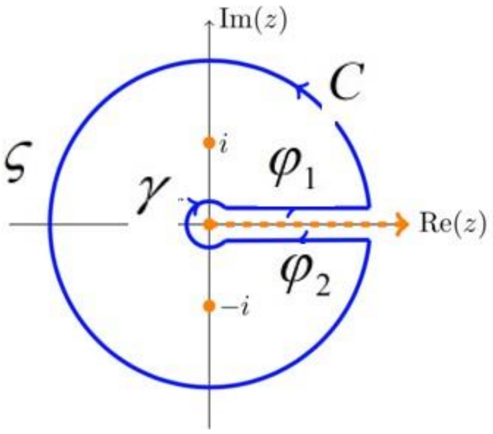

Figure. 3 It shows the four paths of the contour C

When $-1<n<0$, one may calculate

S= $\int_0^\infty \frac{x^n}{x+1}dx$ and $f\left(z\right)=\frac{x^n}{x+1}$

Solution. Our general method is suing the Residue Theorem, so first let us find out the poles of the function $f\left(z\right)$:

$a_1=1$ $a_2=0$

Although it is easy to find pole of 1, the function also has a hidden pole of naught. That's because $n<0$, so when x=0, the denominator is also naught. Next, one should define the range of the argument of z and branch cut, according to Definition 9 $f\left(z\right)$ is a multi-valued function, it actually is not a function but relation, which means one need turn it to a single-valued function. Notice that segments φ_2,φ_1 and γ are pretty closed to the branch cut. In Figure 4, one can separate the contour C in to four parts, and one would calculate the integral of each of them respective

$\arg\left(z\right)\in\left(0,2\pi\right]$ branch cut: real axis

$$\int_C f\left(z\right)dz=\int_{\varphi_1} f\left(z\right)dz+\int_{\varsigma} f\left(z\right)dz+\int_{\varphi_2} f\left(z\right)dz+\int_{\gamma} f\left(z\right)dz$$

(1) $\int_C f\left(z\right)dz$

$$\int_C f\left(z\right)dz=2\pi i\operatorname{Re}s\left(f\left(z\right)\right)=2\pi i\lim_{z\to -1}\left(z+1\right)\frac{z^n}{z+1}=2\pi ie^{i\pi n}$$

(2) One assumes that the radius of arc ς is R and the radius of γ is α. It is convinced that as φ_1 approaching branch cut, when $\alpha\to 0$, it is correct to think the length of hypotenuse and opposite side

are approaching 0 as well. As one can see in Figure 4 and Figure 5, the points on φ_1 build a triangle with the branch cut.

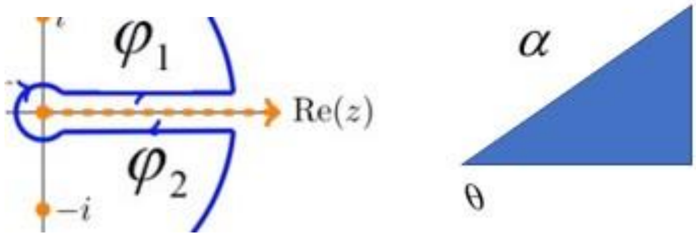

Figure 4. They show the cases in (2) and (3).

$\int_{\varphi_1} f(z)dz$

$\varphi: z = t + i\alpha$ and $t \in (\alpha, \mathbb{R})$ $\quad dz = dt$

$$\int_{\varphi_1} f(z)dz = \lim_{\mathbb{R}\to\infty\alpha\to 0} \int_\alpha^{\mathbb{R}} \frac{(t+i\alpha)^n}{1+t+i\alpha} dt = \int_0^\infty \frac{t^n}{1+t} dt = S$$

(3) $\int_\varsigma f(z)dz$

Assume that $\varsigma : z = \mathrm{Re}^{i\theta}, \theta \in [a,b]$ and $dz = i\,\mathrm{Re}^{i\theta}\,d\theta$

So $\int_\varsigma f(z)dz = \int_a^b \frac{\mathrm{R}^n \mathrm{e}^{i\theta n}}{1+\mathrm{Re}^{i\theta}} i\,\mathrm{Re}^{i\theta} d\theta = \int_a^b \frac{R^{n+1} e^{i\theta(n+1)}}{1+\mathrm{Re}^{i\theta}} id\theta$

Using integral inequality

$$\int_a^b \frac{\mathrm{R}^{n+1}\mathrm{e}^{i\theta(n+1)}}{1+\mathrm{Re}^{i\theta}} id\theta \le \int_a^b \frac{|R^{n+1}||e^{i\theta(n+1)}|}{|1+\mathrm{Re}^{i\theta}|} d\theta = \int_a^b \frac{|R^{n+1}|}{|\mathrm{Re}^{i\theta}+1|} d\theta \le \int_a^b \frac{|R^{n+1}|}{||\mathrm{Re}^{i\theta}|-|-1||} d\theta = \int_a^b \frac{|\mathrm{R}^{n+1}|}{|R-1|} d\theta = \frac{|R^{n+1}|}{|R-1|} J$$

Suppose R approaches naught:

$$|\lim_{R\to\infty} \int_a^b \frac{\mathrm{R}^{n+1}\mathrm{e}^{i\theta(n+1)}}{1+\mathrm{Re}^{i\theta}} id\theta| \le \lim_{R\to\infty} \frac{|R^{n+1}|}{|R-1|} J = 0$$

So $\int_\varsigma f(z)dz = 0$

(4) $\int_{\varphi_2} f(z)dz$

Note: it is incorrect to think the value is $-I$ the conclusion in (2), since in this case the value of complex numbers is different due to argument approaching 2π but not zero. One should transform this integral approaching from the top of branch cut.

Assume $\varphi_2 : z = t - i\alpha$ and $t \in (\mathbb{R}, \alpha)$ $\quad dz = dt$

$$\int_{\varphi_2} f(z)dz = \int_{\varphi_2} \frac{(t-i\alpha)^n}{t+1-i\alpha} dt$$

Then one changes $t - i\alpha$ and $t + 1 - i\alpha$ in natural logarithm form:

$$(t-i\alpha)^n = e^{n\ln(t-i\alpha)} = e^{n(\ln|t+i\alpha|+i(2\pi-\theta)}$$

Since θ approach 0, so $\theta = \tan^{-1}(\frac{i\alpha}{t}) = 0 = \alpha$:

$$(t-i\alpha)^n = e^{n\ln(t-i\alpha)} = e^{n(\ln|t+i\alpha|+i(2\pi-\theta)} = |t+i\alpha|^n e^{ni(2\pi-\alpha)}$$

Next, one uses the similar method for $t+1-i\alpha$:

$$\mathrm{t+1\text{-}i}\alpha = e^{\ln(t+1-i\alpha)} = e^{\ln|t+1-i\alpha|+i(2\pi-\alpha)} = |t+1-i\alpha| e^{i(2\pi-\alpha)}$$

Then:

$$\int_{\varphi_2} f(z)\,dz = \int_{\varphi_2} \frac{|t+i\alpha|^n e^{ni(2\pi-\alpha)}}{|t+1-i\alpha| e^{2\pi i} e^{-i\alpha}} dt = \frac{-e^{in(2\pi-\alpha)}}{e^{i\alpha}} \int_a^{\infty} \frac{|t+i\alpha|^n}{|t+1-i\alpha|} dt$$

when $\mathbb{R} \to \infty \alpha \to 0$ one may obtain

$$\int_{\varphi_2} f(z)\,dz = -e^{2\pi i n} S$$

(5) $\int_{\gamma} f(z)\,dz$

$$\gamma : z = ae^{i\theta} dz = aie^{i\theta} d\theta \qquad \theta \in [2\pi-\delta, \delta]$$

$$\int_{\gamma} f(z)\,dz = \int_{2\pi-\delta}^{\delta} \frac{\alpha^n e^{i\theta n}}{ae^{i\theta}} aie^{i\theta} d\theta = \int_{2\pi-\delta}^{\delta} ia^{n+1} e^{i\theta(n+1)} d\theta$$

as $\alpha \to 0$:

$$\int_{\gamma} f(z)\,dz = 0$$

Conclusion:

$$2\pi i e^{i\pi\theta} = S + 0 - e^{2\pi i n} S + 0$$

So: $\int_C \frac{x^n}{x+1} = \frac{-\pi}{\sin(n\pi)}$ [3]

3. Conclusion

This paper examined a question for the difficulty or even impossible to tackle some real integrals that may not be calculated by classic methods in calculus. it will be easier to solve the integrals. One need to use the residue theorem for this one called Cauchy's residue theorem for this one will be easy to look at the point 3. For this reason, one need to look at formulas to solve the difficult for the integral Calculus. It is easy to find pole 1 if x=0 for Figure 2 for multi-valued function but this will be not get the function, now one need to see answer 2 for the answer is $\arg(z) \in (0, 2\pi]$. Authors believe that our results help explore the field of more branches like pure look like math and physical. Because one may need math to get more problem in the physics, if one has the wrong answer do not give up and country to finish the problems. One get the right answer one can check more time.

References

[1] Greg Rainwater, Residue Theorems and their applications: computing integrals once thought impossible to evaluate analytically, 2019.

[2] Kaihang Liu, Liuyixin Shao, A summary on Two types of Real Integral Using the Residue Theorem, 2011.

[3] Jeremy Orloff, Topic 9 Notes, 2018.

[4] Joseph Bak, Donald J. Newman, Applications of the Residue Theorem to the Evaluation of Integrals and Sums, 2017.

[5] James Ward Brown, Ruel V. Churchill, Complex Variable and Applications, 2011.

[6] Seiichi Nomura, Residue, 2018.

[7] S Ponnusamy, H Silverman, Laurent Series and the Residue Theorem, 2018.

[8] Partial fraction, 2009, https://www.mathcentre.ac.uk/resources/uploaded/mc-ty-partialfractions-2009-1.pdf.

[9] Greg Rainwater, Residue Theorems and their applications: computing integrals once thought impossible to evaluate analytically, 2000.

[10] Anthony Osborne, Complex Variables and their Applications, 2018