Cauchy Residue Theorem’s Application in Improper integrals

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Abstract. Definite integrals are an essential tool for understanding and calculating many aspects of the natural world. An improper integral, one type of definite integral, has either an infinite interval or an integrand that is not defined at one or more points within the interval of integration. In this research, improper integrals are the main concepts that will be discussed. The function can be expressed in terms of its Laurent series expansion–a general form or representation for analytic functions that includes both negative and positive power series of (z – singularity)–about each of its isolated singularities within the contour. And then, finding the singularities that are inside the contour based on the contour function. Calculating the residue of the singularities. Then, substituting the residue with the calculated value, adding all the residue together. Lastly, the result multiplies by, which is the result, namely the integral of the original functions. With the help of the other two methods, Keyhole Contour, and principal values, it is possible to evaluate the integrals and determine the domain of the function. Keyhole Contour separates the function’s domain and evaluates real integrals. And principal values can be used for determining the specific range of the function, so a single value can be chosen.

Keywords: Cauchy residue theorem; Principal value; Definite integrals; Improper integrals.

1. Introduction

The origin of calculus can be traced back to the 17th century when many mathematicians like Isaac Newton and Gottfried Leibniz independently developed the concepts of calculus [1]. This subject plays a significant role in the history of mathematics, and it revolutionized the way people understand and analyze math problems. Integrals, a fundamental concept in mathematics, are widely used in math and other fields like physics, engineering, economics, and statistics [2]. It deals with the areas of the functions. In physics, integrals are used to calculate the work done by a force and the energy stored in a system and the energy stored in a system. In engineering, integrals are used to calculate the stresses in structures and the flow rates in fluids. In economics, its application is calculating the total revenue and total cost functions, which are used to find the profit-maximizing output level for a firm. What’s more, it is efficient to calculate probability density functions, expected values, and variances in statistics with the help of integrals [3].

After seeing how important the integral to daily life is and its applications, it is crucial to find the solving method or the methods of integrating. The Fourier transform is a mathematical method that allows us to analyze a signal in terms of its frequency components [4]. It provides a way to transform a time-domain signal, which represents the variation of a signal over time, into a frequency-domain representation, which represents the relative strength and phase of each frequency component that makes up the signal. The Feynman integral trick is a powerful tool for people to solve complex problems. The Feynman integral trick simplifies complex integrals by introducing an auxiliary parameter that can be differentiated with respect to that parameter [5]. This derivative can often be expressed in terms of the original integral, allowing us to convert a difficult integral into a more general form that can be solved by differentiation and integration. As a matter of fact, not only Fourier Transform and Feynman Integral Trick can be used for calculating integrals, but also Cauchy Residue Theorem can do that. The Cauchy Residue Theorem, one of the integrating methods, is a powerful tool for evaluating the integral of complex functions around closed contours through the analysis of the singularities of the function enclosed by the contour [6].

2. Methods

This section aims to introduce several methods that are frequently used throughout the paper. Sometimes, it could be easier to use Tylor series to calculate the integral. One can change a complex integral that are difficult to cope with into the form that can be handled with the help of series. Take $\int_0^\infty e^{-x^2}dx$ as an example**,** an easier way to do so is [7]

$$\int_0^\infty e^{-x^2}dx = \int_0^\infty \sum_n \frac{\left(-x^2\right)^n}{n!}dx = \sum_n \frac{(-1)^n}{n!}\int_0^\infty x^{2n}dx, \tag{1}$$

where the integral is calculated first while the summation is calculated afterward.

On the other hand, the principal value is a mathematical concept that refers to a specific value chosen from a cluster of possible values. For example, there is the expression of $n + 2\pi$, which means that the result of n is equal to $n + 2\pi, n + 4\pi$, and so on. In this case, there are a lot of angles for one data, so it is necessary to refine the scope into specific groups like $0 \leq x \leq 2\pi$ so as to get one angle. Similarly, in complex analysis, the principal value of a complex logarithm or complex inverse tangent function is the value that lies within a specific range of angles, usually between $-\pi\ and\ \pi$ radians. This is done to avoid discontinuities or multiple-valued functions. In general, the principal value is the preferred or standard value chosen from a set of possible values, based on certain criteria or conventions, to ensure consistency and avoid confusion or ambiguity.

The Cauchy's Residue Theorem gives the generic method for computing the value of a complex tour integral by converting the different kinds of analytic and continuous functions into a general form that enables the function's singularities to be identified and their residues to be calculated:

$$\oint_\gamma f(z)dz = 2\pi i \sum \mathrm{Res}\,(f, a). \tag{2}$$

For different order singularities, it has different calculating methods. For a first-order singularity, the formula for calculating the residue is $\mathrm{Res}(f, z_0) = \lim_{z\to z_0}[(z - z_0)f(z)]$, where z_0 is the first-order singularity of the function $f(z)$ [8]. This formula states that when the function $f(z)$ has a first-order singularity at z_0, its residue is equal to the limit value of that point. The value of the residue can be determined by Laurent expansion of the function. For a second order singularity, the residue is calculated by $\mathrm{Res}(f, z_0) = \lim_{z\to z_0}\left[d/dz\left((z - z_0)^2 f(z)\right)\right]$, where z_0 is the second-order singularity of the function $f(z)$. This formula says that when the function $f(z)$ has a second order singularity at z_0, the residue is equal to the limit of the second derivative at that point divided by two. Similarly, the value of the residue can be determined by Laurent expansion of the function.

3. Applications

3.1. Examples I

The first example is

$$I = \mathrm{P.V.}\int_{-\infty}^{\infty} \frac{e^{ax}}{e^{2x}-1}dx, \tag{3}$$

which has a singular point at $x = 0$. By using variable substitution, which means that substitute e^x with e^{-x} so the upper limits become 0, and lower limits become $-\infty$, it follows that

$$I = \int_0^\infty \frac{e^{ax}}{e^{2x}-1}dx + \int_0^\infty \frac{e^{-ax}}{e^{-2x}-1}dx. \tag{4}$$

After that, set x with $-\ln t/2$ because it can replace the logarithm and the form of e with the simple form: t. Since having the formula $e^{\ln x} = x$, and when changing the independent variable, the upper and lower limits should be changed as well. The Eq. (4) can be rewritten as

$$I = -1/2\int_1^0 \frac{t^{1-a/2}+t^{a/2}}{1-t}dt = -\frac{1}{2}\int_0^1 \frac{1+t^{-\frac{a}{2}}}{1-t}dt - \int_0^1 \frac{1-t^{1+\frac{a}{2}}}{1-t}dt. \tag{5}$$

By introducing Harmonic numbers and Euler Reflection Formula, the integral can be transformed as

$$I = -\frac{1}{2}\left(H_{-\frac{a}{2}} - H_{\frac{a}{2}-1}\right) = -\frac{\pi}{2}\cot\left(\frac{\pi a}{2}\right). \tag{6}$$

Another method based on Cauchy Residue Theorem can be implemented as well. Hence,

$$I = \lim_{\varepsilon\to 0}\int_{|x|>\varepsilon}\frac{e^{ax}}{e^{2x}-1}dx = \int_{\varepsilon}^{\infty}\frac{e^{ax}}{e^{2x}-1}dx + \int_{-\infty}^{-\varepsilon}\frac{e^{ax}}{e^{2x}-1}dx\,. \tag{7}$$

Here, ε is an infinitesimal number. First, change the upper limits and the lower limits to keep the limits same on the two integrals,

$$I = \lim_{\varepsilon\to 0}\int_{\varepsilon}^{\infty}\frac{e^{ax}-e^{(2-a)x}}{e^{2x}-1}dx = \int_{0}^{\infty}\frac{e^{ax}-e^{(2-a)x}}{e^{2x}-1}dx. \tag{8}$$

Namely, the integrand on the right can be extended by continuity at $x = 0$. After then, it is convenient to change the denominator into Tylor series form and finish the remaining parts:

$$I = \int_{0}^{\infty}\frac{e^{ax}-e^{(2-a)x}}{e^{2x}-1}dx = \int_{0}^{\infty}\left(e^{ax} - e^{(2-a)x}\right)\sum_{n=1}^{\infty} e^{-2nx}dx. \tag{9}$$

After calculating the last integral, it is found that

$$I = \sum_{n=1}^{\infty}\frac{1}{2n-a} - \frac{1}{2n+a-2} = -\frac{1}{a} - \sum_{n=1}^{\infty}\frac{a}{a^2-4n^2} = -\frac{\pi}{2}\cot\left(\frac{\pi a}{2}\right), \tag{10}$$

Which is the same as that shown in Eq. (6).

3.2. Examples II

The second example is [9]

$$I = \mathrm{P.V.}\int_{0}^{\infty}\frac{\cos(\ln x)}{x^2+1}dx, \tag{11}$$

which is singular at $x = 0$. This integral can be calculated by virtue of the Residue theorem. To proceed, it is useful to introduce the function $f(z) = \frac{e^{i\log z}}{z^2+1}$. The integrand contour used for $f(z)$ is shown in Fig. 1.

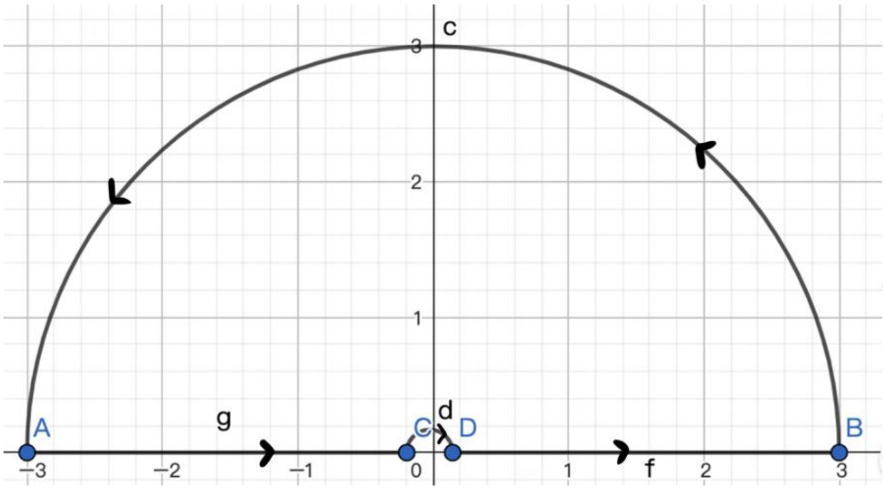

Fig 1. Integrand contour used for $f(z)$

In the upper half circle, the function $f(z)$ only has a simple pole at $z = i$. Integrating $f(z)$ around the contour, it is easily to have

$$\left(\int_{-R}^{-\varepsilon} + \int_{C\varepsilon} + \int_{\varepsilon}^{R} + \int_{CR}\quad\right) f(z)dz = 2\pi i\mathrm{Res}(f, i). \tag{12}$$

Here, the residue is found to be $\mathrm{Res}(f;i)=\frac{e^{-\pi/2}}{2i}$. On the other hand, since

$$\int_{-R}^{-\varepsilon}\frac{e^{ilog\,z}}{z^2+1}dz=-\int_{R}^{\varepsilon}\frac{e^{i(logx+\pi i)}}{x^2+1}dx=e^{-i\pi}\int_{\varepsilon}^{R}\frac{e^{ilogx}}{x^2+1}dx, \tag{13}$$

It is obtained that

$$\left(\int_{-R}^{-\varepsilon}+\int_{\varepsilon}^{R}\quad\right)f(z)dz=(1+e^{-\pi})\int_{\varepsilon}^{R}\frac{e^{ilogx}}{x^2+1}dx. \tag{14}$$

According to the Jordan's lemma, the integral around two circles are $\lim_{\varepsilon\to 0}\int_{C_\varepsilon}f(z)dz=0$, $\lim_{R\to\infty}\int_{C_R}f(z)dz=0$. Based on it, it is easily to get the expression $(1+e^{-\pi})\int_{\varepsilon}^{R}\frac{e^{ilogx}}{x^2+1}dx=\pi e^{-\pi/2}$. Taking real parts of both sides, it is concluded that

$$I=\int_0^\infty\frac{\cos(\ln x)}{x^2+1}dx=\frac{\pi}{e^{\frac{\pi}{2}}+e^{-\frac{\pi}{2}}}=\frac{\pi}{2\cosh\left(\frac{\pi}{2}\right)}. \tag{15}$$

3.3. Examples III

The third example is

$$\mathrm{I}=\mathrm{P.V.}\oint_{\mathrm{C}}\frac{\mathrm{dze}^{2\mathrm{z}}}{\cosh\pi\mathrm{z}}. \tag{16}$$

By using of the relation $\cosh(\mathrm{x})=\cos(\mathrm{ix})=0$, it found that the integrand has singular points at $\mathrm{z}=\pm\mathrm{i}/2$. Considering a rectangle integral contour, this integral is equal to the integral about the individual legs of the contour [10]

$$\mathrm{I}=\int_{-p}^{p}\frac{dxe^{2x}}{\cosh\pi x}+i\int_0^1\frac{dye^{2(p+iy)}}{\cos\pi(p+iy)}+\int_{p}^{-p}\frac{dxe^{2(x+i)}}{\cosh\pi(x+i)}+i\int_1^0\frac{dye^{2(-p+iy)}}{\cos\pi(-p+iy)}, \tag{17}$$

where $p\to\infty$. It should be clear that the 2nd and 4th integrals, i.e., those over the vertical legs of C, will vanish in this limit. That leaves the 1st and 3rd integrals over the horizontal sections, whose combination gives rise to

$$\mathrm{I}=\left(1+\mathrm{e}^{2\mathrm{i}}\right)\oint_{-\infty}^{\infty}\frac{\mathrm{dxe}^{2\mathrm{x}}}{\cosh\pi\mathrm{x}}. \tag{18}$$

This equals, by the residue theorem, the residue at the only pole $z=i/2$ within C:

$$I=\left(1+e^{2i}\right)\cdot 2\pi i\lim_{z\to i/2}\frac{e^{2z}}{\cosh\pi z}=\left(1+e^{2i}\right)\cdot\frac{2\pi ie^{i}}{\pi\sinh\left(\frac{\pi i}{2}\right)}=\frac{2e^{i}}{1+e^{2i}}=\sec(1). \tag{19}$$

4. Conclusion

To conclude, this paper uses several distinct methods to solve the principal value of improper integrals. For the first example, it can be solved by two methods. One is using variable substitution two times, substituting e^x with e^{-x} and setting x with $-\ln t/2$. By using two substitutions, the expression can be simplified a lot because the upper limits and lower limits are the same for two integrals, and the logarithmic can be eliminated. Then, by introducing harmonic numbers and Euler reflection formula, the integral can be transformed into a simple form. The other one is using Cauchy residue theorem. For this purpose, it is necessary to find the singularity and use the Tylor series form to calculate the targeted series. For the second example, it is convenient to introduce a half keyhole-like contour, and the integral can be separated into four parts. By exchanging the upper and lower limits and using Jordan's lemma, the integral can be calculated easily. For the third integral, the trick to finding the singular points is by using the identity $\cosh(x)=\cos(ix)$, and the integrand contour utilized is a rectangle. In a word, the paper demonstrates that the residue theorem is a powerful tool to find the principal value of the improper integral.

References

[1] Ablowitz M. J., Fokas A. S. Complex Variable. Cambridge University Press, 2008.

[2] Arfken G., Weber H. Mathematical Methods for Physicists. Elsevier Academic Press, 2005.

[3] Dai N., Zhang Y. Applying Residue Theorem to Compute Real Definite Integral. Journal of Physics: Conference Series, 2021, 1903: 012022.

[4] Liu K., Shao L. A Summary on Two Types of Real Integrals Using the Residue Theorem. Journal of Physics: Conference Series, 2021, 1903: 012017.

[5] Huang Z., Jin Y., Yang M., Yuan, Y. On the calculation of several definite integrals by residue theorem. Highlights in Science, Engineering and Technology, 2023, 38: 317-322.

[6] Rainwater G. Residue theorems and their applications: computing integrals once thought impossible to evaluate analytically. Capstone Projects and Master's Theses, 2007.

[7] Kalins B., Ayapradha A. J. A Study on Different Approaches for Solving Definite Integrals. International Journal of Recent Technology and Engineering, 2019, 7(6): 736-738.

[8] Shen Y. Residue Theorem and Its Applications. Journal of Tonghua College, 2017, 267(38): 24-26.

[9] Li Q., Lin S. A generalization of the residue theorem. Journal of Qiongzhou University, 2013, 20(5): 30-35.

[10] Taylor J. Complex variables. American Mathematical Society, 2011.