Burnside's Lemma and Its Applications in Combinatorics Problems

Tian Hao

Sendelta International Academy of Shenzhen, Shenzhen, China

3180200240@caa.edu.cn

Abstract. The problems of discriminating the faces of a die using several colors and finding the number of ways a necklace can form are extremely famous in combinational analysis. The Burnside's lemma, which is also called Burnside's counting theorem or other names, is usually necessary in taking account of symmetries when enumerating nonequivalent patterns. In this paper, starting from the definition and clarifications of important concepts used in the proof of Burnside's lemma, it presents a clear proof to this significant formula used across combinatorics, group theory, recreational and contest mathematics. Moreover, these definitions are explained in simpler terms, which provide better intuition to the application of the Burnside's lemma. Next, some famous examples of the application of this lemma are discussed. A simple coloring problem and a simple necklace problem was given in the start of the corresponding sections, allowing readers to understand how this lemma is applicable in these scenarios. Afterwards, Burnside's lemma is applied to solve more general problems, which are more significant.

Keywords: Group theory, Burnside's lemma, Combinatorics, Orbit-stabilizer theorem.

1. Introduction

Since the discovery of group theory by the French mathematician Galois, it has been a powerful tool in mathematics. In the beginning, it was used to investigate the solvability of quintic polynomials. Later, group theory has proven its power in many different fields, including combinatorics [1]. As group theory is especially helpful when applied on problems involving symmetry, it is natural to believe that it can be used on two major types of problems in combinatorics: coloring problems and necklace problem, both involving rotational or reflectional symmetry [2].

Group theory in and of itself though, is not enough. A few important results are required to aid people in the process of dealing with these problems. Specifically, the Burnside's lemma is extremely important to handle a series of issues, and the orbit-stabilizer theorem is necessary to prove this lemma. Burnside's lemma manages to connect the number of cases that remain different under rotations, reflections, or other operations that can be described using a group with the number of fixed points under the operations. The latter is usually relatively easy to determine in these problems, thus Burnside's lemma can be a great helper.

2. Burnside's Lemma

2.1. Clarification of Definitions

The orbit of an element x in a set X under the group G is denoted $Orb(x_G)$, and is defined to be $Orb(x_G) = \{gx \in X: g \in G\}$. In other words, this is the set of elements that can be obtained, or reached, by the element x when the elements in G act upon it. The stabilizer of an element x in a set X under the group G is denoted $Stab(x_G)$, and is defined to be $Stab(x_G) = \{g \in G: gx = x\}$. In other words, this is the set of group elements that do not influence x at all. The fixed points of an element g in a group G when acting on a set X is denoted $fix(g_X)$, and is defined to be $fix(g_X) = \{x \in X: gx = x\}$. In other words, this is the set of elements that are mapped to themselves under the group action g.

The Burnside's lemma states that [3]

$$|Orb(x_G): x \in X| = \frac{1}{|G|} \Sigma_{g \in G} |fix(g_X)| \quad (1)$$

Putting in simpler terms, this means that the number of distinct orbits in X is equal to the total number of fixed points in over all the elements in G divided by the number of elements in G, or the average number of fixed points in G.

2.2. Proof of Burnside's Lemma

The following function $f: G \times X \rightarrow \{0,1\}$ is constructed, where $f(g,x) = 0$ if $gx \neq x$, and $f(g,x) = 1$ if $gx = x$. Then it is found that $|fix(g_X)| = \Sigma_{x \in X} f(g,x)$, because both of them count the number of elements that are fixed under g. Therefore, $\Sigma_{g \in G} |fix(g_X)| = \Sigma_{g \in G} \Sigma_{x \in X} f(g,x) = \Sigma_{x \in X} \Sigma_{g \in G} f(g,x)$. Moreover, note that the stabilizer of x counts the number of group elements g such that $gx = x$, so $\Sigma_{x \in X} \Sigma_{g \in G} f(g,x) = \Sigma_{x \in X} |Stab(x_G)|$ [4]. By the famous orbit-stabilizer theorem which states that $|Stab(x_G)| = \frac{|G|}{|Orb(x_G)|}$, so $\Sigma_{x \in X} |Stab(x_G)| = \Sigma_{x \in X} \frac{|G|}{|Orb(x_G)|}$. Obviously, the term $|G|$ is not influenced by the choice of x at all, thus the term $\Sigma_{x \in X} \frac{1}{|Orb(x_G)|}$ is needed to care about.

Now, if the orbit of x is replaced by x', then by definition $x' = gx$, so $Orb(x'_G) = \{g'gx: g' \in G\} = \{g''x: g'' \in G\}$ since the product of elements within a group is still in the group. Therefore, elements in the same orbit as x has the same orbit as x. Moreover, orbits are disjoint. This is because that if two orbits A and B share an element x, other elements in A or B must have the same orbit as well, so they must also be shared by A and B, thus they are the same orbit. Hence, the set X can be partitioned. Denote X/G as the set of orbits in X, then $X = \bigcup_{Orb(x_G) \in X/G} Orb(x_G)$. Notice that they are disjoint, then [5]

$$\Sigma_{x \in X} \frac{1}{|Orb(x_G)|} = \Sigma_{Orb(x_G) \in X/G} \Sigma_{x \in Orb(x_G)} \frac{1}{|Orb(x_G)|} \quad (2)$$

Obviously, the inner sum is 1, so $\Sigma_{x \in X} \frac{1}{|Orb(x_G)|} = |X/G|$. And that finishes the proof of Burnside's lemma, since $RHS = \frac{|G|}{|G|} \cdot |X/G|$, and $|X/G| = |\{Orb(x_G): x \in X\}|$ by definition.

3. Applications of Burnside's Lemma

In general, Burnside's lemma is a useful tool when dealing with a wide range of counting problems that involve some kind of rule on what to count, typically rotations, reflections, or such operations related to symmetry [6]. This section will introduce two important genres of problems that can be solved or simplified with Burnside's lemma.

3.1. Coloring Problems

Burnside's lemma can be used to solve a vast number of coloring problems involving certain restrictions. A nice example of such problem would be how many ways are there to color a cube's six faces with six distinct colors [7]. If two colorings can be obtained by each other through rotating the cube, they are viewed as the same coloring. The reader can attempt this problem using traditional methods not involving Burnside's lemma.

Burnside's lemma manages to simplify this problem down largely. The group G would be the group of rotations that can act on a cube, while the set X is the set of all possible colorings of the cube. It can be seen that $|\{Orb(x_G): x \in X\}|$ is actually precisely what the question is looking for. This is the number of distinct colorings, since all the colorings that can be obtained by each other through rotation would be in $Orb(x_G)$, so they only contribute once to the total sum.

On the right-hand side, notice that, under rotations, a cube with its six faces colored with distinct colors can never be rotated back to itself, unless the rotation is equivalent to the identity element in

G. Therefore, the only meaningful term in the summation $\Sigma_{g\in G}|fix(g_X)|$is $|fix(1_X)|$, where 1 is the identity element in G. This is obviously just $|X|$, because any coloring, when not moved, is definitely fixed. As there are 6 faces of the cube and 6 colors, one can choose 6 colors for the first face, 5 for the second, and so on, so $|X| = 6! = 720$.

Finally, it is necessary to find the cardinality of G. Considering the four long diagonals of the cube, it can be seen that every rotation corresponds to a different configuration of the long diagonals. Moreover, every configuration of the long diagonals can be obtained through rotation. That is to say, there exists a bijection between G and the configurations of the long diagonals of the cube. Clearly, the latter has cardinality $4!$ as there are 4 objects to freely permute. Therefore, the answer to the original question is $720/24 = 30$ different colorings are possible.

Another example involving a varying number of colors is that how many ways can one color the four vertices of a square with n colors to choose from? Here, colorings that can be obtained from another through rotation are seen as the same. Similarly, one will set G to be the group of all rotations and reflections that can act on a square, and X to be the total number of ways to color the square, regardless of rotations [8].

To count $\Sigma_{g\in G}|fix(g_X)|$, it is vital to note that there are only 4 possible elements that can act on X: the identity, 90º rotation, 180º rotation, or 270º rotation (clockwise or counterclockwise doesn't matter much). In the identity case, the four vertices are obviously all fixed, so $|fix(g_x)| = |X| = n^4$, as there are n colors and 4 vertices. In both the 90º and 270º rotation case, all four colors are switched around, so the only way that for it to be a fixed point is for all the colors to be the same, that is, n ways to color. Finally, in the 180º rotation case, every vertex is switched with the one opposite to it, so as long as the two pairs of opposite vertices are of the same color, the coloring will be fixed under the 180º rotation. There are two pairs, so there are n^2 fixed points. Therefore, $\frac{1}{|G|}\Sigma_{g\in G}|fix(g_X)| = \frac{1}{4}(n^4 + n^2 + 2n)$. To the right, an example of the colouring of the corner of squares with two colors is shown (Colorings in the same row are only counted once, since they can be obtained from each other through rotation). It is seen from Fig. 1 that it agrees with the result, since $\frac{1}{4}(16 + 4 + 4) = 6$ and there are 6 different columns in the figure.

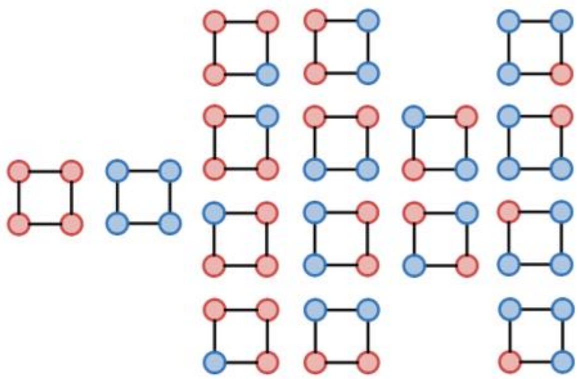

Figure 1. The colouring of four corners of a square with two colors

3.2. Necklace Problems

Another important application of Burnside's lemma is on solving necklace problems [9]. These problems typically ask for the number of ways a necklace can be formed given n beads, and k different types of beads, where necklaces are considered the same if they can be formed by each other through rotations.

When the length of the necklace is a prime, this is relatively easy to tackle. If the beads are numbered from 1 through n, every rotation through m beads would correspond to adding m onto every term, then taking the modulo n of each term. For example, if the necklace originally had beads labelled $(1, 2, 3, 4, 5)$, the necklace becomes $(4, 5, 1, 2, 3)$ after a rotation of 3 beads. For a fixed point to exist, every bead must have the same color as the bead that it moves to after the rotation, or a and $a+m$ must have the same colors. From the same logic, $a+lm$ for any arbitrary natural l must have the same color as a. As n is a prime, $\{a+lm \bmod n\}=\{1,2,3,\ldots,n\}$, so the only fixed point occurs when all beads are of the same color, or when the rotation is just the identity. Therefore, $\Sigma_{g\in G}|fix(g_X)|=k^n+kn-k$. This is because there are k^n different bead combinations (and each of them can undergo the identity rotation while remaining fixed). Besides, there are k necklaces where the beads are of the same color, so kn fixed points as there are n rotations, while subtracting k which is the intersection of the two. Therefore, the number of different necklaces is $\frac{1}{|G|}\Sigma_{g\in G}|fix(g_X)|$, which evaluates to be $\frac{k^n+k(n-1)}{n}$.

Next, the paper will look at the more general case with n beads, and n can take on any value. From the same line of reasoning, $a+lm$ for any natural number l must have the same color as a. However, when m is a factor of n, $\{a+lm \bmod n\}=\{a,a+m,a+2m,\ldots,a+n-m\}$, because a is rotated back to itself after $\frac{n}{m}$ rotations (For simpler notation, the paper will later denote $\frac{n}{m}$ as d, also note that d is the number of different rotations that give a fixed point). Therefore, it is necessary to note that there are m beads apart from each other to be of the same color in order for a fixed point to exist, so $|fix(g_X)|=dk^{\frac{n}{d}}$. This is because that there are d rotations (rotations that move beads by a multiple of m) and m different colors [10].

However, $\Sigma_{g\in G}|fix(g_X)|\neq\Sigma_{d|n}dk^{\frac{n}{d}}$ since there are several colorings that are counted multiple times. For example, when $d=1$, the colorings where all beads are of the same color overlaps with the case where $d=n$, and beads are also all of the same color. A method to counter this is to only count the unique rotations of a coloring. For example, when $n=12$ and $k=2$, when counting the case $d=4$, it is not necessary to count configurations that are fixed points under rotations of 6 units, because those configurations are also counted in the case $d=2$. More generally, configurations in the case d will also be counted in the prime factors of d. Therefore, to prevent over-counting, all the configurations that share prime factors with d and are smaller than d are needed to get rid of. To achieve this effect, one can simply introduce φ, so-called Euler's totient function, to sieve out all the numbers that are co-prime to d.

Therefore, $\Sigma_{g\in G}|fix(g_X)|=\Sigma_{d|n}\varphi(d)k^{\frac{n}{d}}$, and so there are $\frac{1}{n}\Sigma_{d|n}\varphi(d)k^{\frac{n}{d}}$ different necklaces. To check that this formula is correct, one can examine a simpler case that is already studied where n is prime. When n is prime, $d|n$ is only true if $d=1$ or $d=n$. Using the formula $\frac{1}{n}\Sigma_{d|n}\varphi(d)k^{\frac{n}{d}}$, it is obtained that the expression $\frac{1}{n}\varphi(1)k^n+\frac{1}{n}\varphi(n)k^1$ is precisely $\frac{k^n+k(n-1)}{n}$.

4. Conclusion

To summarize, this paper firstly defines the concepts of orbits, stabilizers, and fixed points. Then by using the orbit-stabilizer theorem, it proves an important result called Burnside's lemma. These concepts are also explained in ways with less rigour but allow more intuition. Afterwards, a few significant applications of this lemma are discussed along with examples for each type of problems. Some of these examples are specific cases that are simpler to understand and will demonstrate how Burnside's lemma can be applied. Others are more elaborate, through examples that give general solutions to specific problems. It was shown that the number of ways to color the corners of a cube at given n colors is $\frac{1}{4}(n^4+n^2+2n)$, and the amount of different necklaces of length n at given k colors is $\frac{1}{n}\Sigma_{d|n}\varphi(d)k^{\frac{n}{d}}$. These formulas are verified through substituting in the specific cases of $n=2$ and n is prime, respectively.

References

[1] Tucker A. Applied combinatorics (4th Ed.). John Wiley and Sons, New York. 2002.

[2] Fleischmann P. Groups and Symmetries, Institute of Mathematics and Statistics, University of Kent at Canterbury, U.K., 2011.

[3] Wright E. M. Burnside's lemma: A historical note. Journal of Combinatorial Theory, 1981, 30(1): 89-90.

[4] Nasseef M. T. Counting symmetries with Burnside's lemma and Pólya's theorem. European Journal of Pure and Applied Mathematics, 2016, 9(1): 84-113.

[5] Tomakin F. Y. The Pólya theory and permutation groups. Chamchuri Journal of Mathematics, 2000, 1(2): 1-23.

[6] Fang Jianying, Wang Diji. Applied examples of Burnside lemma and Polya theorem. Journal of Xinjiang Normal University (Natural Sciences Edition), 2003, 22(2): 12-15.

[7] Cao Xiwang. An extension of Burnside's lemma. Journal of Huanggang Teachers College (Natural Sciences Edition), 1994, 14(1): 24-27.

[8] Wang Lijie, Wang Qingxian. A teaching improvement of solving counting problems by group theory. College Mathematics, 2022, 38(2): 93-97.

[9] Xu Baogen. On the formula of enumeration in necklace problem. Journal of East China Jiaotong University, 2003, 20(5): 113-114.

[10] Case Joshua, Koban Lori, LeGrand Jordan. Counting set classes with Burnside's lemma. Involve, 2015, 8(2): 337-344.