Study on the Space Geodesy

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Abstract. Geodesic is a kind of shortest path among all curves in a metric space, which originally appeared in the Gaussian period. Since then, it was extensively applied in various branches of mathematics and physics, such as Riemannian geometry, digital geometry, Einstein's relativity, etc. This thesis mainly discusses geodesic definitions in Euclidean space and those in smooth manifolds after introducing the basic theory of smooth manifolds. At first, the thesis applies three ways to define geodesics on a surface, including the geodesic curvature method, the shortest distance method, and the relation of the principal normal of a curve and the normal vector of a surface. Then, the paper introduces some basic concepts in Riemannian manifolds. Finally, it gives a general definition of geodesics in a smooth Riemannian manifold.

Keywords: geodesic curvature, geodesic, differential manifold, connection.

1. Introduction

1.1 Research Background and Significance

Geodesic is one of the most important and basic concepts in differential geometry, and it is also the most basic research object of geometry. However, from the etymological point of view, geodesic comes from Geodesy instead of geometry. This discipline was developed by Gauss, the "prince of mathematics", when he drew maps for the Hanover Kingdom, so it was called geodesy. At that time, geodesic was one of the objects Gauss often met in surveying and mapping.

The study of geodesics has greatly promoted the development of geometry, and geodesic research has spread all over the main branches of differential geometry. In classical differential geometry, geodesic research has given birth to a series of important research objects such as Geodesic curvature, the Liouville formula, and a series of important conclusions, for example, the first variational formula of arc length. Geodesic is also an important research object in Riemann geometry. Geodesic field, geodesic flow and other concepts can be defined by a geodesic. In addition, the influence of geodesics is not limited to the field of geometry, and many physical phenomena and theorems are related to it or can be explained by it.

1.2 Research Status of Geodesic

Because Euclidean space is measurable and symmetrical, which is an abstraction and extension of real space, the earliest geodesic research was carried out in Euclidean space. Based on the classical differential geometry of Euclidean space, geodesic was discussed in the intrinsic geometry of curved surfaces, and a definition of a geodesic in Euclidean space was given by Geodesic curvature, and a complete set of theories was developed based on this. This part can be found in reference [1].

Since then, according to the available data, the research on geodesics has mainly dispersed into two directions. One is to stay in Euclidean space, and discuss the number of closed geodesics on surfaces and which surfaces will have closed geodesics. Among them, some classical studies on closed geodesics on simple closed surfaces in the last century can be found in reference [2]. The other is to destroy some properties of Euclidean space. To study how to define and calculate geodesics in these situations, such as geodesics defined on smooth (topological) manifolds, Sun Xinrui of Huazhong University of Science and Technology gave a detailed overview of this research in her paper [3].

In recent decades, with the rise of complex geometry and symplectic geometry, geodesics have been extended to Riemannian complex manifolds in two ways, one is with the help of Holomorphic metric and affine connection, and the other is with the help of symplectic geometry theory, and the research in this respect can refer to reference [4].

2. Definition of Geodesic in Euclidean Space

2.1 Three Definitions of Geodesic in Euclidean Space

In Euclidean space, we can give different definitions of geodesic by grasping its different geometric characteristics. In this section, we will sort out the definition ideas and summarize these definitions. Considering the smooth surface that can be embedded with $\mathbb{R}^3$, we naturally think about how to define geodesic in Euclidean space by extending the conclusion that "between two points, the line segment is the shortest" on the plane flexibly.

2.1.1 Geodesic Defined by Geodesic Curvature

Projection is a natural idea to extend the conclusion on the two-dimensional plane to the three-dimensional space. Therefore, we naturally get the first definition of the geodesic. The idea is: by projecting the curve C on a smooth surface S point by point onto the tangent plane of each point on the surface, for one point$p$, if the projection of the curve is a line segment in a small enough neighbourhood of this point, a special one can be obtained by splicing infinite neighbourhoods of each point on the curve.

At this point, the problem is to determine whether the projection is a straight line. Similar to the curvature of a curve on a plane, we need to introduce a new curvature to the original space curve $C$to determine, which should meet the following two conditions:

(1) completely determined by the space curve and surface,

(2) related to the curvature of the curve projected on the tangent plane.

This curvature is called Geodesic curvature, and the following is its strict definition (refer to reference [5] for this definition).

Give a surface$S$

$r = r(u^1, u^2)$,

CIs a curve on a curved surface.

$u^i = u^i(s), i = 1,2,$

Among them, s is the natural parameter of C.

Let p be a point on the surface, $\alpha$is the unit tangent vector of C at the point p, βbe the principal normal vector and γbe the second normal vector, where k is the curvature of the curve C at the point p, n be the unit normal vector of surface S at p, and θ be the angle between $\beta$and n, then the normal curvature of the surface S in the tangent direction $\alpha$of the point p is

$k_n = k \cos\theta = k\beta \cdot n = \ddot{r} \cdot n.$

Let $\varepsilon = n \times \alpha$, then α, ε, n are orthogonal to each other and form a right-handed system, as shown in Figure 1.

Definition 2.1 (Geodesic curvature): the projection of curve C on the surface εon the curvature vector $\ddot{r} = k\beta$of the point p (that is, the projection on the tangent plane Πat the point p on the surface$S$).

$$k_g = \ddot{r} \cdot \varepsilon = k\beta \cdot \varepsilon$$

It is called the Geodesic curvature of the curve C at the point$p$.

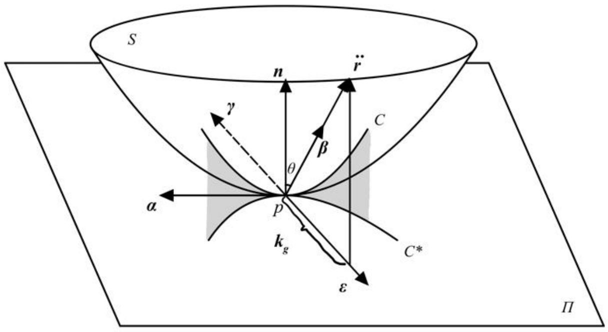

Figure 1 Right Rotation System

This definition well meets the two requirements we put forward above. First, it is easy to know from Geodesic curvature's expression that it is only related to curve C and surface$S$, and does not depend on their embedding ways in space $\mathbb{R}^3$. In fact, we call such quantities intrinsic quantities. In the next chapter, we will discuss this conclusion strictly.

For the second point, reference [5] gives the following theorem.

Theorem 2.2: The absolute value of the Geodesic curvature at point p of curve C on a surface S is equal to the curvature of the orthographic curve $C*$ on the tangent plane $\Pi$of the point p.

The proof of this theorem is given by reference [5], which will not be repeated here.

Theorem 2. 2 fully explains the relationship between Geodesic curvature and the curvature of its orthographic projection curve to the tangent plane. According to our analysis above, in order to make the projection curve of geodesic point by point to the tangent plane a straight line, the curvature of the projection curve of any point on the geodesic line should be zero. Then, according to Theorem 2.2, a geodesic definition is given.

Definition 2.3 (Geodesic): For the curve C on a curved surface$S$, if the Geodesic curvature of each point on the curve is zero, it is called geodesic.

Starting from definition 2.3, we can directly get to the following inference.

Inference 2.4: If there is a straight line in space $\mathbb{R}^3$ on the surface$S$, it must be a geodesic line on the surface.

It is proved that for a straight line (l) on a curved surface S, choose any point $p$on the straight line. Because it is a straight line, its projection on the tangent plane at the point p is the straight line itself, that is, the curvature of the orthogonal projection curve ($l*$) is zero. According to Theorem 2.2, its absolute value of Geodesic curvature is zero, that is, its Geodesic curvature is zero.

2.1.2 Geodesic Defined by Short-Range

At the beginning of this chapter, we mentioned that geodesic can also let us get another definition of geodesic from its short range nature. We can first look at how geodesic short-range is defined.

Proposition 2.5: For any two points p and q on a surface, the geodesic line segment between two points p, q is the curve segment with the shortest arc length among the curve segments on the surface connecting two points.

It must be pointed out that this proposition is a false one. We can simply cite a counterexample (as shown in Figure 2).

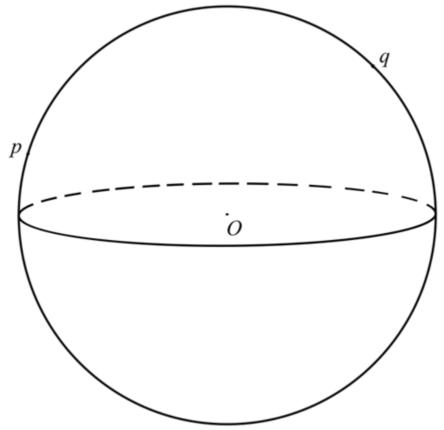

Figure 2. Counterexample

It is not difficult to find that, for two points p, q on the ball O, according to definition 2.3, the superior arc $\widehat{pq}$ and the inferior arc $\widehat{pq}$ on the great circle passing through the centre of the ball are also geodesic line segments connecting the two points p, q, but the superior arc $\widehat{pq}$ is obviously not the shortest. This shows that the direct extension of an axiom is flawed and must be limited.

Theorem 2.6 (Shortness of geodesics): For two points p, q on a surface S located in the same small enough neighbourhood, the geodesic line segment between the two points p, q has the shortest arc length among all curve segments on the surface connecting the two points.

However, if geodesic is to be defined by this theorem in turn, we need to prove its existence, that is,

Theorem 2.7 (Existence of Shortcut Line): For any two points p, q in a sufficiently small neighborhood on any smooth surface S, the curve segment with the shortest arc length always exists.

Here, the advantage of our choice of studying in Euclidean space is revealed, and we can directly use Euclidean metrics to calculate. This proof refers to a part of the contents of the reference [1].

Prove: Let curve C: $u^i = u^i(s)$ be a directional smooth curve on the surface S starting from point $p$and ending at point q, where $s$is its arc length parameter satisfying $a \le s \le b$. Along the curve$C$, the radius vector is $r(s)$, the tangent vector is $T(s) = \frac{dr}{ds}$ and the unit normal vector of the surface is $n(s)$.

By keeping the two endpoints unchanged, a cluster of such smooth curve C_λ can be obtained, starting at the point p and ending at the point q. Its parameter equation can be expressed as

$$C_\lambda : u^i = u^i(s, \lambda),$$

In which, s is the parameter of C_λ, $a \le s \le b$, λ is the parameter of the curve family and $\lambda \in \Lambda$, Λbeing the index set, as shown in Figure 3.

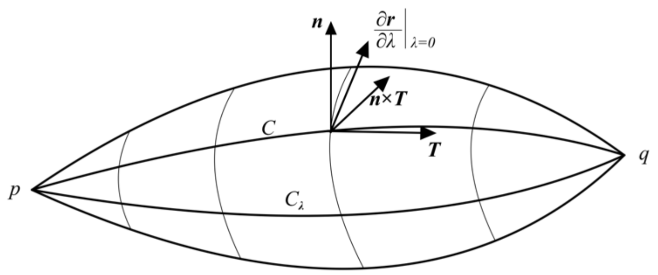

Figure 3: Curve Family

Using the formula for calculating arc length proposed in reference [6], the functional is obtained.

$$L(C_\lambda) = \int_a^b \left|\frac{\partial r}{\partial s}\right| ds = \int_a^b \sqrt{\frac{\partial r}{\partial s} \cdot \frac{\partial r}{\partial s}}\, ds, \tag{2.1}$$

The problem is transformed into proving that the functional is minimal.

In order to make use of the variational method systematically introduced in reference [7], we will first introduce two lemmas, the proofs of which are given in reference [7].

Lemma 2.8 (Necessary condition of functional extremum): If functional

$$v[x(t)] = \int_a^b F\big(t, x(t), x'(t)\big)\, dt$$

has a variation, and it reaches a maximum or a minimum at $x_0(t) = x_0$. (The point of $x_0(t) = x_0$ is inside the functional definition area), then the variation of the function at $x_0(t) = x_0$ is 0.

Lemma 2.9 (Lagrange Lemma): If for each continuous function, there is$\eta(t)$,

$$\int_a^b \Phi(t)\,\eta(t)\,dt = 0,$$

And if $\Phi(t)$ is continuous in the integral interval, $\Phi(t) \equiv 0$ in this interval.

Turn a problem into a form.

$$L(C_\lambda) = \int_a^b F\big(s, r(s), r'(s)\big)ds \tag{2.2}$$

The extremum of this kind of function is required to meet the conditions $r|_p = r_1, r|_q = r_2$ where r_1, r_2 being constant vectors.

If the variational sign is $\boldsymbol{\delta}$, it is obtained according to Lemma 2.8,

$$\delta L = \int_a^b \left(F_r - \frac{\partial F_{r'}}{\partial s}\right)\delta r\, ds = 0, \tag{2.3}$$

Obtained from Lemma 2.8

$$F_r - \frac{\partial F_{r'}}{\partial s} = 0,$$

Extend it to

$$F_r - F_{r's} - F_{r'r}\frac{dr}{ds} - F_{r'r'}\frac{d^2 r}{ds} = 0, \tag{2.4}$$

That is, a boundary value problem of a second-order partial differential equation is obtained.

$$\begin{cases} F_r - F_{r's} - F_{r'r}\dfrac{dr}{ds} - F_{r'r'}\dfrac{d^2 r}{ds} = 0, \\ r|_p = r_1, \quad r|_q = r_2. \end{cases} \tag{2.5}$$

In fact, this equation does not necessarily have a solution, but the functional (2.1) involved in the geodesic definition has its particularity, that is, the integrand F is independent of the parameter s, so the original equation is reduced to

$$\begin{cases} F_r - F_{r'r}\dfrac{dr}{ds} - F_{r'r'}\dfrac{d^2 r}{ds} = 0, \\ r|_p = r_1, \; r|_q = r_2. \end{cases}, \tag{2.6}$$

Obviously, the partial differential equation in formula (2.6) can be transformed into an appropriate equation by multiplying the integral factor r'

$$\frac{d}{ds}(F - r'F_{r'}) = 0, \tag{2.7}$$

Therefore, (2.6) there must be an analytical solution to this boundary value problem, that is, there must be a geodesic defined reversely by Theorem 2.6.

Definition 2.10 (Geodesic): For two points p, q located in the same small enough neighbourhood on a surface S, the curve segment with the shortest arc length among all curve segments on the surface connecting the two points p, q is called geodesic segment.

It is worth mentioning that, when using Lemma 2.7 to calculate the condition of the functional extremum, we didn't indicate whether it was the maximum or the minimum. This is because we gave a condition in definition 2.6, that is, in a small enough neighbourhood, so we only need to judge the extremum, so in fact, the functional extremum we got is not necessarily the minimum. Thus, we got an equivalent definition of definition 2.6, which was given by reference [1]

Definition 2.11 (Geodesic): When the length of a curve C on a curved surface S reaches the staying value, the curve C is geodesic.

Note: The proof of equivalence between definitions 2.10 and 2.11 is given by reference [1], and the equivalence between definitions 2.3 and 2.10 is given by reference [8], which will not be repeated here.

2.2.3 Geodesic Defined by a Coincidence of Two Normals

In fact, this definition is strongly related to Figure 1 used in the definition 2.1. Using Figure 1, let's recall the Geodesic curvature we once defined.

$$\boldsymbol{k_g} = \ddot{\boldsymbol{r}} \cdot \boldsymbol{\varepsilon} = \boldsymbol{k\beta} \cdot \boldsymbol{\varepsilon},$$

Among them, by substituting $\varepsilon = n \times \alpha$ into the above formula, we can get

$$\boldsymbol{k_g} = \boldsymbol{k\beta} \cdot \boldsymbol{\varepsilon} = \boldsymbol{k}(\boldsymbol{\beta}, \boldsymbol{n}, \boldsymbol{\alpha}) = \boldsymbol{k}(\boldsymbol{\alpha} \times \boldsymbol{\beta}) \cdot \boldsymbol{n} = \boldsymbol{k\gamma} \cdot \boldsymbol{n}, \tag{2.8}$$

which means

$$\boldsymbol{k_g} = \boldsymbol{k} \cos\left(\frac{\pi}{2} \pm \boldsymbol{\theta}\right) = \pm \boldsymbol{k} \sin \boldsymbol{\theta},$$

Where k is the curvature of the curve C at a point p, θ is the angle of β with n. As the curve C on the surface is arbitrarily selected, that is, k is not always zero. Referring to Definition 2.3 for the definition of geodesic, and reference [5] gives the following definition.

Definition 2.12 (Geodesic): A non-linear curve on a curved surface is geodesic if and only if the main normal of the curve coincides with the normal of the curved surface except the point with zero curvature.

Although this definition is based on a simple geometric relationship, its significance is not limited to this. It has at least two important meanings: one is related to computational mathematics, and the formula (2.8) gives another method for calculating Geodesic curvature, which can effectively improve the computational efficiency when the ε calculation is difficult and the γ calculation is relatively simple; Secondly, it is related to Riemann geometry, and this definition of a geodesic is a special case in $\mathbb{R}^3$ of covariant differentiation. Therefore, a definition of geodesic equivalent to definition 2.8 is given in reference [8].

Definition 2.13 (Geodesic): If the covariant differential along the tangent vector of a curve on a surface is zero, the surface is called geodesic on the surface.

The definition of covariant differential involves tensor analysis, connected differential geometry, smooth manifold and other disciplines, which is far beyond the scope of this section, and will not be expanded here. Readers who are interested and have the foundation of the differential manifold can refer to the discussion in Chapter 4 of this article or read the reference [9] by themselves.

The equivalence between definitions 2.3 and 2.12 is given by reference [5], and the equivalence between definitions 2.3 and 2.13 is given by reference [8], which will not be repeated here.

3. Brief Introduction of Geodesic on a Manifold

3.1 Introduction to Basic Knowledge of Manifolds

In order to define geodesics on manifolds, we need to know some basic knowledge of manifolds. Here, we assume that readers have already learned point set topology. Recalling section 2.2.1, the author once proposed a method of projecting separately in a minimal neighbourhood, and then splicing these projections to obtain "line segments" to understand the process of defining geodesics by Geodesic curvature. The inspiration for this method actually comes from the definition of differential manifolds in reference [10].

Definition 4.1 (Manifold): M is assumed to be a topological space if it satisfies the following 3 conditions,

(1) M is a Hausdorff space,

(2) M is the second countable, that is, M has a countable topological basis,

(3) M has a topologically homeomorphism with an open subset of n dimensional Euclidean space.

From definition 4.1, we can see that a manifold is actually the product of the second countable piece of Euclidean space "bonded" by certain rules. The author has been thinking about such a question when studying *Differential Manifold:* Why do we have to break a good Euclidean space to study? In fact, breaking the Euclidean space can play an important role in dimension reduction. For example, the sphere is a two-dimensional manifold, but we will discuss it in the three-dimensional Euclidean space in the second chapter. Although the amount of calculation in this example is not large, when the dimension is larger, the complexity of the space will increase dramatically, and the analysis will be extremely difficult.

The definition of manifold 4. 1 is enough for us to study the topological properties of manifolds. However, to define geodesics, we also need to study the analytical properties of manifolds. Therefore, we also need to define the differential structure of manifolds.

Definition 4.2 (Differential Manifold): Let a coordinate chart on the topological manifold M be $\mathcal{A}$, and for any two coordinate charts $(U,\varphi),(V,\psi)\in\mathcal{A}$, where two intersecting open sets U,V on the manifold M. φ,ψ are two homeomorphism maps from an open subset of the n dimensional Euclidean space. If $\psi\circ\varphi^{-1}:\varphi(U\cap V)\to\psi(U\cap V)$ is a differential homeomorphism map and $\mathcal{A}$ contains all coordinate charts on the manifold, the coordinate chart set $\mathcal{A}$is called differential structure on the manifold M. A manifold M with differential structure $\mathcal{A}$is called differential manifold, and it is denoted as $(M,\mathcal{A})$, shortened as M.

With the differential structure, we have noticed another problem: to get the definition of geodesics on differential manifolds, we lack a very important tool: inner product. Without an inner product, space cannot be measured, and there are no series of geometric quantities such as length, angle and area. It is worth mentioning that although manifolds are "spliced" by Euclidean spaces, we can't simply adopt Euclidean inner products. Nevertheless, it is correct to think in this direction that inner products on manifolds should retain some important properties of Euclidean inner products, such as positive property and symmetry.

Definition 4.3 (Riemannian Manifold): Let M^nbe a n dimensional differential manifold, where p is the point on M^n. For a (0,2)- tensor field g on the tangent space T_pM of a point p, if g is satisfied by

(1) Symmetry: $\forall p\in M$, satisfied for any two tangent vectors $X_p,Y_p\in T_pM$

$$g_p(X_p,Y_p)=g_p(Y_p,X_p),$$

(2) Positive definiteness: $\forall p\in M$, satisfied for any two $X_p,Y_p\in T_pM$

$$g_p(X_p,Y_p)\geq 0,$$

Among which $g_p(X_p,Y_p)=0$, if and only if $X_p=\vec{0}$, then the tensor field g is called the Riemann metric on the manifold M^n, and the manifold M^n is called Riemann manifold, which can be recorded as (M,g) [11].

Therefore, the metric on the manifold can indeed be regarded as an extension of Euclidean metric, and the definition of Riemann metric is actually very similar to the definition of the inner product in Euclidean space.

In addition, the following theorem is given in reference [12].

Theorem 4.4 (existence theorem of Riemannian metric): Any differential manifold M^n can define a Riemannian metric.

This theorem enables us to use the Riemann metric to study any differential manifold, which is of great significance. The proof requires interested readers to read the reference [13], so it will not be repeated here.

Finally, in order to achieve the effect of using the Frenet frame to describe the concepts of tangent vector and the normal vector of some geometric sets on manifolds, as in Chapter 2, we need to introduce the concept of the tangent bundle, so as to achieve the goal of extending the concept of vector field to manifolds.

Definition 4.5 (Tangent bundle): Set M^n as a n dimensional differential manifold, and define the set of all ordered binary groups on the manifold like (p, X_p)as the tangent bundle of the manifold M^n, and remember this set as TM, where $p \in M^n$,$X_p \in T_pM$.

With the tangent bundle, we can easily extend the common concept of a vector field in mathematical analysis to the manifold.

Definition 4.6 (Vector Field on Manifold): Set M^n as a n dimensional differential manifold, if there is a continuous mapping

$$X: M \to TM$$

$$p \to X_p,$$

And satisfies $\pi \circ X = \mathrm{Id}_M$, where $\pi\colon\ TM \to M$ is projection, X is called the vector field on the manifold M. After the coordinate system on the manifold is given, the vector field can be calculated according to the formula given in reference [11], which is not repeated here.

3.2 Geodesic on Manifold

Section 2.2.3 of Chapter 2 of this paper defines geodesic in three-dimensional Euclidean space by using the special case of covariant differential in three-dimensional Euclidean space. This section will extend this definition to Riemannian manifold by using the concept of covariant differential, but to give the definition of covariant differential, we have to start with the geometry of connected differential.

4.2.1 Affine Connection

There is a recognized fact in mathematics: in most cases, linear problems are better to solve than nonlinear problems. Because linear problems have two powerful tools, namely linear space and linear transformation. Therefore, it is a very effective means to solve problems by transforming nonlinear problems into linear ones, and this is fully reflected in the study of differential manifolds. Even if geometric sets on manifolds, such as curves, are parameterized, after parameterization, the function representing the curve is still not linear, more likely not even a polynomial function. At this point, the problem related to this curve is nonlinear. However, by using the tangent space of the points on the curve, we can transform the nonlinear curve into the relationship between linear space and linear space, and by using a linear transformation, we can express the relationship between these linear spaces. In this way, we can transform the nonlinear problem into a linear problem.

Connection means that it establishes the connection between different tangent spaces, while Affine means that this connection brings some properties similar to affine geometry to the manifold.

Definition 4.7 (Affine Connection): Set M as a n dimensional differential manifold, $\mathcal{X}(M)$ as a set of all smooth vector fields on the manifold M, and $\mathcal{D}(M)$as a set of all smooth real function defined on the manifold M. If there is a continuous mapping

$$\nabla: \quad \mathcal{X}(M) \times \mathcal{X}(M) \to \mathcal{X}(M)$$

$$(X,Y) \to \nabla_X Y,$$

And satisfy.

(1),$\nabla_{fX+gY}Z = f\nabla_X Z + g\nabla_Y Z$,

(2),$\nabla_X(Y+Z) = \nabla_X Y + \nabla_X Z$, (3),$\nabla_X(fY) = f\nabla_X Y + X(f)Y$,

Among them, $X, Y, Z \in \mathcal{X}(M)$ and $f, g \in \mathcal{D}(M)$, then ∇ is called affine connections on manifold M.

If readers are familiar with the content of mathematical analysis, they can see that the definition method of affine connection is very similar to the directional derivative. With affine connection, we can define the most important concept in this section, covariant differential.

Definition 4.8 (Covariant Differential): Set M as a n dimensional differential manifold, ∇ as an affine connection on the manifold M. If there is a vector field generated along a differentiable curve $C: I \to M$that corresponds one-to-one to the vector field $\frac{DV}{dt}$ generated along the curve, and satisfies

(1) $\frac{D}{dt}(V + W) = \frac{DV}{dt} + \frac{DW}{dt}$;

(2) $\frac{D}{dt}(fV) = \frac{df}{dt}V + f\frac{DV}{dt}$where W is the vector field generated along the differentiable curve $C: I \to M$, f is the differentiable function on I;

(3) If V can be deduced from $Y \in \mathcal{X}(M)$, that is: $V(t) = Y(C(t))$, $\frac{DV}{dt} = \nabla_{dc/dt}Y$can be obtained. Then vector field $\frac{DV}{dt}$ is called covariant differential along the differentiable curve $C: I \to M$, also called covariant differential.

After the coordinate system of variable differential on manifold is given, it can be calculated according to the formula given in reference [13], which will not be repeated here.

The definition of covariant differentiation comes from our need to describe the change of the vector field along the curve, so when covariant differentiation is 0, it means that the vector field V does not change along the differentiable curve $C: I \to M$. Geometrically speaking, if an object moves along this curve $C: I \to M$ in a higher dimension, the magnitude and direction of the velocity will not change.

Besides, to depict geodesics on Riemannian manifolds, we have to prove at least two things: one is to give the concept of parallelism on manifolds, and the other is to prove that any Riemannian manifold has an affine connection that is symmetrical and compatible with Riemannian metrics.

4.2.2 Riemannian Connection

Definition 4.9 (Parallelism): Set M as a n dimensional differential manifold, ∇ as an affine connection on the manifold M. If a vector field generated along a differentiable curve $C: I \to M$ satisfies $\frac{DV}{dt} = 0$ for $\forall t \in I$, it is said that the vector field V has parallelism.

With this concept, we can define the compatibility of connection and Riemannian metric on the Riemannian manifold.

Definition 4.10 (Compatibility): Let (M, g) be a n dimensional Verriemann manifold, and ∇ is an affine connection on the manifold M. If any two vector fields P, P' generated along the differentiable curve $C: I \to M$ satisfy $g(P, P') \equiv c$, where c is a constant, it is said that the affine connection ∇ is compatible with Riemann metric g.

So reference [13] gives the following theorem,

Theorem 4.11 (Theorem of the Existence and Uniqueness of Riemannian Connection): Let (M, g) be a n dimensional Riemannian manifold, and there is a unique affine connection ∇, which is symmetric and compatible with Riemannian metric. This connection is called the Riemannian connection, also known as Levi-Civita connection. The proof of this theorem is given by reference [12], so it will not be repeated here.

4.2.3 Definition of Geodesic on The Manifold

With the foreshadowing of the previous sections, we can finally understand the definition of geodesic on a manifold.

Definition 4.12 (geodesic on manifold): Let (M, g) be a n dimensional Riemannian manifold, and its Riemann connection is ∇, if there is a parametric curve $\gamma : I \to M$ on the manifold satisfies $\nabla_{\dot{\gamma}}\dot{\gamma} = 0$ on $\forall p_0 \in I$, it is called geodesic on the manifold, where $\nabla_{\dot{\gamma}}\dot{\gamma} = \frac{D}{dt}\left(\frac{d\gamma}{dt}\right)$.

It can be seen that $\nabla_{\dot{\gamma}}\dot{\gamma}$ is the covariant differential of the vector field defined along the curve γ, which is the definition 2.13 mentioned in the second chapter.

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