Variance Reduction Techniques Based on Binary Option Pricing

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Abstract. Aiming at the binary option pricing model, control variates and antithetic variates methods are focused on comparing the reasons and influencing factors for the difference of variance reduction effect between them. By adjusting the independent variable of the control variates method and the range of binary options indication function, and by sensitivity analysis of four parameters of the pricing model, the study finds a way to improve the variance reduction ability. Through the detailed mathematical derivation of the antithetic variates method to confirm the law of the 4 parameters in the Black-Scholes model and the binary option price found by the former. And construct more accurate theoretical values to provide better direction for setting options prices in the real world.

Keywords: Control Variates; Antithetic Variates; Binary Option Pricing; Monte-Carlo Method.

1. Introduction

1.1. Background

With the rapid development of the international financial market, option pricing attracted more and more attention, which also leads to a great deal of research. In recent years, a large number of new options derived from standard options have emerged in the international financial derivatives market. The binary option is one of them. To better calculate the Binary options price, numerical calculation methods like the Monte-Carlo method are widely used. The Monte-Carlo method's core idea is to set up a statistical model. The estimated values and statistics of the parameters are derived through repeated random sampling, and the problem's approximative solution is obtained. However, the method will generate some calculation errors affecting the pricing results. The control variates and antithetic variates methods can be used to reduce the variance for pricing the binary option. Due to the difference in the mathematical structure and parameter composition of the two methods, the effect of reducing variance is also different.

1.2. Related Research

In the field of option pricing, Black and Scholes first adopted the method of dynamic replication to conduct pricing and found that the difference in returns and risk of different stocks can be hedged against each other, meaning risk can be ignored in option pricing. They presumption that the underlying stock exhibits continual volatility and drift while following the geometric Brownian motion. [1]. After a few years, Cox and Ross developed the risk-neutral pricing theory based on the Black-Scholes model, which states that if the market arbitrage doesn't exist, the derivative price will depend on the tradable securities, which means their price can be independent of investors' risk aversion [2]. Based on risk-neutral pricing theory, Cox, Ross, and Robinstein constructed a binomial method that greatly optimizes the numerical solution of option pricing. The valid date of the option in the model can be divided into sufficiently small times and the period return of the asset only takes two values: S_0u and S_0d. The probability of moving to a higher value S_0u is p and the lower value S_0d than the current price is 1-p [3].

In the study of binary options, John first defined the binary option as an exotic call option that pays a preset amount if the underlying asset price at maturity exceeds the strike price [4]. Due to the ignorance of the uncertainty related to the underlying asset price at maturity when the traditional binary option model determines the expected return, Thavaneswaran et al. took the underlying asset price at maturity S_T into account with the uncertainty by using fuzzy set theory to price binary options [5]. Based on the economic theory of the British options, Gao defined a new type of binary option that holders have the right to early exercise like American binary options and its payoff will be the best prediction of the European binary payoff under the supposition that a contract drift equals the genuine drift [6].

To better calculate the option price, many researchers use the Monte-Carlo method to get the numerical solution. In addition, many variance reduction techniques on the basis of the Monte-Carlo method like the control variates method and antithetic variates method are wildly developed. Mu compared the effectiveness of the Monte Carlo method and quasi-Monte Carlo method regarding the European call option. The comparison experiment showed that the quasi-Monte Carlo method will be more useful, especially in high dimensional derivative securities pricing [7]. On the basis of the Monte-Carlo simulation, Liu and Gu determine the ideal drift term for the significance sampling of American options while taking into account the variance reduction effect of the importance sampling method, which merely modifies the Brownian motion of the drift term on the simulation pricing of American options [8]. Reuven, Rubinstein, and Ruth considered the relationship between the variance reduction results obtained by using the control variable method and the optimal control matrix in one-dimensional and multidimensional cases. They finally get the result that qualifies the variance reduction loss caused by the optimal control matrix estimation [9]. To solve the error problem in the pricing process of an American put option, Cao proposed a composite variance reduction technique combining the control variates method and antithetic variates method [10].

1.3. Objective

Much research focus on the theory and mathematical derivation of the control variates and antithetic variates method or the variance reduction results after applying this method to the traditional options like the European option and American option. However, few researchers have studied the results of using the two methods of variance reduction mentioned above using binary options from a variety of sets and the control variates method with various independent variables. Therefore, the paper will study the multiple factors affecting control variates and antithetic variates methods and the effect of variance reduction on pricing binary options.

2. Definition in Financial Mathematics

2.1. Black-Scholes Model

The Black-Scholes model is the most traditional mathematical model used for option pricing and hedging which allowed the underlying assets' price to move continuously. The experiment only focuses on the BS pricing instead of clarifying the Black-Scholes model in detail. In the Black-Scholes model, the stock price $S(T)$ at maturity T is as follows:

$$S(T) = S(0)\exp\left(\left(r - \frac{\sigma^2}{2}\right)t + \sigma Z\right), \tag{1}$$

where S (0) is the initial stock price, r is the risk free rate, σ is a parameter representing the stock's volatility and Z has the standard normal distribution.

Assume that the risk-free rate of interest represents the expected return, the paper gets

$$E[S(T)] = e^{rT}S(0). \tag{2}$$

2.2. Binary Option

In [5], the binary option (asset-or-nothing call option) will pay S_T if the stock price S_T is at or larger than the strike price K and it pays zero otherwise. In the experiment, to simplify the calculation of the option pricing and better focus on the effect of variance reduction, two 0-or-1 call option are set as a binary option, the formula is as follows:

$$H(S_T)_i = \mathbf{1}_{K_i} = \begin{cases} 1, & K_i \\ 0, & \text{elsewhere} \end{cases} (i = 1{,}2)$$
$$K_i = \begin{cases} \{S(T) \geq S(0)\}, & i = 1 \\ \left\{\frac{S(0)}{2} < S(T) < S(0)\right\}. & i = 2 \end{cases} \quad (3)$$

When i=1, $H(S_T)_1$ means if the stock price $S(T)$ at time T is larger than $S(0)$, the binary option pays 1 and it pays 0 otherwise. When i=2, $H(S_T)_2$ refers that if the stock price $S(T)$ at time T is in the range of $\frac{S(0)}{2}$ to $S(0)$, the binary option pays 1 and it pays 0 otherwise.

2.3. Monte-Carlo Method

Boyle detailly analyzed the mathematic theorem of the Mont-Carlo method and its application in option pricing [11]. With the aid of computer techniques, Podlozhnyuk and Harris completed the algorithm implementation of the Monte-Carlo method [12]. Here will not dwell too much on MC methods, and the study will use the Monte-Carlo method to calculate the option price based on 2 binary payoff function $H(\mathrm{S_T})$.

3. Method

3.1. Control Variates Method

To estimate $E[X]$, a related variable Y and a constant value b need to be found of it. Besides, X_i and Y_i are independent and identically distributed sequences of random variables. The function can be defined as follows

$$X_{CV}(b) = X_{MC} - b(Y_{MC} - E[Y]), \quad (4)$$

Where X_{CV} is the unbiased estimator of X_i based on control variates method, X_{MC} is the Monte-Carlo estimator of the sequence of X_i, Y_{MC} is the Monte-Carlo estimator of the sequence of Y_i and $E[Y]$ is the average value of Y_i.

The variance of the X_{CV} is as follows

$$Var(X_{CV}(b)) = Var(X_{MC}) - 2bCov(X_{MC}, Y_{MC}) + b^2 Var(Y_{MC}). \quad (5)$$

By setting the derivative in b equal to zero, the study can get the value b^* which minimizes the variance $Var\big(X_i(b)\big)$. Thus, the result of b^* is as follows:

$$b^* = \frac{Cov(X_{MC}, Y_{MC})}{Var(Y_{MC})}. \quad (6)$$

After substituting b^* for b in Equation (5), the study gets the biggest variance reduction, implying that

$$\frac{Var(X_{CV}(b))}{Var(X_{MC})} = 1 - \frac{Cov(X_{MC}, Y_{MC})^2}{Var(X_{MC})Var(Y_{MC})} = 1 - \rho_{XY}^2, \quad (7)$$

Where ρ_{XY} is the correlation between random variable X and Y.

Equation (7) reflects that the stronger the correlation between X and Y, the more significant the variance reduction. Thus, the choice of variable Y is largely important. The experiment will calculate 4 control variates estimator of binary option price $H(\mathrm{S_T})$. Table 1 shows the independent and dependent variables in CV estimation.

Table 1. The Independent and Dependent Variables in Control Variates Estimation

X_{CV}	Y
$H(S_T)_1$	S_T
$H(S_T)_2$	S_T
$H(S_T)_1$	Z
$H(S_T)_2$	Z

3.2. Antithetic Variates Method

3.2.1 Theorem

The antithetic variates method can lead to variance reduction by applicating the negative correlations between variables [13].

If $U \sim \mathcal{U}(0,1)$, U and $1-U$ are antithetic variates. Creating a random variable whose CDF is $F(X)$, by using the universality of the uniform, the research can get $F(X)=U$, where $U \sim \mathcal{U}(0,1)$. Then $X=F^{-1}(U)$ and $X'=F^{-1}(1-U)$ can be written. X and X' are antithetic variates. In the process of option pricing, random variable X is often the payoff of the option.

The antithetic variates method can be combined with option pricing. In Equation (1), there is random valuable $Z \sim \mathcal{N}(0,1)$, and $Z=\phi^{-1}(U)$ where U is follow the uniform distribution. Based on the symmetry of $\mathcal{N}(0,1)$, $\Phi^{-1}(1-U)=-\Phi^{-1}(U)$. Thus Z and Z' are antithetic variates.

Finally, the research considers the payoff function $H(S_T)_1 = \mathbf{1}_{\{S_T \geq S_0\}}$ and stock price in Black-Scholes model. On the basis of Equations (1) and (2), let $X_i = e^{-rT}H\big(S_T(Z_i)\big)$ and $X_i' = e^{-rT}H\big(S_T(-Z_i)\big)$. Both X_i and $X_i{}'$ are antithetic variates, and the antithetic variates estimator of the binary option price is as follows:

$$X_{AV} = \frac{1}{n}\sum_i^n \frac{X_i + X_i'}{2}. \tag{8}$$

3.2.2 More accurate estimation in AV

Focus on the payoff function $H(S_T)_1$, the research makes a deeper mathematical analysis to get a more accurate result of the binary option price. The more accurate antithetic variate estimator of binary options price at maturity is as follows:

$$\begin{aligned} E[H(S_T)_1] &= E\left[\mathbf{1}_{\{S_T \geq S_0\}}\right] \\ &= P(S_T \geq S_0) \\ &= P\left(S_0 \exp\left(\left(r-\frac{\sigma^2}{2}\right)T + \sigma\sqrt{T}Z\right) \geq S_0\right) \\ &= P\left(\left(r-\frac{\sigma^2}{2}\right)T + \sigma\sqrt{T}Z \geq 0\right) \\ &= \Phi\left(\left(\frac{r}{\sigma}-\frac{\sigma}{2}\right)\sqrt{T}\right), \end{aligned} \tag{9}$$

Therefore, the more accurate binary option price is $V_0 = e^{-rT}\Phi\left(\left(\frac{r}{\sigma}-\frac{\sigma}{2}\right)\sqrt{T}\right)$, which has no correlation of S_0. Assuming $c=\left(\frac{r}{\sigma}-\frac{\sigma}{2}\right)\sqrt{T}$ and the mathematic approximation of $\Phi(c)$ is as follows

$$\Phi(c) = \begin{cases} \frac{1}{2} + \int_0^c \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx, & c \geq 0 \\ \frac{1}{2} - \int_0^{-c} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx, & c < 0 \end{cases}. \tag{10}$$

Take $c \geq 0$ as an example that substitutes the variable c in the integral and transforms the result to the expectation.

$$\int_0^c \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx = \frac{1}{\sqrt{2\pi}} \int_0^1 ce^{-\frac{(cu)^2}{2}} du = \frac{1}{\sqrt{2\pi}} E_U\left(ce^{-\frac{(cU)^2}{2}}\right) \tag{11}$$

The result of Equation (11) is plugged in Equation (10), and the antithetic variates are as follow

$$X_i = \frac{e^{-rT}}{2} + \frac{e^{-rT}}{\sqrt{2\pi}} \cdot ce^{-\frac{(cU_i)^2}{2}}, \quad X_i' = \frac{e^{-rT}}{2} + \frac{e^{-rT}}{\sqrt{2\pi}} \cdot ce^{-\frac{(c(1-U_i))^2}{2}}$$
$$X_{AVop} = \frac{1}{n}\sum_{i=1}^{n} \frac{X_i + X_i'}{2}. \tag{12}$$

With the same parameters in MC and CV method, the research will calculate the optimized option price X_{AVop} based on $H(S_T)_1$.
Result

3.3. The result of the binary option price

First, the research fixed suitable parameters for $S(0)$,σ,r and T, which is $S(0)$=100, σ=0.2, r=0.05, T=0.5. Then, python is used to calculate the binary option price based on 3 methods mentioned in the previous part of the article. The total results of the option price and variance reduction are shown in Table 2.

Table 2. The Results of Binary Option Price and Variance Reduction Based on 3 Methods

NO.	Method	Estimate	Std. deviation	Variance Reduction
1	MC for $H(S_T)_1$	0.5286	0.0015	-
2	MC for $H(S_T)_2$	0.4467	0.0015	-
3	CV for $H(S_T)_1$ on S_T	0.5290	0.0010	0.6143
4	CV for $H(S_T)_1$ on Z	0.5289	0.0009	**0.6340**
5	CV for $H(S_T)_2$ on S_T	0.4463	0.0010	0.6143
6	CV for $H(S_T)_2$ on Z	0.4464	0.0009	0.6340
7	AV for $H(S_T)_1$	0.5288	0.0004	0.9222
8	AV for $H(S_T)_1$ optimized	0.5288	0.0000000545	0.9999

3.4. Sensitivity Analysis

In addition, to comprehend the connection between the influence of variance reduction and different parameters in the Black-Scholes model, making the sensitivity analysis based on 4 parameters (T, S_0, r, σ), and observing how the variance reduction of binary options price $H(S_T)_1$ change under the CV estimators X_{CV}.

Changing time T from a certain value 0.5 to a range [0.5,10], which step is 0.5. The variance reduction of X_{CV} is negatively correlated with time T.

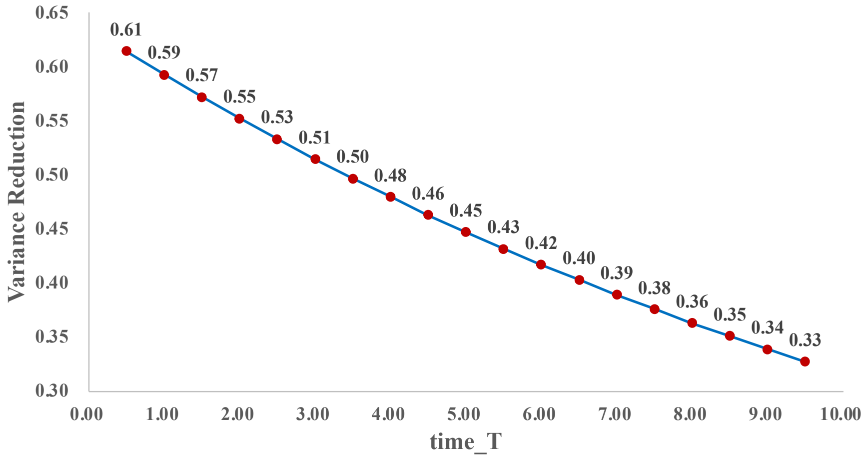

Fig 1. The Relationship between Variance Reduction and time T

Changing stock price S (0) from a certain value of 100 to a range [90,110], which step is 5. The result of variance reduction of X_{CV} is no change.

Changing r from a certain value 0.05 to a range [0.01,0.1], which step is 0.01. The variance reduction is negatively correlated with r.

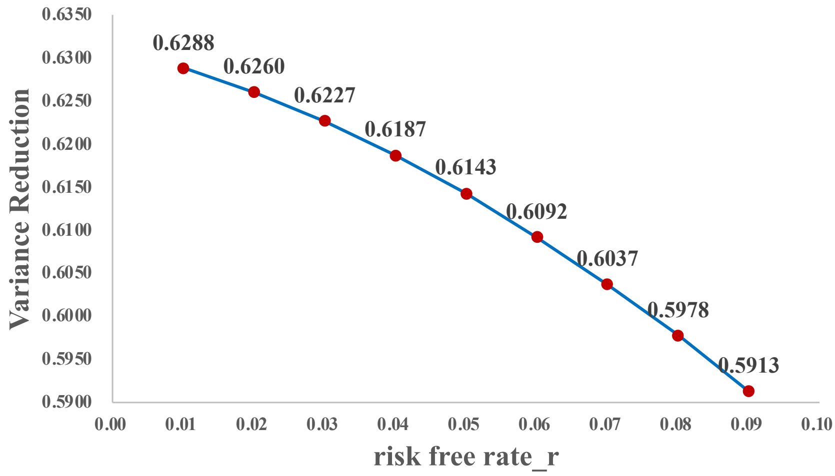

Fig 2. The Relationship between Variance Reduction and risk-free rate r

Changing volatility of stock σ from a certain value of 0.2 to a range [0.1,0.5], which step is 0.01. As σ increases, variance reduction first increases and then decreases. variance reduction is maximized when σ is equal to 0.02.

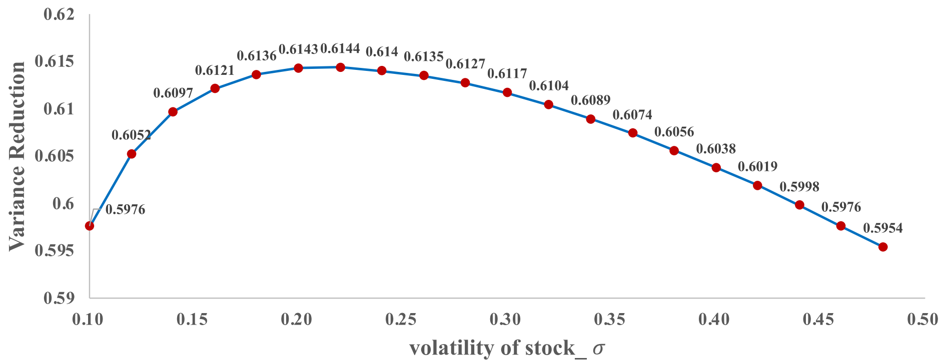

Fig 3. The Relationship between Variance Reduction and volatility of stock σ

Therefore, the result of sensitivity analysis shows that the value of the 3 parameters (T, r, σ) the research first choose is suitable to reduce the variance in the control variates method.

Combining the analysis in section 3.2.2, the binary option price of $H(S_T)_1$ is positively correlated to the parameter $c = \left(\frac{r}{\sigma} - \frac{\sigma}{2}\right)\sqrt{T}$, which means the variance reduction based on antithetic variates method falls when $\boldsymbol{r}$ and $\boldsymbol{T}$ increases, but shows a unimodal trend (first rise then fall) when $\boldsymbol{\sigma}$ increases, and it has no relationship with $S(0)$. These findings are the same as the result of sensitivity analysis on the control variates method

4. Discussion

4.1. Control Variates Method

First, changing the independent variable. The result of No.3 and 4 in table 2 find that the CV variance reduction on Z is larger than on S_T. On the basis of section 3.1, the efficiency of control variates method to reduce variance is based on the correlation between argument S_T or Z and dependent variable $H(S_T)_1$. In the experiment, binary option payoff $H(S_T)_1$ is more related to Z which follows the standard normal distribution than S_T including many other factors like period T, volatility σ and interest rate r. Therefore, the control variates result on Z has a larger variance reduction. This result is also proved in No.5 and 6 based on the second binary payoff function $H(S_T)_2$. Combing the finding to the application of option pricing, Boyle and Broadie confirm that the security to be studied in the application process must have a high degree of positive correlation with its associated securities. This is a crucial requirement to ensure the effectiveness of control variable technology [11,14]. As a result, the control variates method can be used to calculate the option price with more precision for some basic derivative securities since it is simpler to identify the securities with high correlation. However, it is not appropriate for some more complicated securities.

Second, changing the dependent variable. The experiment uses control variates to calculate 2 binary payoff functions based on 2 independents. Comparing No.3 and No.5, the binary price for the 2 payoff functions is different, but the efficiency for variance reduction is the same. No.4 and No.6 also show the same results. These findings reflect a change in the range of indicative functions will not affect the efficiency of variance reduction for the control variates method. In reality, the form of the binary payoff function is not the limit for choosing the control variable method to price the option.

4.2. Antithetic Variates Method

In [12] and [15], Podlozhnyuk et.al proved that the dual variable technique can be effective in studying the price of financial derivative instruments only when the payoff function of the derivative securities is the monotone function of random variables. In the research, the Black-Scholes Model of S_T is a monotonic function and the result in table 2 shows that the variance reduction by AV is greatly effective, which has the same consequence as the study before. In reality, European options, American options, and other simple derivative securities are monotonous, thus using the AV method can get more accurate results. However, this method may not be used for complex new derivative securities such as barrier options and Asian options.

In this study, the monotonicity of Equation (1) is very strong, which is more obvious than the connection between the independent variable and the dependent variable. Therefore, the utility of the AV method is higher than the CV method.

In addition, the AV method can be improved by further mathematical analysis. AV for $H(S_T)_1$ optimized is a true value for the binary option. The final result of AV estimator is just related to $c = \left(\frac{r}{\sigma} - \frac{\sigma}{2}\right)\sqrt{T}$, which means people can achieve their desired option price by artificially adjusting these three parameters. In reality, people often change the risk-free rate and maturity to achieve their target price because the volatility is difficult to change. However, no matter what value people designed, the variance reduction is always close to 100%, because it presents a definite value by the mathematic analysis but not the value simulated by the computer like No.7 from table 2.

5. Conclusion

The paper focuses on the application of control variates and antithetic variates methods to binary options prices. The study compares the reduction value from the control variate method based on 2 independent variables. It also changes the range of indicative functions of binary options. The results show that the effect of variance reduction from the control variates method is based on the degree of correlation between the binary option price and the independent variable chosen. It also reflects that the effectiveness of variance reduction is irrelevant to the range of indicative functions. Additionally, the research makes a sensitivity analysis on 4 parameters from the control variates method and finds out the variance reduction is not affected by S_0, decreases when r and T increase, but shows a unimodal trend when σ increases. It means the experiment can control the effect of variance reduction by changing the value of the parameters related to the Black-Scholes model. Finally, the research finds that the effect of reducing variance from the antithetic variates method is better than the control variates method. The paper also constructs a more accurate estimator by further computing the theoretical value from the antithetic variated method. The results show that the standard deviation of the improved estimator is reduced to a large extent and the estimated value can be just treated as the true value. It is worth investigating whether AV reduces variance better than CV for more binomial options.

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