Proving combinatorial identities using complex numbers

Zichao He[1, *, †], Zhuoxi Hou[2, †], Yining Zhang[3, a, *, †]

[1]Spartanburg Day School, Spartanburg, SC, 29307, USA

[2]The High School Affiliated to Renmin University of China, Beijing, 100086, China

[3]Cape Henry Collegiate, Virginia Beach, VA, 23454, USA

*Corresponding author's e-mail: zichao.he@sdsgriffin.org

[a]yzhang22@student.capehenry.org

†These authors contributed equally.

Abstract. The creation of complex numbers addresses a variety of mathematical problems. Complex numbers are the building blocks of more complicated and advanced math, such as algebra. Complex numbers also are important in many other research fields, especially in electronics and electromagnetism. Complex numbers have various applications in both the scientific and engineering world, such as signal processing, electromagnetism, control theory, vibration analysis. Complex numbers are always used as mathematical tools to describe signals that vary periodically. Despite being an extension of the real number system, complex numbers are largely self-contained. The geometric meaning and algebraic structure of complex numbers can help us solve a variety of geometric problems. This paper summarizes the basic knowledge of complex numbers and gives several fascinating results proved by the geometric and algebraic properties of complex numbers.

Keywords: complex numbers, De Moivre' theorem, combinatorial identities.

1. Introduction

Complex numbers have various applications in both the scientific and engineering world, such as signal processing, electromagnetism, control theory, vibration analysis[1-10]. Complex numbers are always used as mathematical tools to describe signals that vary periodically. To analyze voltages and currents, complex numbers are also inevitable tools. In fluid dynamics, functions involving complex variables are utilized to analyze potential flow. Complex numbers are the building blocks of more complicated and advanced math, such as algebra and geometry. A complex number is an ordered pair of real numbers in (a,b), where a and b are all real numbers. In this paper, we inherit the standard notation R to denote the set of all real numbers. The symbol **C** is used to represent the set of all complex numbers. That is,

$$\boldsymbol{C} = (a, b): a \in R, b \in R.$$

We start our investigation of complex numbers with the language of the coordinate pair. We discuss in detail both the algebraic and geometric structures of complex numbers. We use these basic properties to prove several meaningful qualities and identities only involved with real numbers.

2. Main Works

2.1. Arithmetic properties of Complex numbers

Two binary operations can be introduced on such set **C.**

Definition 2.1.1 Assume $(a, b), (c, d)$ are two arbitrary complex numbers. The addition of these two complex numbers is defined as follows

$$(a, b) + (c, d) = (a + c, b + d).$$

Definition 2.1. 2 Assume $(a, b), (c, d)$ are two arbitrary complex numbers. The multiplication of these two complexes is defined as follows

$$(a,b)\cdot(c,d)=(ac-bd,ad+bc).$$

Remark 2.1. 3 Both addition and multiplication are defined as Definition 2.1.1.and Definition 2.1. 2.1.2 satisfy the law of commutativity and associativity.

Remark 2.1. 4 The complex number (0, 0) is the zero elements in the sense that for any complex number (a,b),

$$(a,b)+(c,d)=(a+c,b+d).$$

Theorem 2.1. 5 The set **C is** an additional group.

Proof. It suffices to show that any element of set **C has** an inverse. It is easy to see that the inverse element of (a,b) under the operation of addition is (-a,-b).

Theorem 1.6 The set **C** is a multiplication group.

Proof. The element (1, 0) is the identity element in the sense that for any element (a,b),

$$(a,b)\cdot(1,0)=(a,b).$$

It is easy to check that for every nonzero element (a,b), there is a unique element $(\frac{a}{a^2+b^2},-\frac{b}{a^2+b^2})$ such that

$$(\frac{a}{a^2+b^2},-\frac{b}{a^2+b^2})(a,b)=\ (1,0).$$

This means every nonzero element has a unique inverse.

Remark 2.1.6 Addition and multiplication of C also satisfy the law of distribution:

$$[(a,b)+(c,d)]\cdot(e,f)=(a,b)\cdot(e,f)+(c,d)\cdot(e,f).$$

Remark 2.1.7 Considering the set $\{(a,0)\colon a\in R\}$, it is easy to check that $\{(a,0)\colon a\in R\}$ is a subfield of C because the projection mapping ϕ: $\{(a,0)\colon a\in R\}\to R$

$$(a,0)\to a$$

is an algebraic isomorphism between $\{(a,0)\colon a\in R\}$ and R. Therefore, the real number field R is a subfield of C algebraically. What makes field C distinct from field R is the special element (0,1). This element satisfies

$$(0,1)^2=(0,1)\cdot(0,1)=(-1,0).$$

We use i to denote the element (0,1) specifically. A complex number (a,b) can be rewritten as

$$(a,b)=(a,0)+(0,b)=a+b\cdot i.$$

From now on, instead of using real pair (a,b) to represent a complex number, we use $z=a+b\cdot i,$ where a is called the real part of z, b is called the imaginary part of z, denoted as $a=Rez$ and $b=Imz$ respectively. Using the present notation, addition and multiplication are defined as:

$$(a+bi)+(c+di)=(a+c)+(b+d)i,$$

$$(a+bi)(c+di)=(ac-bd)+(ad+bc)i.$$

Subtraction and division are defined as the inverse operation of addition and multiplication, respectively:

$$(a+b\cdot i)-(c+d\cdot i)=(a-c)+(b-d)\cdot i,$$

$$\frac{a+bi}{c+di}=(a+bi)\left(\frac{c-di}{c^2+d^2}\right)$$

$$=\frac{ac+bd}{c^2+d^2}+\frac{bc-ad}{c^2+d^2}i\ .$$

Let $z=a+bi$ be a complex number. Define:

$$|z|=\sqrt{a^2+b^2},$$

$$\bar{z}\text{=a-bi},$$

Where $|z|$ is called the modulus or absolute value of z, $\bar{z}$ is called the complex conjugate of z.
Remark 2.1.10 Let z and w be two complex numbers, then

(i) Rez= $\frac{1}{2}(z+\bar{z})$, Imz=$\frac{1}{2i}(z-\bar{z})$;
(ii) $z\bar{z}=|z|^2$;
(iii) $\overline{z+w}=\bar{z}+\bar{w}$, $\overline{zw}=\bar{z}\ \bar{w}$;
(iv)$|zw| = |z||w|$, $\left|\frac{z}{w}\right| = \frac{|z|}{|w|}$;
(v) $|z|=|\bar{z}|$.

2.2. Geometric interpretation

After establishing a cartesian coordinate system on the complex plane, any complex numbers can be viewed as a point on the plane. Given a complex number $z = a + bi$, it can be viewed as a point with x-coordinate a and y-coordinate b. The polar coordinate of this complex number can be represented as (r, θ), which means

$$a = rcos\theta,\ b = rcos\theta.$$

So that complex number $z = a + bi$ can also be represented as

$$z = r(cos\theta + isin\theta),$$

where $r = |z| = \sqrt{a^2 + b^2}$ is the module of z, and θ is called the argument of z. Notice that if θ is the argument of z, angle $\theta + 2k\pi$ (k could be any integer) can also be an argument of z. It leads to the conclusion that a complex number can have an infinite number of arguments. We use $Argz$ to denote all arguments of the complex number **z.** The argument in the interval $[-\pi, \pi]$ is denoted as arg(z).

We can also see the complex number $z = a + bi$ as a projection on x and y-axis of vectors a and b. That's why we can use Complex numbers and Euclidean vectors to illustrate the same thing. We know that all the vectors received through the parallel movement of a certain vector represent the same complex number, and if the start and end of a vector is $Z_1$and Z_2, the complex number of this vector is $Z_1 - Z_2$ so that $|Z_1 - Z_2|$represent the distance between $Z_1$and Z_2. Especially when the start of a vector is the origin, the complex number its terminal point indicates is the same complex number the vector does.

Now we can know that what we defined previously, the addition of complex number is the same as the addition of vectors: set the start of two not repeated non-zero vector $Z_1$and Z_2 to be the origin, use the side of Z_1 and $Z_2$to set a parallelogram, then the vector drawn along the diagonal with the origin as the starting point is expressed as $z1+z2$; the vector starts at Z_2 and end at Z_1can be represented as $Z_1 - Z_2$. And now we can go back to see the inequation in 1.1: $|z_1 + z_2| \le |z_1| + |z_2|$It is just a simple geometric proposition talking about how the sum of two sides is bigger than the third line in a triangle.

In order to demonstrate the geometrical meaning of complex number multiplication, we implement the triangular expression of complex numbers. Let,

$$z_1 = r_1(cos\theta_1 + isin\theta_1),$$

$$z_2 = r_2(cos\theta_2 + isin\theta_2),$$

then

$$z_1z_2 = r_1r_2(cos(\theta_1 + \theta_2) + isin(\theta_1 + \theta_2)).$$

thus, it can be valid that

$$|z_1 + z_2| = |z_1||z_2|,$$

$$Arg(z_1 z_2) = Argz_1 + Argz_2.$$

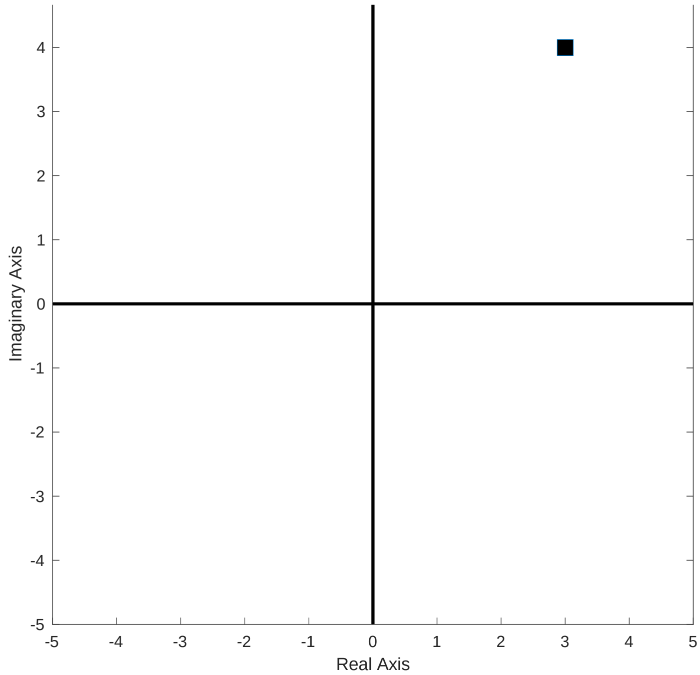

Figure 1. Plot Number (3+4i) on the complex plane

The first equation has already been proved in section 1.1, and the second equation can be interpreted as an equivalence of two sets. This means the multiplication of two complex numbers results in a complex number that contains a module that is the multiplication of the modules of the two complex numbers and contains an argument that is the sum of the argument of the two complex numbers. From a geometrical perspective, multiplying complex number w by complex number z can be compared to rotating z counterclockwise by $argw$ degrees, and extend its longevity by $|w|$ times. Especially, if w is a unit vector, then the product of w multiplying z would be rotating z counterclockwise by $argw$ degrees. For example, if i is a unit vector and its argument is $\frac{\pi}{2}$, iz is therefore equivalent to the vector received from rotating z counterclockwise by $\frac{\pi}{2}$. This type of geometric intuitiveness is helpful when approaching problems.

Now take a look at the division of complex numbers. Considering the equation,

$$\frac{z_1}{z_2} = \frac{r_1}{r_2}[\cos(\theta_1 - \theta_2) + isin(\theta_1 - \theta_2)].$$

that can mean

$$\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|},$$

$$Arg\left(\frac{z_1}{z_2}\right) = Argz_1 - Argz_2.$$

Here, the second equation can also be interpreted as the equivalence of two sets. This indicates that the included angle between vectors $z1$ and $z2$ can be represented with $Arg(\frac{z_1}{z_2})$, this simple fact can be very useful when discussing various geometrical problems. For example, with this, it is easier to prove that $Re(z_1\widetilde{z_2}) = 0$ is a necessary and sufficient condition to the vectors $z1$ and $z2$ being proved to be perpendicular. This is due to the fact that if $z1$ is perpendicular to $z2$, their included angle is equal to $\pm\frac{\pi}{2}$, that $arg\left(\frac{z_1}{z_2}\right) = \pm\frac{\pi}{2}$, which demonstrates the fact that $\frac{z_1}{z_2}$ is a pure imaginary number. Thus $z_1\widetilde{z_2} = \frac{z_1}{z_2}|z_2|^2$is also a pure imaginary number, and that$Re(z_1\widetilde{z_2}) = 0$ In the same way, it can be proven that the necessary and sufficient conditions to z_1 and z_2 being parallel is $Im(z_1\widetilde{z_2}) = 0$.

Theorem 2.2.1 (Rules for division)
Assume $z_1= r_1$ $(\cos\theta_1+\mathrm{i}\sin\theta_1)$, $z_2=r_2$ $(\cos\theta_2+\mathrm{i}\sin\theta_2)$, $z_2\neq 0$.
Then $\frac{z_1}{z_2}=\frac{r_1}{r_2}$ $(\cos(\theta_1-\theta_2)+\mathrm{i}\sin(\theta_1-\theta_2))$.
Proof.

$$\frac{z_1}{z_2}=\frac{r_1(\cos\theta_1+\mathrm{i}\sin\theta_1)}{r_2(\cos\theta_2+\mathrm{i}\sin\theta_2)}$$
$$=\frac{r_1}{r_2}\cdot\frac{(\cos\theta_1+\mathrm{i}\sin\theta_1)(\cos\theta_2-\mathrm{i}\sin\theta_2)}{(\cos\theta_2+\mathrm{i}\sin\theta_2)(\cos\theta_2-\mathrm{i}\sin\theta_2)}$$
$$=\frac{r_1}{r_2}\cdot\frac{\cos\theta_1\cos\theta_2-\mathrm{i}\cos\theta_1\sin\theta_2+\mathrm{i}\sin\theta_1\cos\theta_2+\sin\theta_1\sin\theta_2}{\cos^2\theta_2+\sin^2\theta_2}$$
$$=\frac{r_1}{r_2}\cdot(\cos\theta_1\cos\theta_2+\sin\theta_1\sin\theta_2)+\mathrm{i}(\cos\theta_1\sin\theta_2+\sin\theta_1\cos\theta_2)$$
$$=\frac{r_1}{r_2}\ (\cos(\theta_1-\theta_2)+\mathrm{i}\sin(\theta_1-\theta_2)).$$

Theorem 2.2.2 [10] De Moivre‘ theorem
We prove De Moivre’s Theorem by invoking mathematical induction for $n\in\boldsymbol{N}^*$
Proof.
For $n = 1$, it is easy to find that $(\cos x+ i\sin x)^1=\cos(1x)+i\sin(1x)=\cos(x)+i\sin(x)$
Assume that formula is true for $n = k$.

$$(cosx + isinx)^k=\cos(kx)+i\sin(kx).\ .\ .(ii)$$

It suffices to prove that result is true for $n = k + 1$.
$(\cos x+i\sin x)^{k+1}=(\cos x+i\sin x)^k(\cos x+i\sin x)$

$$= (\cos(kx) + i\sin(kx))(\cos x + i\sin x)$$

$= \cos(kx)\cos x\ x\sin(kx)\sin x + i(\sin(kx)\cos x + \cos(kx)\sin x)$
$= \cos\{(k + 1)x\} + i\sin\{(k + 1)x\}$
Hence the result is proved.

2.3. Several inequalities and identities

Theorem 2.3.1: $\forall n\in N^*$, $\sin 1 + \sin 2 + \sin 3 + \cdots + \sin n \le \frac{1}{\sin\frac{1}{2}}$

Proof.
For simplicity, we assume z=cos1+isin1,
then according to $|Imz|\le|z|$,
$|\sin 1+\sin 2+\sin 3+\ldots+\sin n|$
$\le|(cos1+isin1)+(cos2+isin2)+(cos3+isin3)+\ldots+(cos\ n+isin\ n)|$
According to the De Moivre‘ theorem that $cos\ n\theta+isin\ n\theta=(cos\ \theta+isin\theta)^n$,
the above formula
$=|(cos1+isin1)+(cos1+isin1)^2+(cos1+isin1)^3+\ldots+(cos\ 1+isin\ 1)^n|$
$=|z+z^2+z^3+\ldots+z^n|$
Using the formula of geometric series $z+z^2+z^3+\ldots+z^n=\frac{z(1-z^n)}{1-z}$,
we get $|z+z^2+z^3+\ldots+z^n|=\left|z\ \frac{z(1-z^n)}{1-z}\right|$,
Then according to $|zw|=|z||w|$, $\left|z\ \frac{z(1-z^n)}{1-z}\right|=|z|\left|\frac{z(1-z^n)}{1-z}\right|$,
we use the De Moivre‘ theorem again, $z^n=cos\ n+isin\ n$
Thus, $|z|\left|\frac{z(1-z^n)}{1-z}\right|=\left|\frac{1-cosn-isinn}{1-cos1-isin1}\right|$
According to trigonometric formula that $cos\ 2\alpha=1\text{-}2sin^2\alpha$ *and* $sin2\alpha=2sin\alpha cos\alpha$,
$\left|\frac{1-cosn-isinn}{1-cos1-isin1}\right|=\frac{|1-cosn-isinn|}{2sin\frac{1}{2}}$
Using $|z+w|\le|z|+|w|$, we get $\frac{|1-cosn-isinn|}{2sin\frac{1}{2}}\le\frac{1+|cosn+isinn|}{2sin\frac{1}{2}}$

$= \frac{2}{2\sin\frac{1}{2}} = \frac{1}{\sin\frac{1}{2}}$

Theorem 2.3.2 $cos^n\theta = \frac{1}{2^n}\sum_{k=0}^{n} C_n^k \, cos(n-2k)\theta \, , n \in \boldsymbol{N}.$

Proof:

By the equation:

$$e^{i\theta} = cos\,\theta + i\,sin\,\theta$$

$$cos\,\theta = \frac{e^{i\theta} - e^{-i\theta}}{2}$$

we can get that:

$$cos^n\theta = \left(\frac{e^{i\theta} - e^{-i\theta}}{2}\right)^n$$

And also, because of:

$$(x+y)^n = \sum_{k=0}^{n} C_k^n x^{n-k} y^k$$

We can get that:

$$cos^n\theta = \left(\frac{e^{i\theta} - e^{-i\theta}}{2}\right)^n = \frac{1}{2^n}\sum_{k=0}^{n} C_k^n \left(e^{i\theta}\right)^{n-k}\left(e^{-i\theta}\right)^k$$

And then it is easy to see that:

$$cos^n\theta = \left(\frac{e^{i\theta} - e^{-i\theta}}{2}\right)^n$$
$$= \frac{1}{2^n}\sum_{k=0}^{n} C_k^n \left(e^{i\theta}\right)^{n-k}\left(e^{-i\theta}\right)^k = \frac{1}{2^n}\sum_{k=0}^{n} C_n^k e^{i(n-2k)\theta} = \frac{1}{2^n}\sum_{k=o}^{n} C_n^k \, cos(n-2k)\theta \; .$$

Theorem 2.3.3 Let $n \in \mathbf{N}, 0 < r \in \mathbf{R}$. Then, then the equation $x^{n+1} + rx^n - r^{n+1} = 0$ has no complex root whose modulo is r.

Proof: Suppose t is the solution of the equation $x^{n+1} + rx^n - r^{n+1} = 0$, and its modulus is r. Then,

$$t^n(t+r) = r^{n+1} \tag{1}$$

Computing modulus on both sides of equation (1), it follows that

$$|t|^n|t+r| = |r|^{n+1}.$$

Notice that $|t| = r$. It follows that $|t+r| = r$.

Thus, the modulus of z simultaneously satisfies two conditions that $|t| = r$ and $|t+r| = r$ Geometrically, the root z can be viewed as the intersection of the circle $|z| = r$ and the circle $|z+r| = r$. As a result, we can obtain two roots that both satisfying $|z| = r$ and $|z+r| = r$

$$z_1 = r\left(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right), z_2 = r\left(\cos\frac{4\pi}{3} + i\sin\frac{4\pi}{3}\right).$$

substituting z_1 into (1) and after simi

After subsitituting z_1 and simplification, the following result can be obtained

$$\cos\frac{(2n+1)\pi}{3} + i\sin\frac{(2n+1)\pi}{3} = 1.$$

By comparing the real part and the imaginary part of the same complex number, the following results can be obtained

$$\cos\frac{(2n+1)\pi}{3}=1 \text{ and } \sin\frac{(2n+1)\pi}{3}=0.$$

$\therefore \frac{(2n+1)\pi}{3}=2k\pi(k\in\mathbf{Z})$, that is $2n+1=6k(k,n\in\mathbf{Z})$, which is a contradiction.

Similarly, it is easy to check z_2 is also not the root.

To conclude, the equation $x^{n+1}+rx^n-r^{n+1}=0$ does not have a complex root whose module is r.

Theorem 2.3.4([10])

$$\binom{n}{1}+\binom{n}{5}+\cdots+\binom{n}{4m-3}=\frac{1}{2}\left(2^{n-1}+2^{\frac{\pi}{2}}\cdot sin\left(\frac{n\pi}{4}\right)\right),$$

where $4m-3$ is the largest integer no bigger than n and where $\binom{n}{k}=\frac{n!}{k!(n-k)!}$.

Proof By binomial theorem, we have the following equities:

$$(1+1)^n=\binom{n}{0}+\binom{n}{1}+\cdots+\binom{n}{n}, \tag{2}$$

$$(1+i)^n=\binom{n}{0}+\binom{n}{1}i+\cdots+\binom{n}{n}i^n, \tag{3}$$

$$(1+i^2)^n=\binom{n}{0}+\binom{n}{1}i^2+\cdots+\binom{n}{n}i^{2n}, \tag{4}$$

$$(1+i^3)^n=\binom{n}{0}+\binom{n}{1}i^3+\cdots+\binom{n}{n}i^{3n}. \tag{5}$$

$(2)+\varepsilon_1^3\cdot(3)+\varepsilon_1^2\cdot(4)+\varepsilon_1\cdot(5)$, it is easy to obtain that

the right-hand side is equal to $4\left(\binom{n}{1}+\binom{n}{5}+\cdots+\binom{n}{4m-3}\right)$,

the left-hand side is equal to $2^n+(1+i)^n\cdot i^3+(1+i^2)^n\cdot i^2+(1+i^3)^n\cdot i$

$$=2^n+i\left((1-i)^n-(1+i)^n\right)=2^n+2^{\frac{\pi}{2}+1}\sin\frac{n\pi}{4}.$$

Thus, $\binom{n}{1}+\binom{n}{5}+\cdots+\binom{n}{4m-3}=\frac{1}{2}\left(2^{n-1}+2^{\frac{\pi}{2}}\cdot sin\left(\frac{n\pi}{4}\right)\right)$.

3. Conclusion

Despite being an extension of the real number system, complex numbers are mainly self-contained. The geometric meaning and algebraic structure of complex numbers can help us solve a variety of geometric problems. This paper summarizes the basic knowledge of complex numbers and gives several fascinating results proved by the geometric and algebraic properties of complex numbers. In the future, we will report more on holomorphic functions. We will show how to use knowledge of the theory of holomorphic functions to prove more complicated identities.

References

[1] B. Blank, An Imaginary Tale Book Review, in Notices of the AMS Volume 46, Number 10, November 1999, pp. 1233-1236.

[2] M. Crowe, A History of Vector Analysis, U. of Notre Dame Press, Notre Dame, 1967.

[3] S. Lang. Complex Analysis. Springer-Verlag, New York, fourth edition, 1999.

[4] B. L. van der Waerden, A History of Algebra, Springer Verlag, NY 1985.

[5] S. Saks and Z. Zygmund. Analytic Functions. Elsevier, PWN-Polish Scientific, third edition, 1971.

[6] E. C. Titchmarsh. The Theory of Functions. Oxford University Press, London, second edition, 1939.

[7] E.T. Copson. Asymptotic Expansions, volume 55 of Cambridge Tracts in Math. and Math Physics. Cambridge University Press, 1965.

[8] P. L. Duren. Univalent Functions. Springer-Verlag, New York, 1983.

[9] A. Erdélyi. Asymptotic Expansions. Dover, New York, 1956.

[10] E.M. Stein and R. Shakarchi, Complex Analysis, Princeton University Press, 2003.