Comparison of the Cauchy Integral Formula and Residue Theorem on the Calculation of Definite Integrals

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Abstract. Integration is a useful mathematical tool which is applied in a wide range of studies and assists people to solve many problems. Despite methods of real analysis, complex integrations in complex analysis are more beneficial and more convenient than real integrations under certain circumstances. Cauchy's Residue Theorem and Cauchy Integral Formula are two methods used to solve complex integrals. Cauchy's Residue Theorem helps to solve complex integrals via the calculation involving singular points, whereas Cauchy Integral Formula is implemented to solve complex integrals by finding out poles enclosed by the contour. Herein, the comparison and analysis of the process of solving both real and complex integrations between Cauchy's Residue Theorem and Cauchy Integral Formula are demonstrated through sample questions involving exponential integrations and polynomial integrations. This passage also includes definitions for Cauchy's Residue Theorem, Cauchy Integral Formula, corollary of Cauchy Integral Formula, steps and solutions of sample contour integral questions.

Keywords: Cauchy's Residue Theorem; Cauchy Integral Formula; Improper Integral; Complex Analysis.

1. Introduction

In recent years, the fast development of mathematics boosts the importance of complex functions, bringing more experts and scholars to devote themselves to researching complex theorems. Through the applications of complex functions [1], complex functions redound to the effectiveness of solving problems met in real life and further promote technological development in various fields [2]. As people study deeper, the demand for solving existing problems in the study of complex functions is in haste.

Cauchy's Residue Theorem (CRT) is an essential part used for analytic functions to calculate integrations enclosed by a closed contour and can be utilized in real integral computations [3]. This calculation process is also called enclosure integration. Through the computation, a real integral is transformed into complex integrals, based on CRT, which is solved in a simplified process. Thereby, finding out all singular points enclosed by the contour to make sure which type they belong to, and correctly calculating the corresponding residues are the hinge to solve contour integrals [4]. The Cauchy Integral Formula (CIF), on the other hand, is an integral expression of functions and also a formula that uses bound values to indicate interior values of an analytic function. The CIF turns to solve enclosure integrations into evaluating functions to simplify many problems met in complex integrations. Higher-order differential equation deduced from the CIF is also crucial formula in complex function theory [5]. Theories including Liouville's theorem, Cauchy's inequalities, and Morera's theorem can also be obtained by CIF [6].

2. Theorem and Formula

2.1. Cauchy's Residue Theorem

Suppose that C is a single closed contour in the nonnegative axis. If f is an analytic function inside and on C except for some singular points denoted as z_k where $k = 1, 2, 3 \dots, n$, then

$$\int_c f(z)\, dz = 2\pi i \sum_{k=1}^{n} \mathrm{Res}(f(z), z) \quad (1)$$

where $\mathrm{Res}(f(z), z)$ are residues of a function at singular points inside of C. A singular point z_0 is defined as a point if the function is not analytic at z_0 but is analytic at some point in every neighborhood of it [7].

2.2. Cauchy Integral Formula

Suppose that f is an analytic function and C is a simple closed contour. If z_0 is any point belonging to C, then [8]

$$f(z_0) = \frac{1}{2\pi i}\int_C \frac{f(z)}{z-z_0}dz. \tag{2}$$

CIF can be extended to get an integral representation for derivatives $f^{(n)}(z_0)$ $(n = 0, 1, 2, \ldots)$ of f at z_0. Namely [9],

$$f^{(n)}(z_0) = \frac{n!}{2\pi i}\int_C \frac{f(z)}{(z-z_0)^{n+1}}dz. \tag{3}$$

3. Applications

Example 3.1. Solve the complex integration

$$I = \int_{|z|=2} \frac{ze^z}{(z^2-1)}dz, \tag{4}$$

where the contour in the complex plane is displayed in Fig. 1.

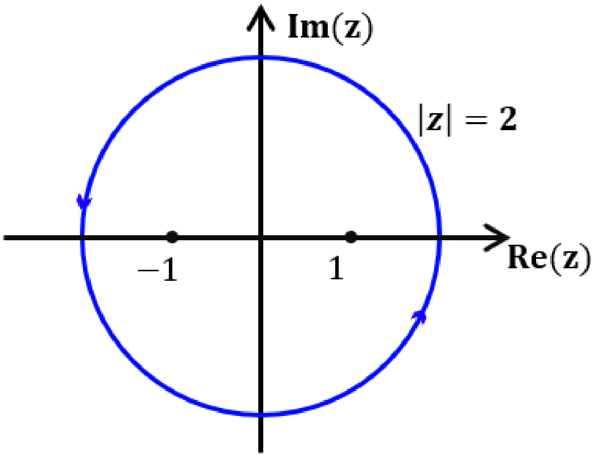

Fig 1. Illustration of the contour

Solution I (CRT). The integrand can be rewritten as $\frac{ze^z}{(z^2-1)} = \frac{A}{z+1} + \frac{B}{z-1}$, where $A = \frac{e^z}{2}$ and $B = \frac{e^z}{2}$. There are two singular points which are $z = 1$ and $z = -1$, see Fig. 1. According to the CRT, the integral is equivalent to

$$I = \int_{|z|=2} \frac{ze^z}{(z^2-1)} = 2\pi i\big(\mathrm{Res}(f(z), 1) + \mathrm{Res}(f(z), -1)\big). \tag{5}$$

By definition, one has $\mathrm{Res}(f(z), 1) = \lim_{z\to 1}(z-1)\frac{ze^z}{z^2-1} = \frac{e}{2}$ and $\mathrm{Res}(f(z), -1) = \lim_{z\to -1}(z+1)\frac{ze^z}{z^2-1} = \frac{e^{-1}}{2}$. Hence,

$$I = \int_{|z|=2} \frac{ze^z}{(z^2-1)}dz = 2\pi i\left(\frac{e}{2} + \frac{e^{-1}}{2}\right) = 2\pi i \cosh(1). \tag{6}$$

Solution II (CIF). The function of complex variables has two poles where $z = 1$ and $z = -1$. It can be written as $\int_{|z|=2} \frac{ze^z}{(z^2-1)} dz = \int_{C_1} \frac{\frac{ze^z}{z+1}}{z-1} dz + \int_{C_2} \frac{\frac{ze^z}{z-1}}{z+1} dz$. By applying the CIF, it is immediately to find that

$$2\pi i \left.\frac{ze^z}{z+1}\right|_{z=1} + 2\pi i \left.\frac{ze^z}{z-1}\right|_{z=-1} = 2\pi i \left(\frac{e}{2} - \frac{e^{-1}}{2}\right) = 2\pi i \cosh(1) \tag{7}$$

Example 3.2. Solve the complex integration

$$I = \int_{|z|=2} \frac{5z-2}{z(z-1)^2} dz \tag{8}$$

where the contour in the complex plane is displayed in Fig. 2.

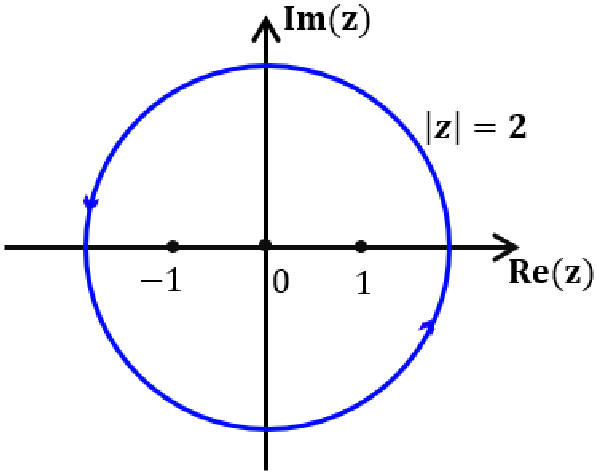

Fig 2. Illustration of the contour

Solution I (CRT). By partical fraction, $\frac{5z-2}{z(z-1)^2} = \frac{A}{z} + \frac{B}{(z-1)^2}$, where $A = -2$ and $B = 2z + 1$. Obviously, $z = 0$ and $z = 1$ are singular points of the integration (see Fig. 2). Its integration can then be written as

$$I = \int_{|z|=2} \frac{ze^z}{(z^2-1)} dz = 2\pi i\big(\mathrm{Res}(f(z), 0) + \mathrm{Res}(f(z), 1)\big). \tag{9}$$

The residues can be be computed by the definition of the CRT: $\mathrm{Res}(f(z), 0) = \lim_{z=0} \frac{5z-2}{(z-1)^2} = -2$ and $\mathrm{Res}(f(z), 1) = \lim_{z=1} \frac{5z-2}{z} = 2$. Therefore, the integration can be solved by substituting the residues into the equation, which implies

$$I = \int_{|z|=2} \frac{ze^z}{(z^2-1)} dz = 2\pi i(-2 + 2) = 0. \tag{10}$$

Solution II (CIF). As the poles are found at $z = 1$, $z = -1$, and $z = 0$, the integral is equivalent to

$$\int_{|z|=2} \frac{5z-2}{z(z-1)^2} dz = \int_{C_1} \frac{\frac{5z-2}{z}}{(z-1)^2} dz + \int_{C_2} \frac{\frac{5z-2}{(z-1)^2}}{z} dz. \tag{11}$$

By applying the CIF,

$$I = 2\pi i \left.\left(\frac{5z-2}{z}\right)'\right|_{z=1} + 2\pi i \left.\frac{5z-2}{(z-1)^2}\right|_{z=0} = 2\pi i(2 - 2) = 0. \tag{12}$$

Example 3.3. Solve the real integral

$$I = \int_{-\infty}^{\infty} \frac{1}{(x^2+1)^2} dx \tag{13}$$

where the contour in the complex plane is displayed in Fig. 3.

This integral can be solved by seperately solving the two complex integrals and subtracting them together. Namely,

$$\int_{-\infty}^{\infty}\frac{1}{(x^2+1)^2}dx=\int_{C_1+C_2}\frac{1}{(z^2+1)^2}dz-\int_{C_2}\frac{1}{(z^2+1)^2}dz \tag{14}$$

Where C_1 denotes $z=Re(z)$, and C_2 denotes $z=Re^{i\theta}$ with $0\le\theta\le\pi$.

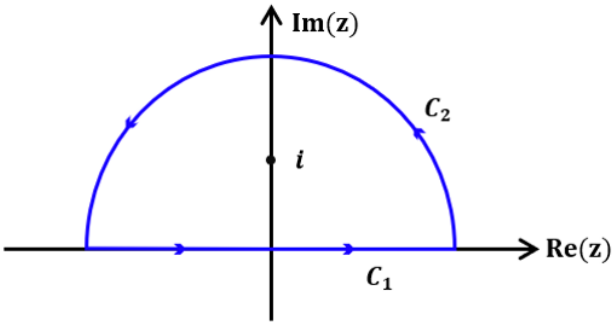

Fig 3. Illustration of the contours

In Eq. (14), the second term can be written as $\int_{C_2}\frac{1}{(z^2+1)^2}dz=\int_0^{\pi}\frac{1}{\left(R^2e^{2i\theta}+1\right)^2}iRe^{i\theta}d\theta$ by substituting z with C_2. By applying the trangle inequality for integrals $(R>1)$,

$$\left|\int_0^{\pi}\frac{1}{\left(R^2e^{2i\theta}+1\right)^2}iRe^{i\theta}d\theta\right|\le\int_0^{\pi}\left|\frac{1}{\left(R^2e^{2i\theta}+1\right)^2}iRe^{i\theta}\right|d\theta. \tag{15}$$

Since $\left|R^2e^{2i\theta}+1\right|\ge R^2$, it holds that $\frac{1}{\left(R^2e^{2i\theta}+1\right)^2}\le\frac{1}{(R^2-1)^2}$. Thus, the $\left|\int_0^{\pi}\frac{1}{\left(R^2e^{2i\theta}+1\right)^2}iRe^{i\theta}d\theta\right|\le\int_0^{\pi}\left|\frac{1}{\left(R^2e^{2i\theta}+1\right)^2}iRe^{i\theta}\right|d\theta\le\int_0^{\pi}\frac{R}{(R^2-1)^2}d\theta=\frac{\pi R}{(R^2-1)^2}$. In other words,

$$\int_{C_2}\frac{1}{(z^2+1)^2}dz=\lim_{R\to\infty}\frac{\pi R}{(R^2-1)^2}=0 \tag{16}$$

Solution I (CRT). The integrant can be written as

$$\int_{c_1+c_2}\frac{1}{(z^2+1)^2}dz=\int_{c_1+c_2}\frac{1}{(z+i)^2(z-i)^2}dz. \tag{17}$$

The only residue inside the area enclosed by c_1+c_2 can be calculated by applying the CRT: $\mathrm{Res}(f(z),i)=\lim_{z\to i}(z-i)^2\left(\frac{1}{(z+i)^2}\right)'=\frac{1}{4i}$. Therfore,

$$\int_{c_1+c_2}\frac{1}{(z^2+1)^2}dz=2\pi i\big(Res(f(z),\,i)\big)=\frac{\pi}{2}. \tag{18}$$

Taken together, the integral in Eq. (13) is

$$\int_{-\infty}^{\infty}\frac{1}{(x^2+1)^2}dx=\int_{c_1+C_2}\frac{1}{(z^2+1)^2}dz-\int_{C_2}\frac{1}{(z^2+1)^2}dz=\frac{\pi}{2}-0=\frac{\pi}{2}. \tag{19}$$

Solution II (CIF). The pole enclosed by the contours is $z=i$. According to CIF, the integration is computed:

$$\int_{C_1+C_2}\frac{1}{(z^2+1)^2}dx=\int_{C_1+C_2}\frac{\frac{1}{(z+i)^2}}{(z-i)^2}dz=2\pi i\left(\frac{1}{(z+i)^2}\right)'=\frac{\pi}{2} \tag{20}$$

Ultimately,

$$\int_{-\infty}^{\infty}\frac{1}{(x^2+1)^2}dx=\int_{C_1+C_2}\frac{1}{(z^2+1)^2}dz-\int_{C_2}\frac{1}{(z^2+1)^2}dz=\frac{\pi}{2}-0=\frac{\pi}{2}. \tag{21}$$

Example 3.4. Solve the real integral [11]

$$I = \int_{-\infty}^{\infty} \frac{1}{x^4+1} dx. \tag{22}$$

This real integral can be calculated by solving these two complex integrals and doing the subtraction

$$\int_{-\infty}^{\infty} \frac{1}{x^4+1} dx = \int_{C_1+C_2} \frac{1}{z^4+1} dz - \int_{C_2} \frac{1}{z^4+1} dz, \tag{23}$$

where C_1 denotes $z = Re(z)$, and C_2 denotes $z = Re^{i\theta}$ with $0 \le \theta \le \pi$. The contour in the complex plane is displayed in Fig. 4. As in the previous Example 3.4, $\int_{C_2} \frac{1}{z^4+1} \mathrm{dz} = 0$.

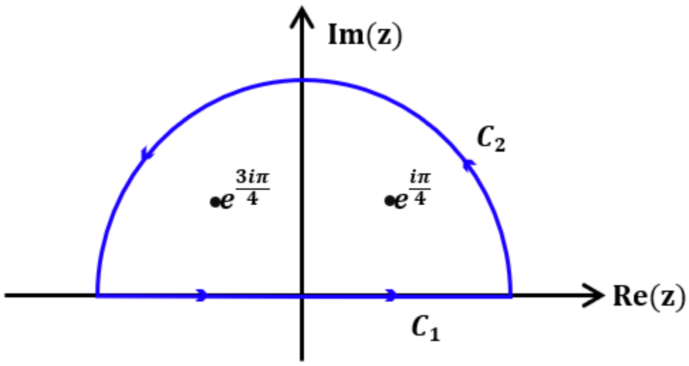

Fig 4. Illustration of the contours

Solution I (CRT). It is found that $\int_{C_1+C_2} \frac{1}{z^4+1} dz$ has singular points: $z_1 = e^{\frac{i\pi}{4}}$, $z_2 = e^{\frac{3i\pi}{4}}$, $z_3 = e^{\frac{5i\pi}{4}}$, $z_4 = e^{\frac{7i\pi}{4}}$, with z_1 and z_2 stay within $C_1 + C_2$. The residues are calculated according to the CRT: $\mathrm{Res}(f(z), z_1) = \lim_{z\to e^{\frac{i\pi}{4}}} \left(z - e^{\frac{i\pi}{4}}\right)\left(\frac{1}{z^4+1}\right) = \frac{e^{-\frac{3i\pi}{4}}}{4}$ and $\mathrm{Res}(f(z), z_2) = \lim_{z\to e^{\frac{3i\pi}{4}}} \left(z - e^{\frac{3i\pi}{4}}\right)\left(\frac{1}{z^4+1}\right) = \frac{e^{-\frac{i\pi}{4}}}{4}$. The integration is computed by substituting the residues into the equation, i.e.,

$$\int_{C_1+C_2} \frac{1}{z^4+1} dz = 2\pi i \left(\frac{e^{-\frac{3i\pi}{4}}}{4} + \frac{e^{-\frac{i\pi}{4}}}{4}\right) = \frac{\sqrt{2}}{2}\pi. \tag{24}$$

Finally, the real integral is found to be $\int_{-\infty}^{\infty} \frac{1}{x^4+1} dx = \frac{\sqrt{2}}{2}\pi$.

Solution II (CIF). According to CIF, the function $\int_{C_1+C_2} \frac{1}{z^4+1} dz$ can be recasted into two parts, namely,

$$I = \int_{C_1+C_2} \frac{\frac{1}{z^4+1}}{z-e^{\frac{i\pi}{4}}} dz + \int_{C_1+C_2} \frac{\frac{1}{z^4+1}}{z-e^{\frac{3i\pi}{4}}} dz = 2\pi i \left(\left.\frac{1}{z^4+1}\right|_{z=e^{\frac{i\pi}{4}}} + \left.\frac{1}{z^4+1}\right|_{z=e^{\frac{3i\pi}{4}}}\right) = \frac{\sqrt{2}}{2}\pi. \tag{25}$$

4. Conclusion

Based on sample questions, CRT and CIF are roughly the same in essence. CRT and CIF help to solve complex integrals by making use of undefined points of the integrand, which are finite singular points in CRT and simple poles inside or on the contour in the CIF. If there is a singular point for calculating residues in CRT, there is a pole for the CIF. For higher-order derivatives, in comparison, the calculation involves differentiation to solve the complex integral if the CIF is used, whereas the process remains the same as calculating the first-order complex integrations by using CRT, which can be regarded as a slight shortcoming of the CIF. Moreover, when there are many singular points or essential singularities, contour integrations can only be solved by using CRT rather than CIF. As an extension of the CIF, CRT is more widely used in various fields, such as Gauss's Law in electrostatics. This is a theorem implemented in the idea of CRT by which people can calculate electric flux enclosed by a Gaussian surface.

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