Applications of Residue Theorem to Four Distinct Kinds of Definite Integrals

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Abstract. Cauchy's residue theorem gives a relative general form for complex integral along a simple closed contour. With the help of Cauchy's residue theorem, appropriate closed contour can be chosen to calculate some abnormal definite integrals that might be very complicated and are difficult to solve by conventional methods. This paper focuses on four distinct kinds of definite integrals, including the integrals involving sine and cosine functions, polynomial functions, exponential functions, and logarithmic functions. The contour chosen are basically a sector of circle that involves one or several isolated singularities of the function. Then the residue at the isolated singularities of the function is calculated. The value of the residues is substituted in the formula that is deducted in the Cauchy's residue theorem. Then the integral along the simple closed contour can be expressed in two parts, in which one is along the real axis while the other is along the circle. This study demonstrates that the Cauchy's residue theorem is superior to the conventional real analysis methods.

Keywords: Cauchy's residue theorem; Improper integral; Special function; Contour integration.

1. Introduction

The residue theorem is a powerful tool in complex analysis that allows people to evaluate certain contour integrals. The content of the residue theorem is relatively straightforward. It states that if $f(z)$ is a function that is analytic everywhere inside and on a closed contour C except for isolated singularities at $z_1, z_2, \cdots, z_k$, then the value of the integral of $f(z)$ around C is equal to $2\pi i$ times the sum of the residues of $f(z)$ at each of the singular points inside C [1]. The development of the residue theorem has been an ongoing process, with contributions from many mathematicians over the years. Some of the key figures in the development of the residue theorem include Cauchy, Laurent, Riemann, Weierstrass, and Mittag-Leffler. It was first introduced by Cauchy in the early 19th century as part of his work on complex analysis. However, it was not until the late 19th century that the residue theorem was fully developed and rigorously proved by the German mathematician Bernhard Riemann. Riemann's proof of the theorem was based on the notion of Laurent series expansions of complex functions, which allows one to express a function as a sum of powers of z, both positive and negative [2].

Today, the residue theorem remains an important tool in complex analysis and has many applications in mathematics, physics, and engineering. It has since been extended and generalized in various ways. For example, the theorem can be extended to functions defined on more general spaces, such as metric spaces or Banach spaces [3]. In addition, there are many variations of the theorem that involve weaker or stronger conditions on the function or the interval. Some of these variations include the Brouwer fixed point theorem, the Schauder fixed point theorem, and the Kakutani fixed point theorem, which have applications in areas such as game theory, economics, and physics. The retention number theorem has also been used to prove other important results in mathematics. For example, it can be used to prove the existence of solutions to certain types of differential equations and integral equations. It has also been used in the study of chaotic dynamical systems, where fixed points play a key role in determining the behavior of the system over time [4]. Overall, the retention number theorem is a fundamental result in mathematics with many important applications and extensions, and it continues to be an active area of research for mathematicians today.

2. Residue Theorem: Definition and Formula

Cauchy's residue theorem is developed by Cauchy's integral formula [5]

$$f(z_0) = \frac{1}{2\pi i}\pi i \oint \frac{f(z)}{z-z_0} dz \tag{1}$$

where $f(z)$ is an analytic function. C is a simple closed contour in the complex plane, and z_0 is an isolated singularity of $f(z)$ and it shall be inside C. As z_0 is the only isolated singularity, this integral should be the same as the one with a small circle with z_0 as a center, suggesting that $z = z_0 + re^{i\theta}$ and $0 \leq \theta \leq 2\pi$. Thus, it holds that

$$\oint_C \frac{f(z)}{z-z_0} dz = \int_0^{2\pi} \frac{f(z+re^{i\theta})}{re^{i\theta}} ire^{i\theta} d\theta = if(z_0) \int_0^{2\pi} d\theta = 2\pi i f(z_0), \tag{2}$$

which is exactly the Cauchy's integral formula. This formula can be iterated to [6]

$$f^n(z_0) = \frac{n!}{2\pi i} \oint_C \frac{f(z)}{(z-z_0)^{n+1}} dz, n \in N. \tag{3}$$

For instance, the Residue theorem can be used to evaluate the integral $\int_C \frac{4z-5}{z(z-1)} dz$, where C is the circle $|z| = 2$ which is described in the counter-clockwise direction, see Fig. 1. The integrand has the two isolated singularities $z = 0$ and $z = 1$, both of which are interior to C. Since the corresponding residues are B_1=5 at $z = 0$ and $B_2 = -1$ at $z = 1$, then the integral $\int_C \frac{4z-5}{z(z-1)} dz = 2\pi i(B_1 + B_2) = 8\pi i$.

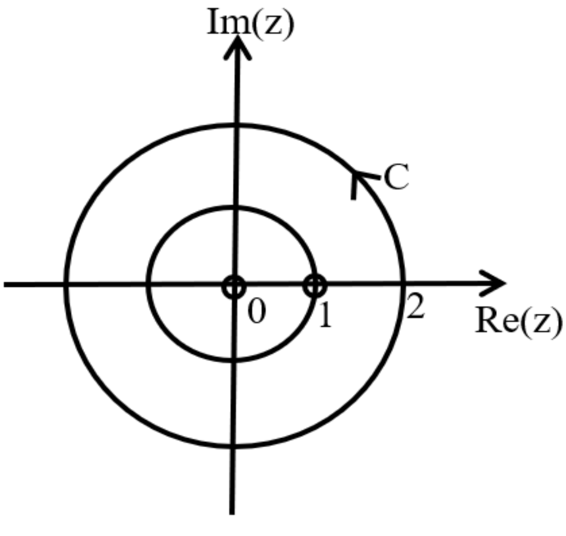

Fig 1. Illustration of the contour in complex plane

3. Applications of Residue Theorem

3.1. $R(\cos\theta, \sin\theta)$-type Integral

Consider the following integral over the interval $[0,2\pi]$ [7]

$$I = \int_0^{2\pi} \frac{d\theta}{1-2p\cos\theta + p^2}, \tag{4}$$

Where $0 < p < 1$. The first step is to introduce a complex variable according to $z = e^{i\theta}$, so that $\cos\theta = \frac{1}{2}(z + \frac{1}{z})$ and $dz = ie^{i\theta}\, d\theta = izd\theta$. Therefore, the Eq. (4) can be rewritten as an integral around a closed contour C which is a unit circle about the origin

$$I = \oint_C \frac{dz}{iz}\frac{1}{1-p\left(z+\frac{1}{z}\right)+p^2} = \frac{1}{i}\oint dz\,\frac{1}{(1-pz)(z-p)}. \tag{5}$$

Although the integrand has two singular points, only the one at $z = p$ is located inside the contour. Therefore, the Residue theorem implies that

$$I = \frac{1}{i}2\pi i \operatorname*{Res}_{z=p} f(z) = \frac{2\pi}{1-p^2}. \tag{6}$$

Another integral arises from the famous Kepler's problem [8]

$$\Phi(\epsilon) = \frac{1}{2\pi}\int_0^{2\pi}\frac{\sin^2\theta d\theta}{(1+\epsilon\cos\theta)^2} \tag{7}$$

where $\epsilon \in [0,1]$. Started with the passage to the complex plane: $\sin\theta = \frac{1}{2\mathrm{i}}(\mathrm{z} - \mathrm{z}^{-1})$, $\cos\theta = \frac{1}{2}(\mathrm{z} + \mathrm{z}^{-1})$, and $d\theta = \frac{dz}{iz}$. So, the above integral is

$$\Phi(\epsilon) = \frac{1}{2\pi}\oint_{|z|=1}\frac{-\frac{1}{4}\left(z-z^{-1}\right)^2 dz}{iz\left[1+\epsilon\frac{(z+z^{-1})}{2}\right]^2} = -\frac{1}{4\pi i}\oint_{|z|=1}\frac{\left(z^2-1\right)^2 dz}{z[2z+\epsilon(z^2+1)]^2}. \tag{8}$$

The poles are at $z_0 = 0$ and when $2z + \epsilon(z^2 + 1) = 0$, that is, $z_1 = \frac{-1-\sqrt{1-\epsilon^2}}{\epsilon}$ and $z_2 = \frac{-1+\sqrt{1-\epsilon^2}}{\epsilon}$. Obviously, z_0 lies into the unit circle. From the relation $1 + \sqrt{1-\epsilon^2} > 1 > \epsilon$, it follows that $z_1 < -1$ and, consequently, $-1 < z_2 < 0$. Namely, z_2 is inside the unit disk and z_1 is outside. Let $f(z) = -\frac{1}{2\epsilon^2}\frac{(z^2-1)^2}{z(z-z_1)^2(z-z_2)^2}$, then $\Phi(\epsilon) = Res(f,0) + (f, z_2)$. Straightforwardly, $\mathrm{Res}(f,0) = \lim_{z\to0} zf(z) = -\frac{1}{2\epsilon^2}$. For the second-order pole at $z = z_2$, $\mathrm{Res}(f,z_2) = \lim_{z\to0}\frac{d}{dz}((z - z_2)^2 f(z))$. Specifically, for $\epsilon = 0$ the integral becomes

$$\Phi(0) = -\frac{1}{4\pi i}\oint_{|z|=1}\frac{\left(z^2-1\right)^2}{2z^3}dz = -\frac{1}{8\pi i}\oint_{|z|=1}\left(z - \frac{2}{z} + \frac{1}{z^3}\right)dz = \frac{1}{2}. \tag{9}$$

3.2. $\frac{P(x)}{Q(x)}$-type Integral

The first integral that belongs to this class is [9]

$$I = \int_{-\infty}^{+\infty}\frac{1}{(1+x^2)^2}dx. \tag{10}$$

For convenience, let $f(z) = 1/(1+z^2)^2$, and it is clear that for large z, $f(z) \approx 1/z^4$. By using of the residue theorem, $I = \int_{C_1+C_R} f(z)dz = 2\pi i\sum_{k=1}^{n}\mathrm{Res}_{z=z_k} f(z)$. As $R \to \infty$, $\lim_{R\to\infty}\int_{C_R} f(z)dz = 0$ and $\operatorname*{Res}_{z=i} f(z) = g'(i) = -\frac{2}{(2i)^3} = \frac{1}{4i}$ where $g(z) = (z-i)^z f(z) = \frac{1}{(z+i)^2}$. To summarize,

$$I = 2\pi i\left(\frac{1}{4i} + 0\right) = \frac{\pi}{2}. \tag{11}$$

Next, it is useful to compute

$$I = \int_{-\infty}^{+\infty}\frac{1}{x^4+1}dx. \tag{12}$$

Likewise, let $\mathrm{f}(z)=1/(1+z^4)$ and it is easy to check that the poles of $f(z)$ are at $e^{i\pi/4}$, $e^{i3\pi/4}$, $e^{i5\pi/4}$, $e^{ir\pi/4}$. Clearly, only $z_1=e^{i\pi/4}$ and $z_2=e^{i3\pi/4}$ are inside the contour. Since $\operatorname*{Res}_{z=z_1} f(z)=\lim_{z\to z_1}(z-z_1)f(z)=\frac{1}{4e^{i3\pi/4}}$ and $\operatorname*{Res}_{z=z_2} f(z)=\lim_{z\to z_2}(z-z_2)f(z)=\frac{1}{4e^{i9\pi/4}}$, the residue theorem predicts that

$$2\pi i\left(\operatorname*{Res}_{z=z_1} f(z)+\operatorname*{Res}_{z=z_2} f(z)\right)=2\pi i\left(-\frac{2i}{4\sqrt{2}}\right)=\frac{\sqrt{2}}{2}\pi. \tag{13}$$

The third integral of this kind is [10]

$$I=\int_0^\infty \frac{x^{1/3}}{1+x^2}dx. \tag{14}$$

To solve this problem, it is necessary to define $f(z)=\frac{z^{1/3}}{1+z^2}$ and introduce a keyhole-like contour. For the outer contour C_R of radius R, $\int_{C_R} f(z)dz=0$. By contrast, For the inner contour C_r of radius r where $r<1/2$ is assumed, $|f(z)|=\frac{|z^{1/3}|}{|1+2^z|}\le\frac{r^{1/3}}{1-r^2}\le\frac{(1/2)^{1/3}}{3/4}$. Suppose that the last number in the above equation is M, then $|f(\mathrm{z})|<M$. Hence,

$$\left|\int_{C_r} f(z)dz\right|\le\int_0^{2\pi}\left|f\left(re^{ie}\right)\right|\left|ire^{i\theta}\right|d\theta\le\int_0^{2\pi} Mrd\theta=2\pi Mr, \tag{15}$$

Which goes to zero as $r\to 0$. For the remaining two paths, $\lim_{r\to 0,R\to\infty}\int_{C_1} f(z)dz=\int_0^\infty f(x)dx=I$ and $\lim_{r\to 0,R\to\infty}\int_{C2} f(z)dz=e^{i2\pi/3}\int_0^\infty f(x)dx=e^{i2\pi/3}I$.. Since $f(z)$ is meromorphic and the poles of $f(z)$ are at $\pm i$, the residue theorem says $\int_{C_1+C_R-C_2-C_r} f(z)dz=2\pi i\left(\operatorname*{Res}_{z=i} f(z)+\operatorname*{Res}_{z=-i} f(z)\right)$, which means that

$$\left(1-e^{i2\pi/3}\right)I=2\pi i\left(\operatorname*{Res}_{z=i} f(z)+\operatorname*{Res}_{z=-i} f(z)\right). \tag{16}$$

Noticing that $\operatorname*{Res}_{z=-i} f(z)=\frac{(-i)^{1/3}}{-2i}=-\frac{1}{2}$ and $\operatorname*{Res}_{z=i} f(z)=\frac{e^{-i\pi/3}}{2}$, the integral is

$$I=\frac{\frac{-1+e^{-i\pi/3}}{2}2\pi i}{1-e^{i2\pi/3}}=\frac{-\pi ie^{i\pi/3}}{1-e^{i2\pi/3}}=\frac{\pi/2}{\sin(\pi/3)}=\frac{\pi}{\sqrt{3}}. \tag{17}$$

3.3. $f(x)e^{imx}$-type Integral

The first example is [11]

$$I=\int_{-\infty}^{+\infty}\frac{e^{imx}}{(a^2+x^2)^2}dx. \tag{18}$$

For the integrand, it has a second-order pole $|a|i$ in the upper half plane. According to the residue theorem, $\operatorname{Res}\left[\frac{1}{(a^2+z^2)^2},|a|i\right]=\lim_{z\to|a|i}\left[(z|a|i)^2\frac{1}{(a^2+z^2)^2}\right]=\lim_{z\to|a|i}-2(z+|a|i)^{-3}=\frac{-i}{4|a|^3}$. Hence, $I=2\pi\, iRes\left[\frac{1}{(a^2+z^2)^2},|a|i\right]=\frac{\pi}{2|a|^3}$.

The second example is

$$I=\int_0^\infty\frac{\cos x}{x^2+a^2}dx=\operatorname{Re}\left\{\int_0^\infty\frac{e^{ix}}{x^2+a^2}dx\right\} \tag{19}$$

For the integrand, it has a second-order pole $|a|i$ in the upper half plane. According to the residue theorem, $\mathrm{Res}[R(z)e^{ix},|a|i] = \lim_{z\to|a|i}(z-|a|i)\frac{e^{iz}}{z^2+a^2} = \frac{e^{-|a|}}{2|a|i}$. Ultimately, $\mathrm{I} = \frac{\pi}{2|a|}e^{-|a|}$.

3.4. $f(x)\ln(x)$-type Integral

For the integral of the form

$$I = \int_0^\infty \frac{\ln x}{(1+x^2)^2}dx \tag{20}$$

Let $f(x) = \frac{\ln^2 x}{(1+x^2)^2} = \frac{\ln^2 x}{(x+i)^2(x-i)^2}$, which has two pole. As x approaches i and $-i$, the residue are $-\frac{\pi}{4}+\frac{\pi^2}{16}i$ and $\frac{3}{4}\pi-\frac{9\pi^2}{16}i$, respectively. According to the residue theorem,

$$\oint f(x)dx = 2\pi i\sum Res(f(x)) = 2\pi i\left(\frac{\pi}{2}-\frac{\pi^2}{2}i\right) = \pi^3+\pi^2 i. \tag{21}$$

On the other hand, the Eq. (20) can be rewritten as

$$I = -4\pi i\int_0^\infty \frac{\ln x}{(1+x^2)^2}dx + 4\pi^2\int_0^\infty \frac{dx}{(1+x^2)^2}. \tag{22}$$

Hence, comparing the real and immaginary parts of Eq. (21) and Eq. (22), one has

$$\int_0^\infty \frac{\ln x}{(1+x^2)^2}dx = -\frac{\pi}{4}. \tag{23}$$

Another example available is

$$I = \int_0^\infty \frac{\ln(x)}{(x+1)^3}dx. \tag{24}$$

For this integral, the branch of the logarithm satisfies the condition $-\pi < \arg(z) < \pi$, and the integrals over the circumferences vanish. Therefore, the residues satisfy

$$2\pi iRes(f(z)) = \int_0^\infty \frac{log^2(-x+i\epsilon)}{(x-i\epsilon+1)^3}dx - \int_0^\infty \frac{log^2(-x-i\epsilon)}{(x+i\epsilon+1)^3}, \tag{25}$$

where ϵ is a small parameter. In fact, the integral is also equal to $4\pi i\int_0^\infty \frac{\log(x)}{(x+1)^3}dx$. By using of the fact that $\mathrm{Res}(f(z)) = -1$, it is readily to infer from Eq. (25) that

$$\int_0^\infty \frac{\log(x)}{(x+1)^3}dx = -\frac{1}{2}. \tag{26}$$

4. Conclusion

To summarize, this paper takes use of the residue theorem to solve four kinds of different definite integrals. The main idea of the residue theorem is to transfer a path integration to the sum of the residues of the points in the region that the path encloses. As the improvement of the Cauchy integration formula and Cauchy integration theorem, the residue formula is a powerful tool for calculating the integration that a analytic function goes along a path and the integration for other more ordinary functions. The integrands that are considered in the paper include trigonometric function, rational expression, logarithmic function and the exponential function. Among the four basic classifications and many instances of each, the ways to solve them can be summarized as follows. For the trigonometric function, the Euler formula $e^{i\theta} = cos\theta + isin\theta$ should be considered initially, and the complex number is then involved. As for the rational expression, factorization should be done first to get the residue. Finally, for the exponential function and logarithmic function, a keyhole-like contour is necessary to handle the singular points. Clearly, this work demonstrates that the residue theorem is indeed a powerful method to solve a lot of different integrals.

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