

Applying the Group Theory to Rubik's Cube

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Abstract. Gradually formed in the process of the study about the formula solution of the root of polynomial equations, group theory integrates some theoretical achievements in geometry and number theory, plays a fundamentally important role in abstract algebra, and is closely related to the intellectual toy named Rubik's cube. Many theorems and positions of it are vividly reflected and verified, and can be applied to the study a kind of problems about the Rubik's cube. These examples include the symmetry of Rubik's cube, the combination and permutation of the colors on Rubik's cube, and the God's number. Besides, various properties and proportions about Rubik's cube could also be analyzed and proved when the perspective of group theory is applied. This work demonstrates that group theory is a versatile method to solve many theoretical problems in nature and practical problems in life.

Keywords: Group theory, Rubik's cube, God's number, Symmetry.

1. Introduction

Group theory has three main historical sources: number theory, geometry, and the theory of algebraic equations. As one of the most important groups, permutation groups were first studied in the work of Lagrange and Abel and so forth on the general solutions of polynomial equations of high degree. The theory of algebraic equation has long been a basic subject to classical algebra, mainly including two problems, the existence of the root of equations, and the relationship between the expression of the root and the coefficients of the equation [1]. As for the first problem, Gauss proved the fundamental theorem of algebra in his doctoral dissertation in 1799, which reveals that any real coefficient polynomial equation has at least one root in the complex field. For the second one, it has been proved that polynomial equations of degree up to four can be solved exactly with algebraic methods by 16th century, which means the roots can be easily obtained by plugging the coefficients of the equation into the formula directly, such as $\frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ for quadratic equations. In view of this, mathematicians believed that there exist formulas providing the solutions to polynomial equations of degree higher than four, but none of them succeeded.

By 18th century, Lagrange began to realize that such formulas did not exist; Cauchy studied the theory of permutation groups systematically; and Abel proved the well-known Abel's theorem in 1824, by which he proved that polynomial equations of degree higher than five cannot be solved by exact formulas generally. Subsequently, based on the achievements of predecessors, Galois developed the idea of group in his study of the solvability of equations in 1930s, proposing a series of significant concepts of 'subgroup', 'maximal normal subgroup' and so on [2].

Integrating contributions from other fields such as number theory and geometry, the theory of group was firmly established around 1870 and quickly became a highly-profile subject in mathematics, recognized as one of the great mathematical achievements of the 19th century. It has by far opened numerous new areas of research and application, and it also has had a huge impact on the development of physics, chemistry, and even the emergence of structuralist philosophy in the 20th century. In recent years, group theory has penetrated into plenty of fields like computer science, cryptography, informatics and so on.

Group theory is a subject which studies numbers, patterns and symmetries. The Rubik's cube possesses in abundance. It is a kind of intellectual toy which was invented by Erno Rubik of Budapest in 1974 as illustrated in Fig. 1 [3]. The cube can be scrambled by rotating one of its six faces casually and consecutively, and the goal is to restore the cube to the solved position where each face only

possesses one same color. Although it seems that the gameplay of Rubik's cube has nothing to do with numbers, it's found that the structure and the law of rotation of the cube are the very reification of the theory of group.



Fig. 1 The left and right panels show the solved and mixed Rubik's cubes, respectively.

Rubik's cube is not merely a fun toy, but also a precise tool to demonstrate a deal of terminology of group theory, such as substitution, action, orbit, transitivity, originality, and homomorphism. By using of Rubik's cube, the concepts of group theory become more vivid and easy to grasp. Under the guidance of group theory, the restoration of Rubik's cube becomes more methodical and systematic, taking as an example the role played by conjugation and commutation in simplifying the process of restoring the cube.

2. Basic Properties of Rubik's Cube

2.1. Symmetry of Rubik's Cube

Symmetry is an eternal concept of modern physics. If a system transforms from one state into another when an operation is executed on it, and the two states are equivalent, the system is said to be symmetric to this operation. Each specific symmetry of a physics system corresponds to a law of conservation. Conversely, every conservation law of a physics system implies that the system has a relative symmetry. In mathematics, the concept symmetry is described by group theory.

As is shown in Fig. 2, Rubik's cube has axial symmetries of four orders [see Fig. 2(a)], two orders [see Fig. 2(b)], and three orders [see Fig. 2(c)], and is symmetric about the body center at the same time [4].

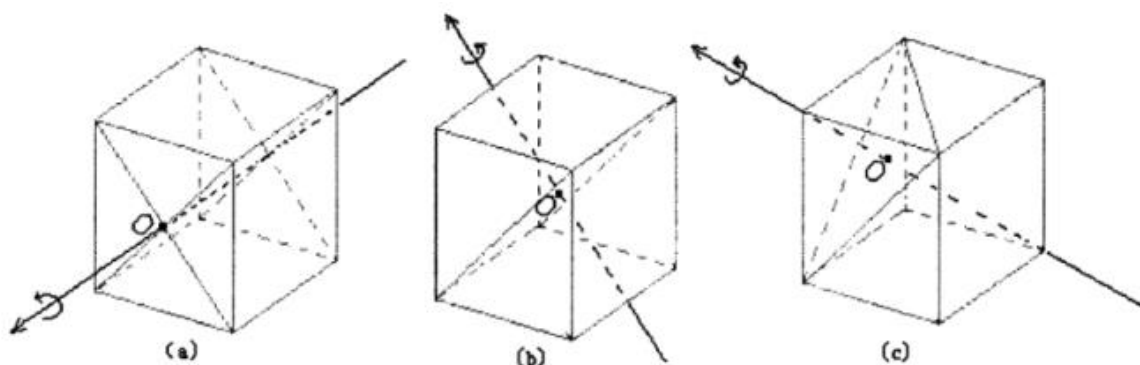


Fig. 2 The axial symmetries of (a) four orders, (b) two orders, and (c) three orders in Rubik's cube.

2.2. Combination Principle of Colors of Rubik's Cube

Every state of the Rubik's cube, whether solved or scrambled, can be achieved by rotating consecutively the six faces of the cube in solved position in finite steps, and can be restored by rotating the faces back step by step. Known from the principles of multiplication and addition in combinatorics, there exist $\frac{8! \times 3^8 \times 12! \times 2^{12}}{2 \times 2 \times 3} = 43252003274489856000$ different states of a Rubik's cube in total.

Besides the 6 central blocks fixed by the shaft, 20 free blocks remain to be shifted. The 8 corner blocks can be placed in the eight corners of the cube, and three out of six faces of every corner block are colored differently, so there are $8! \times 3^8$ cases. Similarly, there are $12! \times 2^{12}$ cases for the 12 edge blocks. Hence, if one takes the Rubik's cube apart and reassembles it at random, there are $8! \times 3^8 \times 12! \times 2^{12}$ cases for it. However, some of the cases are illegal, which means that cubes reassembled that way by no means be restored to its solved position.

Conventionally, the six colors of each face of the solved cube are denoted with their orientation on it: U(Up), F(Front), R(Right), D(Down), B(Back) and L(Left) [5]. With the help of these symbols, each simple move of the cube can be recorded in the Lagrange of group theory.

Take the basic move F as an example, the permutation can be written as (Uf Rf Df Lf)(Fu Fr Fd Fl)(Fur Fdr Fdl Ful)(Luf Ufr Rfd Dfl)(Ufl Ruf Dfr Lfd), which is obviously an odd permutation. However, if one just concentrates on the permutation of the positions of the corner and edge blocks, and ignores the rotation of each block itself, the move can be denoted as (FL UF FR DF)(ULF UFR DRF DFL), it turns out to be an even permutation. Hence, it is impossible to switch the positions of two different blocks without any other blocks moved, otherwise an odd permutation must be conducted. Now that every basic move of a Rubik's cube is an even permutation, when one takes a cube apart and reassembles it, the possibility is 50% for the cube to be partly restorable, in which case every block of the cube can be restored to its due position with the faces of blocks might be disordered. Nevertheless, the fact that the blocks can be positioned correctly does not mean that it is solvable, thus additional constraints are required.

Hereafter, edge blocks is taken firstly. With the body-center of the Rubik's cube as the origin and the rays passing through the centers of three faces of the cube from the origin as the positive directions of the x , y and z axis, the space rectangular coordinate system on the Rubik's cube is established. Regarding any of the three coordinate axes, four edge blocks could be found positioned on the edges of the cube parallel to the axis. To recognize all these edge blocks, it is denoted that each of them with an arrow parallel to the axis the block parallel to, pointing in the positive direction of the axis. Then every basic move of the cube would cause the arrows of two edge blocks to point in the opposite direction. Since any move of the cube can be split into several consecutive basic move, if there are an odd number of edge blocks that are reassembled upside down, the partly restorable cube is still unrestorable. Therefore, the possibility is 50%. When considering the corner blocks, the situation is exactly the same, except that each of them has three faces colored. Ergo, the possibility is one third. Therefore, the conclusion that there exist 43252003274489856000 different states of a Rubik's cube in total can be drawn safely.

3. God's number and Rubik's Cube Group

When playing a Rubik's cube, a natural question raised is the amount of turns at least requires to restore a cube from an arbitrary starting position. In other words, the diameter of the Cayley diagram of a Rubik's cube is regarded as God's number. To solve the problem, Morwen Thistlethwaite, a professor at the University of Tennessee came up with an idea of viewing the colors of the two opposite sides of the Rubik's cube as one color, so that it could be studied with three colors rather than six. Through the dedegroup reduction method, he tried to find solution of the three-color Rubik's cube first, based on which he further dealt with the six-color cube. This multi-stage algorithm permits the Rubik's cube to be handled in fewer steps. By now, such method is still a main method for computer to calculate the least steps to restore the Rubik's cube. He supposed all stages of the Rubik's cube to be G_0 groups, and can be reduced to be G_1 groups, and finally to be G_4 groups, the solved position. In this way, Thistlethwaite proved that God's number is fewer than 52 in 1981 [6]. After this, in December 1990, Hans Kloosterman changed the state of the intermediate groups G_2 and G_3 , and limited the God's number to 42 [7]. Since Thistlethwaite proposed the multi-stage algorithm, almost all the subsequent researches on God's number were carried out on the basis of it, trying to achieve optimization on the number of intermediate groups and the state of the stages. In 1992,

German mathematician Herbert Kociemba reduced the process to two stages, so that a shorter solution can be theoretically obtained, but at the same time the computation is increased. It was not until January 1995 that he calculated the number, and then optimized it to 29 or less [8].

While most mathematicians were working on the upper limit, Michael Reid broke through the lower limit and proved it required at least 20 steps to restore. Thus, the God's number is less than 29 and more than 20. Finally, with the help of computer science, Rokicki showed that this number equals 20 by exhaustive method. This computation used a bank of computers at Google and required a total of 35 CPU-years [9].

Considering the close connection between Rubik's cube and group theory, plenty of meaningful mathematical problems and conclusions can be formed with the knowledge and idea of group theory applied to the Rubik's cube. Here are some examples. One is that the set O consisting of all states of the Rubik's cube where every block of the cube is positioned correctly with random orientation, is a subgroup of the Rubik's cube group. The other is that the group O consisting of all states of the Rubik's cube where only all states in the direction of the corner block are scrambled, is isomorphic to the group $(C_3)^7$ [10].

4. Conclusion

The group theory is extremely essential to the study of mathematics, and its application to the Rubik's cube is a vital part of it. By applying the group theory, many propositions and properties of the Rubik's cube are discussed. From the perspective of solid geometry, the Rubik's cube is excellently spatial symmetric, and can be appropriately described and analyzed by the group theory. With the help of the group theory, the combination principle of colors on the Rubik's cube is further discussed and calculated. Moreover, the exact result to the minimum number of steps to restore a Rubik's cube is obtained. Apart from this, several interesting and inspiring problems around the Rubik's cube are put forward, together with the finding of some new properties.

Notably, the third-order Rubik's cube discussed in the main text is the most classic type of magic IQ cube. Considering the close relationship between it and the group theory, further study is highly anticipated to unveil the application of group theory to many other kinds of spatial-symmetric magic IQ cubes, like Rubik's cube with even order, rice dumpling magic IQ cube, pyramid magic IQ cube, inequilateral magic IQ cube, and mirror cube and so on.

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