# Homomorphism, Isomorphism, and Their Applications in Group Theory

Baixiang Cheng*

Department of Mathematics, Applied Mathematics, and Statistics, Case Western Reserve University, Cleveland, Ohio, USA

*Corresponding author: bxc468@case.edu

**Abstract.** The main study of modern algebra is the so-called algebraic systems, i.e., sets with operations. Modern algebra has important applications in other branches of mathematics and natural science. In modern algebra, homomorphism and isomorphism are relatively elementary but extremely important concepts that are interrelated and different. A homomorphism is a mapping that preserves the constitution of a mathematical format and is a generalization of a homomorphism. it is a structurally invariant mapping between two algebraic structures in abstract algebra. The conditions for a homomorphism to be a homomorphism in different algebraic systems are different, and the conditions for a homomorphism to be a homomorphism are described here. Some applications of homomorphisms and homomorphisms to different algebraic systems are discussed, from which the importance of homomorphisms and homomorphisms is illustrated.

**Keywords:** Group theory, Homomorphism, Isomorphism, Group homomorphism.

## 1. Introduction

In the 19th century, many new mathematics theories were born, including group, ring, domain, pattern, and Riemannian geometry. The notion of the group originated from the research of polynomial equations pioneered by E. Galois in the 1830s. After receiving contributions from other fields, such as number theory and geometry, the group concept was established around 1870 [1]. To study groups, mathematicians have invented diverse concepts to decompose groups into smaller and better-understood parts. For instance, quotient groups, permutation groups, and cyclic groups.

The concept of the group has been universally recognized as one of the most basic concepts in mathematics, and it has many applications. It plays an essential role by pervading branches of mathematics such as topology and functional theory. Regarding the scientific content, group theory belongs to mathematics and has applications in many branches of mathematics. In abstract algebra, group theory research is known as the algebraic structure of groups. Groups are fundamental in abstract algebra. The notion of groups occurs in a lot of branches of math, and the group-theoretic method has made a major impact on other related topics.

Homomorphism and isomorphism are relatively elementary but critical fundamental theories of mathematics, which are related but different [2]. A homomorphism is mapping one mathematical set to another set or to itself in such a way that the result obtained from an operation on the elements of one group is mapped to the result obtained from a corresponding operation on the related images in the other set. In abstract algebra, isomorphism is a bijection in the structure that doesn’t change [3]. In general category theory, isomorphism involves the existence of a state projection and another state projection such that the composite of the two is a constant state projection [4]. An isomorphism is a class of mappings defined between mathematical objects that reveal the relationship between the properties or operations of these objects [5]. This article will cover the definition of isomorphism and homomorphism together with the fundamental theorem, the definition of group homomorphism, and the fundamental theorem, semigroups, fuzzy semigroups, and fuzzy quotient groups. Homomorphism is a mapping of algebraic structures. One part of this paper introduces the definition of homomorphism, the fundamental theorem, and some corollaries. Another part presents the applications of homomorphism in mathematics, including semigroups, fuzzy subgroups, regular subgroups, and fuzzy quotient groups. Some applications of homomorphism and isomorphism in algebra are also discussed. It can be seen that homomorphism has an essential place in group theory and is crucial for developing the group theory branch of mathematics.

# 2. Homomorphism and Isomorphism

## 2.1. Fundamental Theorems

*Theorem 2.1*. Let $\varphi$ be the group homomorphism in group $G$ to that of $\overline{G}$. The set of all original images of the unit element of $\overline{G}$ under $\varphi$, called the kernel of $\varphi$, is denoted as $Ker\varphi$. Then the set of images of all elements in $G$ under $\varphi$, called the set of images of $\varphi$, that is denoted as $Im\varphi$, or $\varphi(G)$ [6].

*Proof*. Since the homomorphism of the group preserves the sub-structure known before, $Im\varphi$ is a subgroup belonging to $\overline{G}$. Structure of the normal subgroup is preserved in the full group homomorphism. In particular, $\varphi$ is a full homomorphism from $G$ to $Im\varphi$, and so pulls the mundane regular subgroup $\{\overline{e}\}$ back to the regular subgroup $Ker\varphi$. In addition, a natural full homomorphism exists between the group and its quotient group, called the natural homomorphism.

*Theorem 2.2*. Let φ be a full homomorphism of group $G$ to that of $G$. Then define $N = Ker\varphi \trianglelefteq G$ and $G/N \cong \overline{G}$. If $\varphi$ is just a homomorphism, then the above theorem can be rewritten as the following expression $G/Ker\varphi \cong Im\varphi$.

*Proof*. If the bijection $\sigma$ is reasonable. Suppose $aN = bN$, then $a^{-1}b \in N$. So, $\varphi(a^{-1}b) = \overline{e}$, and $\overline{a}^{-1}\overline{b} = \varphi(a^{-1})\varphi(b) = \varphi(a^{-1}b) = \overline{a^{-1}b} = \overline{e}$. Then $\overline{a} = \overline{b}$, which is $\sigma(a) = \sigma(b)$.

Suppose that the bijection $\sigma$ is surjection. Due to $\varphi$ is epimorphism, $\forall\overline{a} \in \overline{G}$, $\exists a \in G$, $\overline{a} = \varphi(a)$, then, $\sigma(aN) = \overline{a}$.

Suppose that the bijection $\sigma$ is injection. if $aN \neq bN$, then $a^{-1}b \notin N$, $a^{-1}b \notin e$. As a result, one can get $\overline{a}^{-1}\overline{b} = \overline{a^{-1}b} \notin \overline{e}$, and $\sigma$ is injection.

Suppose that bijection $\sigma$ is homomorphism. It is readily to find that $\sigma(aN \cdot bN) = \sigma(abN) = \overline{ab} = \varphi(ab) = \varphi(a)\varphi(b) = \overline{a} \cdot \overline{b}$

Taken together, $\sigma$ is homomorphism, $G/N \cong \overline{G}$.

*Theorem 2.3*. Set $G$ and $\overline{G}$ as two groups and $G \sim \overline{G}$. Suppose that $G$ is a cyclic group, hence $\overline{G}$ is also a cyclic group. The homomorphic picture of a cyclic group should be cyclic as well.

*Proof*. This theorem only needs to prove that the generating element $a$ of the cyclic group $G$ is like the generating component of the group $G$ full homomorphic image $\overline{G}$.

## 2.2. Corollary

*Corollary* 1: Let $G$ and $\overline{G}$ be two finite groups. If $G \sim \overline{G}$, then $|\overline{G}|$divides $|G|$.

*Corollary* 2: The quotient group of a cyclic group is also a cyclic group.

# 3. Group homomorphism

## 3.1. Theorems

*Theorem 3.1*. Let $\varphi$ be a full homomorphism which connects the group $G$ to that of $\overline{G}$, $Ker\varphi \subseteq N \trianglelefteq G$. $\overline{N} = \varphi(N)$. Then $G/N \cong \overline{G}/\overline{N}$ [7].

*Proof*. First of all, $N \trianglelefteq G$ and $\varphi$ is epimorphism implies that $\overline{N} = \varphi(N) \trianglelefteq \overline{G}$. Define $\tau: G/N \to \overline{G}/\overline{N}$, $\tau(xN) = \varphi(x)\overline{N}$, $\forall x \in G$. The following four cases should be considered individually.

• $\tau$ is bijection: $aN = bN \Rightarrow a^{-1}b \in N$. Therefore, $\tau(a^{-1}b) = \tau(a)^{-1}\tau(b) \in \overline{N} \Rightarrow \tau(a)\overline{N} = \tau(b)\overline{N}$.

•$\tau$ is surjection: $\forall \overline{a}\overline{N} \in \overline{G}/\overline{N}$, $\overline{a} \in \overline{G}$, $\exists a$ make $\varphi(a) = \overline{a}$, then, $\tau(aN) = \varphi(a)\overline{N} = \overline{a}\overline{N}$.

•$\tau$ is injection: Suppose $\tau(aN) = \tau(bN) \Rightarrow \varphi(a)\overline{N} = \varphi(b)\overline{N}$, then $\varphi(a^{-1}b) = \varphi(a)^{-1}\varphi(b) \in \overline{N} = \varphi(N)$.Besides, for $\exists c \in N$ and $\varphi(a^{-1}b) = \varphi(c)$, $\varphi(c^{-1}a^{-1}b) = \overline{e}$. Taken together, one has $c^{-1}a^{-1}b \in Ker\varphi \subseteq N. \Rightarrow a^{-1}b \in N \Rightarrow aN = bN$.

•$\tau$ is homomorphism: $\tau(aNbN) = \tau(abN) = \varphi(ab)\overline{N} = \varphi(a)\varphi(b)\overline{N} = \varphi(a)\overline{N}\varphi(b)\overline{N} = \tau(aN)\tau(bN)$.

*Theorem 3.2*. Let $G$ be a group, $H \leq G$, and $N \trianglelefteq G$. It is found that $H \cap N \trianglelefteq H$ as well as $HN/N \cong H/(H \cap \mathrm{N})$.

*Proof*. $\varphi$: $h \to hN(h \in H)$ can be treated as a full homomorphism resulting from $H$ to $HN/H$. By using $Ker\varphi = H \cap K$ and the first isomorphism theorem mentioned earlier are sufficient to prove the statement.

*Theorem 3.3*. Suppose that $G$ is a group, $N \trianglelefteq G$ and $\overline{H} \leq G/N$ $\overline{H} \leq G/N$, then (1) there is a unique subgroup $H \supseteq N$ belonging to $G$ with $\overline{H} = H/N$ ; (2) when $\overline{H} \trianglelefteq G/N$, one has a unique $H \trianglelefteq G$ so as to $\overline{H} = H/N$ and $G/H \cong (G/N)/(H/N)$.

*Proof*. Using the subgroup correspondence theorem and group homomorphism preserving subgroups, the full homomorphism preserving regular subgroups is established.

### 3.2. Corollary

*Corollary* 1: Let $H, N$ be two regular subgroups belonging to a group $G$ and $N \subseteq H$. Then $G/H \cong (G/H)/(H/N)$.

*Corollary* 2: Let $H, K$ be two regular subgroups belonging to a group $G$. It can be inferred that $G/HK \cong (G/H)/(HK/N)$.

## 4. Application

### 4.1. Semigroups and Fuzzy Sub-semigroup

In algebra, the semigroup is a special form in which the binary operator is involved. It does not necessarily have an identity element and conversely, its elements do not necessarily have inverses in this semigroup [8]. Nevertheless, semigroup $S$ can have an identity element and there is $e \in S$, $ex = x$, $xe = x$, where $\forall\, x \in S$. When there is no element in a semigroup, it is termed empty semigroup. It can be proved that the amount of non-isomorphic semigroups of order 1 is 1, of order 2 is 5, of order 3 is 24, and so forth. Taking the order 2 case as an example, there are sixteen distinct mappings from $\{a, b\} \times \{a, b\} \to \{a, b\}$. They form ten equivalent classes of non-isomorphic binary operations, among which 5 of them are associative.

On the other hand, semigroup is an easy algebraic form since it has solely one binary operation called the associative. The set $A \subseteq S$ is called a sub-semigroup if $A$ is a semigroup and follows a same binary operator in $S$. Suppose that $S$ is a semigroup and define projection $a: S \longrightarrow [0,1]$ as a fuzzy sub-semigroup as long as $a(xy) \geq \min\{a(x), a(y)\}$ for $\forall x, y \in S$. Relationships among semigroups, fuzzy sub-semigroups, and others can be found in literature [9].

### 4.2. Fuzzy Quotient Groups

The homomorphism of fuzzy subgroups is well established [10]. The properties of homomorphism originate from abstract algebra, specifically a mapping between two algebraic structures that remain structure invariant. A fuzzy quotient subgroup is constructed while group homomorphism theorem has been proved. If $f$ is the homomorphic full projection of group $X$ to group $X'$, then:

•If $H$ is a fuzzy subgroup of $X$, then $f\,(H)$ is a fuzzy subgroup of $X'$.

•If $N$ is a fuzzy normal subgroup of $X$, then $f\,(N)$ is a fuzzy normal subgroup of $X'$ with $X/N \sim X'/f(N)$.

•If $H'$ is a fuzzy subgroup of $X'$, then $f^{-1}(H')$ is a fuzzy subgroup of $X$.

• If $N'$ is a fuzzy normal subgroup of $X$, then $f^{-1}(N')$ is a fuzzy normal subgroup of $X$ with $X/f^{-1}(N') \sim X'/N'$.

• If $A$ is a fuzzy $S-$fitting normal subgroup of $G$, $K$ and $N$ are fuzzy normal subgroups of $G$ with $K \subseteq N \subseteq A$. Hence, $A/N$ is fuzzy isomorphic to $(A/K)/(N/K)$, $A/N \cong (A/K)/(N/K)$.

*Remark*. Since $K$ is a fuzzy normal subgroup of $G$, then $A/K$ and $N/K$ are meaningful. Since $K \subseteq N \subseteq A$, then $N/K$ is a fuzzy regular subgroup of $A/K$. likewise, $(A/K)/(N/K)$ is meaningful and can be determined from the set of fuzzy quotient groups by the set of fuzzy quotients. Namely, $\cdots \leq (A/K)_\lambda/(N/K) \leq (A/K)_\mu/(N/K) \leq \cdots \leq (A/K)_\omega/(N/K) \leq \cdots$, $\forall\lambda \in [0,1]$, $(A/K)_\lambda = A_\lambda/K$, and $(A/K)_\lambda = A_\lambda/K$. $A_\lambda/K \cong A_\lambda/K_e$. Here, $K_e$ is the kernel of $K$),$\forall\lambda \in [0,1]$, and $(A/K)_\lambda/(N/K) \cong (A_\lambda/K_e)/(N/K) \cong (A_\lambda/K_e)/(N/K)_e \cong (A_\lambda/K_e)/(N_e/K) \cong (A_\lambda/K_e)/(N_e/K_e) \cong A_\lambda/N_e$. Moreover, since $\forall\lambda \in [0,1]$, $(A/K)_\lambda = A_\lambda/K \cong A_\lambda/K_e$, then $\forall\lambda \in [0,1]$, $(A/K)_\lambda/(N/K) \cong A_\lambda/N_e$. That is, $A/N \cong (A/K)/(N/K)$.

## 5. Conclusion

The concepts of homomorphism and isomorphism are essential in group theory, and they have plenty of applications in several research areas. In this paper, the applications of homomorphic homomorphism in group theory and mathematics have been analyzed above. The critical role of homomorphic homomorphisms in theoretical and applied research is found. The definition and the fundamental theorem of a homomorphism can be seen in this paper, which is a mapping describing the degree of structural similarity between two homomorphic graph libraries. Isomorphism is a type of homomorphism, which is a bijective homomorphism. In applying homomorphic isomorphism, semigroups, fuzzy semigroups, fuzzy subgroups, fuzzy normal subgroups, and fuzzy quotient groups are studied analytically. These theories are the basis of mathematics, and similarly, the theory of homomorphic homomorphism is widely used among other disciplines. Thus, homomorphism, isomorphism, and group theory have a vital role in mathematics and other disciplines.

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