Nonexistence of Surjective Lie Homomorphism Between Complex Special and General Linear Groups

Jiajun Fu*

Department of math, Duke University (Kunshan), Suzhou, China

*Corresponding author: jf348@duke.edu

Abstract. Lie groups are very important objects in algebra, topology and geometry, as it is naturally endowed with two special structures: the algebraic structure of group and the geometric structure of differential manifold. Hence, it is quite meaningful to better know the structure of Lie groups and the actions, and the transformations between them. Two specific theorems about Lie groups $GL(n;\mathbb{C})$, $SL(n;\mathbb{C})$ and their homomorphism and Lie homomorphism are proved in this paper. First, the fact that there does not exist a surjective Lie group homomorphism between $GL(n;\mathbb{C})$ and $SL(n;\mathbb{C})$ is introduced. This is done by first having a trial of constructing Lie group surjective homomorphism but fails. Next, this result is further proved by applying topological method that to construct a covering map between $PGL(n;\mathbb{C})$ and $SL(n;\mathbb{C})$, followed by computing the fundamental group of $PGL(n;\mathbb{C})$. Finally, by applying that $PSL(n;\mathbb{C})$ is a simple group, the fact that there does not exist a surjective group homomorphism between $SL(n;\mathbb{C})$ and $GL(n;\mathbb{C})$ is demonstrated, together with the statement that $PSL(n;F)$ is a simple group.

Keywords: Lie homomorphism, Lie group, Classical group, Fiber bundle.

1. Introduction

Group theory was first developed by the study of Galois. Later on, the Norwegian mathematician Lie found the special structure or Lie groups, a kind of groups which is endowed with both the structure of groups and the structure of differential manifolds. Matrix Lie groups is a special kind of Lie groups, they are linear and elements in groups can be expressed as matrices. After defining the groups and Lie groups, it is natural to study their properties, especially about the general group homomorphism and Lie homomorphism. $GL(n;\mathbb{C})$ and $SL(n;\mathbb{C})$ are both classical groups which are matrix Lie groups and it is necessary to consider the Lie homomorphism and homomorphism between them. For the first problem, it is known that $SL(2;\mathbb{C})$ has a property that the only non-trivial normal subgroup of it is $\{\pm I\}$, which must experience the process of computing the matrix in the proof. Suppose that $g: G \longrightarrow K$ is a surjective group homomorphism, then the normal subgroup containing Kerg of G has an one-to-one correspondence to the normal subgroup in K. In addition, by using the surjective group homomorphism $g: GL(n;\mathbb{C}) \longrightarrow SL(n;\mathbb{C})$, then it is easy to think about considering the problem as the classification of Jordan normal form. Specifically, it is easier to talk about Jordan normal form on a general linear group which seems quite promising.

Moreover, the surjective group homomorphism seems trivial. For example, take $\mathrm{A} \longmapsto (\det A)^{-1}A$ but this is actually not true because it is not a homomorphism to a special linear group. Also, taking the square root of a complex number will destroy its multiplicative structure. Actually, there doesn't exist such a group homomorphism if it requires that the homomorphism is continuous under the natural topology of two linear groups. The next key point is to compute the fundamental groups that obey formulas $\pi_1(GL(n;\mathbb{C})) = \mathbb{Z}$, $\pi_1(SL(n;\mathbb{C})) = 0$ and $|\pi_1(PGL(n;\mathbb{C}))| = |\pi_1(PSL(n;\mathbb{C}))| = n$ by constructing fiber bundles and long exact sequences. For the second problem, on the basis of the proof of the first problem, it further has to prove that $PSL(n;\mathbb{C})$ is a simple group. This would be difficult, but there has already been a proof of more general case where $PSL(n;F)$ is simple. This result is a wonderful one in the study of finite groups theory. The proof of this result will involve Iwaswa's criterion and will involve multiple of computation of matrices. The specific result would not be given, but the most general ideas and the main steps in the proof will be highly mentioned throughout this paper.

2. The Lie Homomorphism Situation

Lemma 2.1. Let $g: G \longrightarrow K$ is a surjective group homomorphism, then the normal subgroup containing Kerg of G has an one-to-one correspondence to the normal subgroups in K .

Lemma 2.2. The center of $SL(n;\mathbb{C})$ is $\{\lambda I: \lambda^n = 1, \lambda \in \mathbb{C}\} \cong \mathbb{Z}/n\mathbb{Z}$.

Proof. First, it is useful to note that

$$\{\lambda I: \lambda^n = 1, \lambda \in \mathbb{C}\} \cong \frac{\mathbb{Z}}{n\mathbb{Z}} \triangleleft Z\big(SL(n;\mathbb{C})\big). \tag{1}$$

Let E_{ij} be the matrix whose places are 0 except 1 at (i,j), one has $\det\big(I + E_{ij}\big) = 1 + \delta_{ij}$. Also, if $A \in Z(SL(n;\mathbb{C}))$ then $\big(I + E_{ij}\big)A = A\big(I + E_{ij}\big)(i \neq j)$. It follows that

$$A \in \{\lambda I: \lambda^n = 1, \lambda \in \mathbb{C}\}]. \tag{2}$$

Hence $Z(SL(n;\mathbb{C})) = \{\lambda I: \lambda^n = 1, \lambda \in \mathbb{C}\} \cong \mathbb{Z}/n\mathbb{Z}$.

Lemma 2.3. $PGL(n;\mathbb{C}) \cong PSL(n;\mathbb{C})$.

Proof. As $SL(n;\mathbb{C})$ is the normal subgroup of $GL(n;\mathbb{C})$ and take any $A \in GL(n;\mathbb{C})$. Suppose that $\lambda \in \mathbb{C}$ satisfies that $\lambda^n = \det A$, then

$$A = \lambda^{-1}A\lambda I \in SL(n;\mathbb{C}) \lhd Z\big(GL(n;\mathbb{C})\big). \tag{3}$$

Besides, it is not difficult to find that the following relations

$$GL(n;\mathbb{C})/Z\big(GL(n;\mathbb{C})\big) \cong SL(n;\mathbb{C})/SL(n;\mathbb{C}) \cap Z\big(GL(n;\mathbb{C})\big), \tag{4}$$

and it is equal to $SL(n;\mathbb{C})/Z(SL(n;\mathbb{C}))$.

Obviously, this also tells a fact that they are homeomorphism as Lie groups. What's following is a topological lemma which can be used to compute the fundamental groups of these classical groups.

Lemma 2.4. Let X be the fiber bundle on $B, p: X \longrightarrow B$, and the fiber is F, they should have the long exact sequence: $\cdots \longrightarrow \pi_{n+1}(B) \longrightarrow \pi_n(F) \longrightarrow \pi_n(X) \longrightarrow \pi_n(B) \longrightarrow \pi_{n-1}(F) \longrightarrow \cdots$.

Next the paper will do the computation of the fundamental groups.

Lemma 2.5. $\pi_1(GL(n;\mathbb{C})) = \mathbb{Z}(n \geq 1), \pi_1(SL(n;\mathbb{C})) = 0(n \geq 2)$ and $|\pi_1(PGL(n;\mathbb{C}))| = |\pi_1(PSL(n;\mathbb{C}))| = n(n \geq 2)$.

Proof. Over $\mathbb{C}^n$, because that any two linear independent unit vectors can be extended to two groups of orthogonal basis, the transformation matrix between them is unitary matrix. Hence, on $\{v \in \mathbb{C}^n: |v| = 1\} \approx S^{2n-1}$, the $U(n)$ has transforming action. Consider $(1,0,\cdots,0)^T \in \{v \in \mathbb{C}^n: |v| = 1\}$, the stable subgroup of it is the combination of all unitary matrices whose $(1,1)$ are 1 and $(1,j)$ and $(j,1)$ are 0, isomorphic to $U(n-1)$. So there is fiber bundle $p: U(n) \longrightarrow S^{2n-1}$ whose fiber is $U(n-1)$. Hence by *Lemma* 2.4, it is naturally to have

$$\pi_2(S^{2n-1}) \longrightarrow \pi_1\big(U(n-1)\big) \longrightarrow \pi_1\big(U(n)\big) \longrightarrow \pi_1(S^{2n-1}). \tag{5}$$

Hence for $n \geq 2$, (as $\pi_2(S^{2n-1}) = 0$ and $\pi_1(S^{2n-1}) = 0$), one has $\pi_1\big(U(n)\big) = \pi_1\big(U(n-1)\big)$. It follows that

$$\pi_1\big(U(n)\big) \cong \pi_1\big(U(1)\big) \cong \pi_1(S^1) = \mathbb{Z}. \tag{6}$$

Because any loop over $GL(n;\mathbb{C})$ is homotopic to the loop in $U(n)$, which is guaranteed by Schmidt orthogonalization on row vectors. So, $\pi_1\big(GL(n;\mathbb{C})\big) = \mathbb{Z}, (n \geq 1)$. While for the $SL(n;\mathbb{C})$, similarly, one has fiber bundle $p: SU(n) \longrightarrow S^{2n-1}$ whose fiber is $SU(n-1)$, resulting

$$\pi_1\big(SU(n)\big) \cong \pi_1\big(SU(2)\big) \cong \pi_1(S^3) = 0(n \geq 2). \tag{7}$$

Similarly, for any loop in $SL(n;\mathbb{C})$ it is homotopic to the loop in $SU(n)$, namely $\pi_1\big(SL(n;\mathbb{C})\big) = 0(n \geq 2)$. By *Lemma 2*.2, it is naturally to have that

$$\big|\pi_1\big(PGL(n;\mathbb{C})\big)\big| = \big|\pi_1\big(PSL(n;\mathbb{C})\big)\big| = n(n \geq 2). \tag{8}$$

and there is fiber bundle $p: GL(n;\mathbb{C}) \longrightarrow PGL(n;\mathbb{C})$ whose fiber is S^1. Hence, it immediately follows

$$\pi_1(S^1) \longrightarrow \pi_1\big(GL(n;\mathbb{Z})\big) \longrightarrow \pi_1\big(PGL(n;\mathbb{C})\big) \longrightarrow \pi_0(S^1). \tag{9}$$

Taken together, $\pi_1(PGL(n;\mathbb{C})) = \pi_1(PSL(n;\mathbb{C})) = \mathbb{Z}/n\mathbb{Z}$.

Proposition 2.1. There does not exist a continuous surjective homomorphism from $GL(n;\mathbb{C})$ to $SL(n;\mathbb{C})$. $(n \geq 2)$.

Proof. Let $\phi: GL(n;\mathbb{C}) \longrightarrow SL(n;\mathbb{C})$ be continuous surjective Lie homomorphism, then $\operatorname{Ker}\phi = \phi^{-1}(I)$ is the closed normal subgroup of $GL(n;\mathbb{C})$, say K. Then Lie (K) would be the real Lie ideal of $\mathfrak{gl}(n,\mathbb{C})$. Besides, the $GL(n;\mathbb{C})$ is the real analytic principal bundle of $SL(n;\mathbb{C})$, and the structure group is K. So for a sufficiently small neighbourhood V of $SL(n;\mathbb{C})$ at its unit, naturally one has $\phi^{-1}(V) \approx V \times K$. Hence in the mean of a real vector space,

$$\mathfrak{gl}(n;\mathbb{C}) = \mathfrak{sl}(n;\mathbb{C}) \oplus \operatorname{Lie}(K) \tag{10}$$

with $\dim_{\mathbb{R}} \operatorname{Lie}(K) = 2$.

Because ϕ is a continuous map, so the image of the connected set $Z(GL(n;\mathbb{C}))$ (denoted as $\mathbb{C}^*$) is connected, so $\mathbb{C}^* \lhd K$. In addition, since $\dim_{\mathbb{R}} \mathbb{C}^* = \dim_{\mathbb{R}} K = 2$, the unit connected component of K is equal to $\mathbb{C}^*$ and K is the at most countable no-intersection union of the cosets of $\mathbb{C}^*$. This is because that K is the Lie subgroup of $GL(n;\mathbb{C})$, which is a analytic manifold and all topological manifolds are second countable. That is to say:

$$\varphi: PGL(n;\mathbb{C}) \longrightarrow SL(n;\mathbb{C}): A\mathbb{C}^* \longmapsto AK \tag{11}$$

is a covering map of Lie groups. But noticing that $SL(n;\mathbb{C})$ is simply connected while $\pi_1(PGL(n;\mathbb{C})) = \mathbb{Z}/n\mathbb{Z}$, then the contradiction can be seen clearly.

3. The situation of group homomorphism

Basically, the homomorphisms that can be thought of in a short time are those who are Lie homomorphism. While for those who are not Lie homomorphism, the map (if actually exist) can probably only be constructed by some extremely tricky skills and may appear in a very ugly form. So naturally one should involve some new tools to handle this problem. Actually, the key point of solving this problem is to prove a very important fact that $PSL(n;\mathbb{C})$ is a simple group. The following are two basic theorems which are necessary to demonstrate this statement.

Theorem 3.1. $PSL(n;\mathbb{C})$ is simple.

The theorem can be generalized to a more general statement as shown below.

Theorem 3.2. $PSL(n;F)$ is simple when $n > 2$.

Proof. To prove this result, a criterion by Iwasawa in 1941 which relates simple quotient groups and doubly transitive group actions, will be utilized. Let G acts doubly transitively on a set X, and assume that: for some $x \in X$ the group $Stab_x$ has an abelian normal subgroup whose conjugate subgroups generate G: $[G,G] = G$. Consequently, G/K is a simple group, where K is the kernel of the action of G on X. To prove PSL $(n;F)$ is simple, it is required to study the action of $SL(n;F)$ on the linear subgroup of F^n which is the projective space $\mathbb{P}^{n-1}(F)$. The action of $SL(n;F)$ on $\mathbb{P}^{n-1}(F)$ is doubly transitive with kernel equal to the center of the group and the stabilizer of some point has a normal subgroup.

To conclude, by Iwasawa's theorem that PSL $(n;F)$ is simple, it remains to show that the subgroups of $SL(n;F)$ that are conjugate to U generate $SL(n;F)$ and $[SL(n;F), SL(n;F)] = SL(n;F)$. This will follow from a study of the elementary matrices $I_n + \lambda E_{ij}$ where $i \neq j$ and $\lambda \in F^{\times}$. The matrix $I_n + \lambda E_{ij}$ has 1's on the diagonal and a λ in the (i,j) position. Therefore its determinant is 1, so such matrices are in $SL(n;F)$. An example of such matrix in U could be

$$I_n + E_{12} = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & I_{n-2} \end{pmatrix}. \tag{12}$$

By some tedious computation, one can get two properties that are about $I_n + \lambda E_{ij}$. For $n > 2$, each $I_n + \lambda E_{ij}$ is conjugate in $SL(n;F)$ to $I_n + E_{ij}$. While for $n > 2$, the matrices $I_n + \lambda E_{ij}$ generate $SL(n;F)$. There properties imply that the conjugates of $I_n + \lambda E_{ij}$ generate $SL(n;F)$. Since $I_n + E_{12} \in U$, the subgroups of $SL(n;F)$ are conjugate to U generate $SL(n;F)$. Finally, the theorem that for $n > 2$, $[SL(n;F), SL(n;F)] = SL(n;F)$ would complete the proof that PSL $(n;F)$ is simple for $n > 2$.

Proposition 2.1. There does not exist a surjective group homomorphism from $GL(n;\mathbb{C})$ to $SL(n;\mathbb{C})$.

Proof. Let g be the surjective group homomorphism from $GL(n;\mathbb{C})$ to $SL(n;\mathbb{C})$. Denote $K = Kerg$, then because

$$g\left(Z\big(GL(n;\mathbb{C})\big)\right) \lhd Z\big(SL(n;\mathbb{C})\big) \cong \mathbb{Z}/n\mathbb{Z} \tag{13}$$

and $Z\big(GL(n;\mathbb{C})\big) = \mathbb{C}^*$ does not exist non-trivial finite-order subgroup, so $\mathbb{C}^* \lhd K$. Since under g, the normal subgroup containing K of G corresponds to the normal subgroups in $SL(n;\mathbb{C})$. Hence, it can be concluded that

$$\mathbb{C}^* \lhd K \lhd C \lhd GL(n;\mathbb{C}), \tag{14}$$

which is subject to $g(C) = Z\big(SL(n;\mathbb{C})\big)$. Therefore, as $GL(n;\mathbb{C})/\mathbb{C}^* = PGL(n;\mathbb{C}), C/\mathbb{C}^*$ is the non-trivial normal subgroup of $PGL(n;c) \cong PSL(n;\mathbb{C})$, it is contrary to the simplicity of $PSL(n;\mathbb{C})$, hence completing the proof.

4. Conclusion

Generally, this paper has proved that the nonexistence of two surjective Lie homomorphism and general group morphism, between the complex coefficients general linear group and special linear group, which are both classical Lie groups. On the algebraic point of view, the paper has tried to construct the kind of morphism by some set theory or algebraic skills from the natural structure of linear Lie groups. One of the most important things is the involvement of the methods from algebraic topology. Assuming that the Lie groups serve as manifolds which are endowed with such properties, then naturally one can construct fiber bundle on that. On the fiber bundle, there are lots of useful tools like homotopic exact sequences that can help people better understand the topological structure of the groups themselves. Another important thing in this paper is about $PSL(n;F)$, whose simplicity is necessary in the proof of the problem. Moreover, this paper briefly covers and introduces the main body of the original idea from Iwsawa and proves that $PSL(n;F)$ is a simple group. Although this result comes from a simple observation, it may be useful in some fields like finite groups, number theory, and topological K theory.

References

[1] Wang efang. Fundamentals of finite group theory. Tsinghua University Press, 2002.

[2] Hall Brian. C. Lie Groups, Lie Algebras, and Representations: An Elementary Introduction, Springer-Verlag, 1997.

[3] Hatcher A. Algebraic Topology. Cambridge University Press, 2002.

[4] Keth Conrad. https://kconrad.math.uconn.edu/blurbs/grouptheory/PSLnsimple.pdf, 2017.

[5] Yu Demin, Zhang Zaiyun, Zhou Liren, Gan Xiangyang. Isomorphisms and Morphism of a Infinite Dimension Allie Algebra. Journal of Qinghai Normal University (Natural Science), 2009, 12(2): 4-10.

[6] Zhang Wei, Guo Jidong, Zhang Liang. The number of endomorphisms and automorphisms of a class of non-abelian groups. Journal of Shandong University (Natural Science), 2023, 58(2): 1-7.

[7] Varadaeajan V. S. Lie Groups, Lie algebras, and Their Representations. Springer-Verlag, 1997.

[8] Wilson Robert A. The Finite Simple Groups. Springer-Verlag, 1997.

[9] Hu Yanxin, Guo Zengxin, Xin Xiangpeng. Lie Group Analysis, Optimal System, Symmetry Reduction and Exact Solution of Generalized Dispersive Equation. Journal of Sichuan Normal University (Natural Science), 2023, 46(1): 66-71.

[10] Chas M., Kabiraj A. The Lie bracket of undirected closed curves on a surface. Trans. Am. Math. Soc., 2022, 375(4): 2365-2386.