

Analysis of the Encoding and Decoding Process of Polar Code

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Abstract. This paper briefly introduces the history of channel coding from the hamming code to the newest polar code. Then it introduces that the polar code is the only one that can reach the Shannon limit. With such a crucial technique, this study set it as the core research target of this article. After pointing out the target and the content of the polar code, this paper explains the core principle, which is the channel polarization theory. It also contains channel polarization's two components, the channel combining and splitting theory. In the main part, this paper explains the encoding and decoding of polar code in detail based on the polarization theory. It analyzes the whole process of polar code using examples with multiple inputs. And at last, the result of the likelihood ratio and channel capacity is obtained, which can be utilized to compute the relevant parameters to evaluate the efficiency of encoding and decoding and other rates.

Keywords: Polar Code, Channel Coding, Binary Erase The Channel.

1. Introduction

The signal is faint, easy to interrupt, has a short propagation range, and has poor coverage in the early stages of communication technology development. Reliable communication via channel coding wasn't possible until Shannon published the famous Shannon Theorem in 1948[1]. To counter interference in the transmission process, the so-called channel coding, also known as error control coding, involves adding redundant information to the original data at the sending end that is related to the original data. The main goals of channel coding are to both prevent and partially rectify errors.

In 1949, the first major development in channel coding took place. R. Hurming and M. Garay proposed the Hamming code, the first usable error control coding system [2]. Fix block codes' flaws till convolutional codes come into existence. Over the following ten years, wireless communication performance grew dramatically because of convolutional coding. Convolutional codes, on the other hand, are also quite complicated, and they fall short of Shannon's theoretical upper bound [3]. Turbo codes were then invented to increase encoding efficiency through repetitive decoding significantly. Turbo codes are frequently employed in the 3G and 4G mobile communication series' encoding technology [4]. Today's topics on the stage include 5G, LDPC, and Polar codes. The oldest channel coding strategy to the Shannon limit is the LDPC code, which Robert Gallager suggested in 1962 [5]. A lecturer at Birken University in Turkey named E. Arikan first proposed the Polar code in 2007 [6]. Although Polar was proposed late as a coding scheme that has been theoretically proved to reach the Shannon limit, it is destined to be a landmark coding method.

In the work that Erdal Arikan wrote in 2009, he went into greater depth about channel polarization. He suggested a brand-new coding technique that was based on it [7]. Polar Code is the name. The only channel coding technique that can be categorically demonstrated to "reach" the Shannon limit is polar code, which has a deterministic building mechanism. The channel polarization theory is the key theory of polar code, comprised of two components—channel combining and channel splitting.

Channel combining is an operation that divides the input bit data into a group in pairs. It performs modular two addition to obtain two results and then performs the same operation after grouping the results of each group in pairs and iterates continuously. Channel splitting is the second step of channel polarization. It is embodied in channel association. Its general process is to split the joint channel into two sub-channels. At this time, the transfer probability of the two sub-channels also changes, so the polarization code completes the channel splitting. The most important thing in channel splitting is the

derivation of the transition probability of the subchannel that is different from the previous one, which will be discussed later in the paper.

Polarization codes can accommodate input and output of varied lengths with the same encoder and decoder structure, with 1 bit as the change step, in order to acquire higher coding gain. Polarization codes offer minimal coding and decoding complexity as well as consistent encoding and decoding structure. Moreover, a polarization code high throughput decoder has been developed. Claude created the 237Gbps polarization code decoder with the highest throughput.

At present, the research of polarization code mainly focuses on four aspects: encoding construction, decoding algorithm, hardware implementation and application of polarization [8]. The Monte Carlo method, first proposed in the literature by Arikan [7], is the oldest construction algorithm, yet, its complexity makes it challenging to apply in engineering. Mori and Tanaka proposed a novel construction method called Density Evolution (DE), which applies the method of density evolution in LDPC codes to the design of polarization codes [9]. It provides good results, but the computational complexity is significantly higher than before. At the same time, the research of polarization code construction in discrete channels is gradually extended to continuous channels, such as the Gaussian channel, Rayleigh channel, relay channel, and so on [10]. In addition, polar codes are gradually applied to multiple access channels, quantum channels, and so on [11].

The continuous deletion list decoding (SC) algorithm first developed by Arikan is the decoding algorithm [7]. In contrast to conventional coding techniques, the SC decoding algorithm performs poorly for codes of finite length. For this reason, researchers attempt to apply various decoding algorithms to decoding polarization codes that have demonstrated great performance in other codes. The BP decoding technique, for instance, is used to decode polarization codes and has had excellent success doing so using LDPC decoding [12]. The LP decoding algorithm and BCJR decoding algorithm are also applied to the polarization codes decoding method. All of the solutions mentioned above have flaws in complexity or application scope, despite the fact that some performance advancements have been made in this area of research. Moreover, some academics have tried to improve the SC decoding algorithm.

For example, the sequential decoding algorithm (SCL) proposed by Tal et al., and the stack decoding algorithm (SCS) proposed by Niu Kai et al., have greatly improved the performance of the decoding algorithm of polarization codes, and played an important role in promoting the application of polarization codes [13,14].

This paper will give a review on polar code. It includes three parts: channel polarization, encoding, and decoding part of polar code. Mostly, it will focus on the channel polarization part, for it's the core of the whole process.

2. Principle

2.1. Channel coding

As shown in figure 1, the data transferred and the data received frequently differ significantly because of outside noise and other interference. As seen in figure 2, the so-called channel coding, also known as error control coding, involves adding redundant information linked to the original data at the sending end to reduce interference during transmission. This correlation allows for detecting and correcting transmission errors at the receiving end. The primary purposes of channel coding are error prevention and partial error correction.

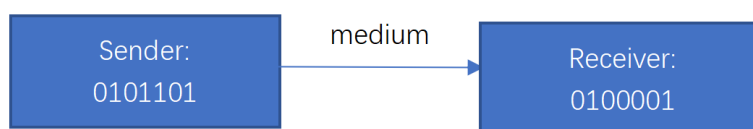


Figure 1. An example of error occurrence

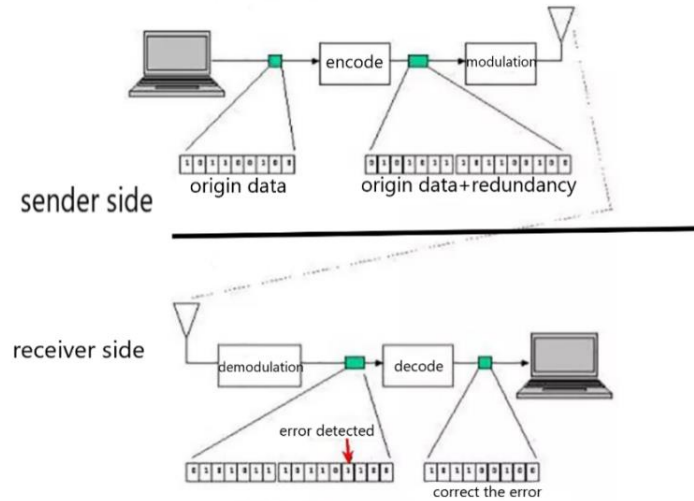


Figure 2. The whole flow of data sending and receiving with channel coding

2.2. Polar code

When the codeword's length is N , the encoding complexity is $O(N \log N)$. The core idea of the Polar code is channel polarization theory. Different channels correspond to different polarization modes. This paper mainly discusses the encoding and decoding method of the binary discrete memoryless channel (B-DMC) proposed by E. Arikan [7].

P.s. The operations of vector and matrix discussed here are operations on the field of bivariate, namely GF (2).

2.3. Channel polarization

Channel polarization is divided into two stages: Channel Combining and Splitting. For any B-DMC W and any $\delta \in (0, 1)$, when $N = 2^n \rightarrow \infty$, in the polarization channel, the proportion of number of channel which satisfy $I(W_N^i) \in (1 - \delta, 1]$ is nearly $I(W)$; on the other hand, the proportion of $I(W_N^i) \in [0, \delta)$ is nearly $1 - I(W)$. For any B-DMC W , $I(W) > 0$, and for any $R < I(W)$, there exist a sequence set $A_N \in \{1, \dots, N\}$, $N \in \{1, 2, \dots, 2^n, \dots\}$. For all $i \in A_N$, $|A_N| \geq NR$, and $Z(W_N^i) \leq O(N^{-\frac{5}{4}})$.

At this stage, the independent N copies of channel W (B-DMC) are combined to produce a vector channel recursively $W_N: X^N \rightarrow Y^N$, $N = 2^n$. Recursion begins at level 0 ($n=0$), using only 1 copy of W , and defined $W_1 \triangleq W$. The first level ($n=1$) recursion combines two independent copies, as shown in Figure 3, to obtain the vector channel $W_2: X^2 \rightarrow Y^2$, whose transfer probability is:

$$W_2(y_1, y_2 | u_1, u_2) = W(y_1 | u_1 \oplus u_2) W(y_2 | u_2) \quad (1)$$

The second level's iteration is shown in figure 4. W_2 is regarded as subchannel. And it combines two independent subchannel to obtain $W_4: X^4 \rightarrow Y^4$.

$$W_4(y_1^4 | u_1^4) = W_2(y_1^2 | u_1 \oplus u_2, u_3, u_4) W_2(y_3^2 | u_2, u_4) \quad (2)$$

The mapping relation from figure 4: $u_1^4 \rightarrow x_1^4$ can be represented by the following formula $x_1^4 = u_1^4 G_4$, G_4 is the generator matrix. $G_4 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix}$. So $W_4(y_1^4 | u_1^4) = W^4(y_1^4 | u_1^4 G_4)$.

The general form is shown in figure 4. It is the reality of the polarization of single channel W_N . The two independent copies are $W_{N/2}$, which combine to the whole channel. u_1^N : Input vector, s_1^N : First translation of input vector, $s_1^N: s_{2i-1} = u_{2i-1} \oplus u_{2i}, s_{2i} = u_{2i}, 1 \leq i \leq N/2$, R_N : Bit reversal

sort operation, extract the input in odd position. The input is s_1^N and the output is v_1^N . And v_1^N will be the input of two $W_{N/2}$

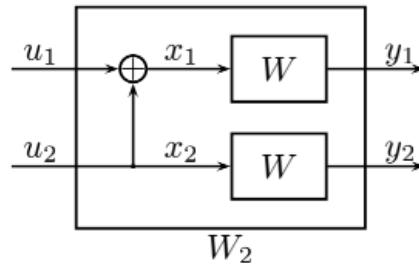


Figure 3. An illustration of single operation in two inputs. P.s. it does not need the bit reversal sort operation.

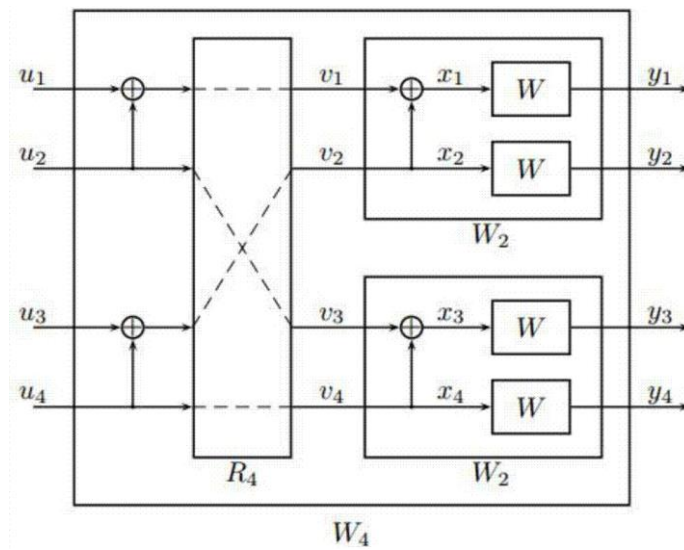


Figure 4. An example of iteration of four input or the second level iteration

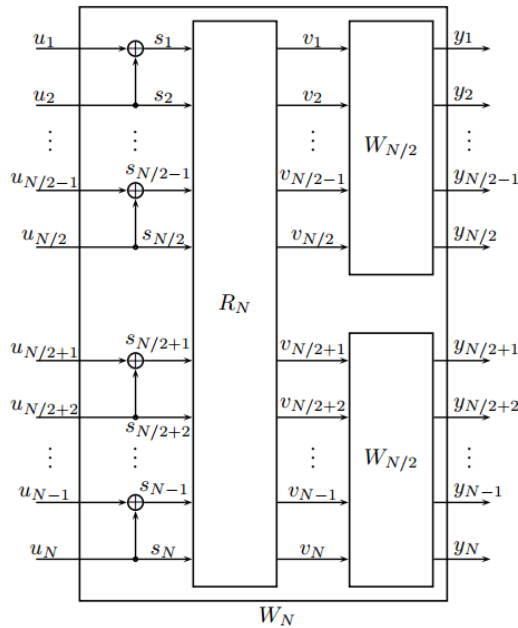


Figure 5. The general process of channel combining with N input and output

It's the second step of channel polarization. To split the W_N that combined by channel combining operation into N coordinate channels with binary input. It's defined on the side of decoding. From the channel combining, it is obvious that the y_n can be considered as output transmitted in n different

channels(theoretical only, not in reality). Also, the n different channels may interact with the others because of the modular binary addition(the meaning of ' $\oplus$ '), which can be reflected in the generator matrix G_N . So on the side of decoding, if it gives the encoding result y_N and aiming to interpret the input sequence u_N , this paper is first going to split the combined channel into subchannels. Shown in figure 5, there is a view of each subchannel. From the character of the encoding process, it can be found that apart from y_n , this study need u_0 to u_{i-1} to interpret u_i . Define the subchannel $W_N^i: u_i \rightarrow Y^N \times X^{i-1}$, $1 \leq i \leq N$, and u_i With a hat means that it is already known.

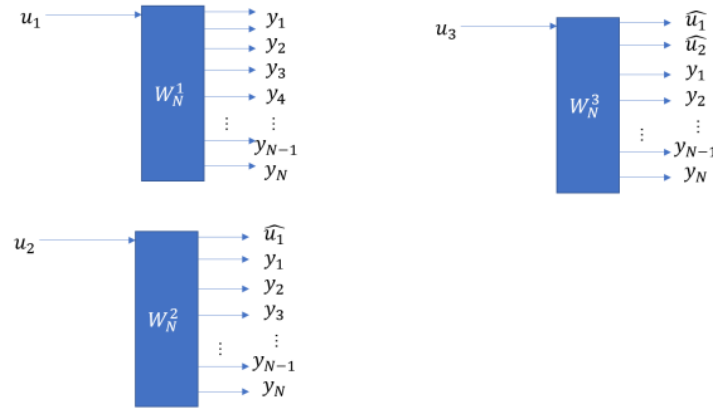


Figure 6. A view of subchannel in channel splitting for single output (u_i) in decoding

There are three examples in figure 6 from u_1 to u_3 that showing how the channel splitting works in the decoding side.

3. Realization approach of polar code

Based on the channel polarization theory, the encoding and decoding part can be devised as below.

3.1. Encoding

As for the communication channel, there are multiple input bits, defined as u_i , $\bar{u} = [u_1, u_2, \dots, u_i]$, N is the number of input bits. Taking the 2-bit input as an example, as shown in figure 3, it is the simplest model of polar code encoding. After the operation to the u_i , this study can obtain the codeword x_i . In the 2-bit example, $\begin{cases} x_1 = u_1 \oplus u_2 \\ x_2 = u_2 \end{cases}$ (1). Using the general method of channel coding, the codeword, in which G_N is the generator matrix. In this example, from (1) this study can obtain $G_2 = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$. So in this 2-bits input polar code based on discrete memoryless channel $(u_1, u_2)G_2 = (x_1, x_2)$. 4-bits input as shown in the figure 4,

$$x_i = \bar{u} \times G_N \quad (3)$$

$$\begin{cases} x_1 = u_1 \oplus u_2 \oplus u_3 \oplus u_4 \\ x_2 = u_3 \oplus u_4 \\ x_3 = u_2 \oplus u_4 \\ x_4 = u_4 \end{cases} \quad \# \quad (4)$$

Similarly, from (3) and (4) this study can obtain the generator matrix $G_4 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix}$.

Using the same principle, this study can obtain $G_8, G_{16} \dots G_N$. The operation flow of G_8 is shown in figure 7. In this structure, it is easier to give explanation to each input.

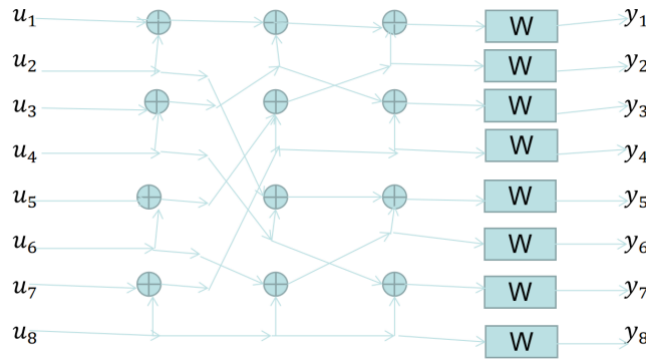


Figure 7. The polar encoding process of 8-bits input

In the second encoding structure, this study try to keep the mod-2 addition operation($\oplus$) at the top of each operation. Taking the 8-bits input as an example, this structure can be shown as figure 8. The comparison can be found between figure 8 and figure 4.

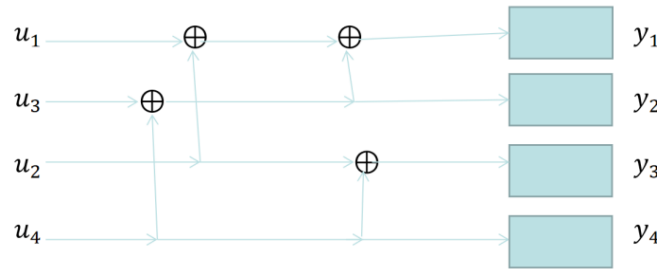


Figure 8. The second kind of structure of encoding of 4-bits input

It can be easily found that the position of the input had been transited before being sent into the process. This study can define that $(\widetilde{u}_1, \widetilde{u}_2, \widetilde{u}_3, \widetilde{u}_4)$ is the input bits after switching the odd and even

position($[u_1, u_3, u_2, u_4]$ in this example). $\widetilde{\mathbf{x}} = \widetilde{\mathbf{u}} \times \widetilde{G}_N \#(4)$.
$$\begin{cases} \widetilde{x}_1 = u_1 \oplus u_2 \oplus u_3 \oplus u_4 \\ \widetilde{x}_2 = u_2 \oplus u_4 \\ \widetilde{x}_3 = u_3 \oplus u_4 \\ \widetilde{x}_4 = u_4 \end{cases} \quad (5).$$
 From

(4) and (5) this study can derive the generator matrix in this structure that is different from what in

the first kind of encoding structure. $\widetilde{G}_4 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix}$. The 2-inputs generator matrix is the same

between two kinds of structure, which is $\widetilde{G}_2 = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$. This study can find that $\widetilde{G}_4 = G_2 \otimes G_2$, $\widetilde{G}_8 = G_2 \otimes \widetilde{G}_4$..." $\otimes$ " is the Kronecker product. From this similarity, this study can draw a general conclusion to any generator matrix: $\widetilde{G}_N = \widetilde{G}_{N/4} \otimes \widetilde{G}_{N/2}$. P.s. In the following part, $\widetilde{G}_N$ will be denoted as G_N for convenience, but the structure will be declared first. A property of generator matrix is $G_N = G_N^{-1} \#(6)$.

Through this method, this study can calculate the target generator matrix and codewords with any number of input. It is defined under the circumstance of the second structure. There are some definition: $u_1^N = [u_1, u_2, u_3 \dots u_N]$, $x_1^N = [x_1, x_2, x_3 \dots x_N]$, $x_1^N = u_1^N \times G_N$, $G_N = B_N F^{\otimes n}$, $N = 2^n$, $F = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$, $B_N = R_N(I_2 \otimes B_{N/2})$, $(S_1, S_2, \dots, S_N)R_N = (S_1, S_3, \dots, S_{N-1}, S_2, S_4, \dots, S_N)$, R_N is used for switching the odd and even bit and the origin $B_2 = I_2$, $B_4 = R_4(I_2 \otimes B_2) = R_4$. For example,

In this way there is no need for the inputs to change the parity order before being processed. While in some articles, the $G_N = F^{\otimes n}$. Under this circumstance, the parity order needs to be changed first. The detailed operation are as figure 9. A practical way to finish this process is shown in figure 10. In

this method, this study need to first give the binary rank of u_1^N . Then it needs to be flipped and the reversed value is equal to the processed output rank in order.

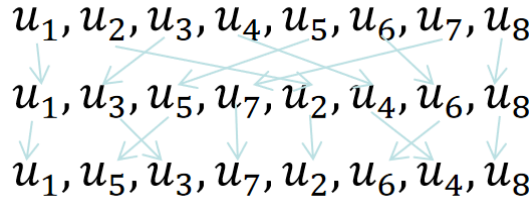


Figure 9. An example of change the parity order of 8-bits input. The bits in odd position will be put forward for $n - 1$ times.

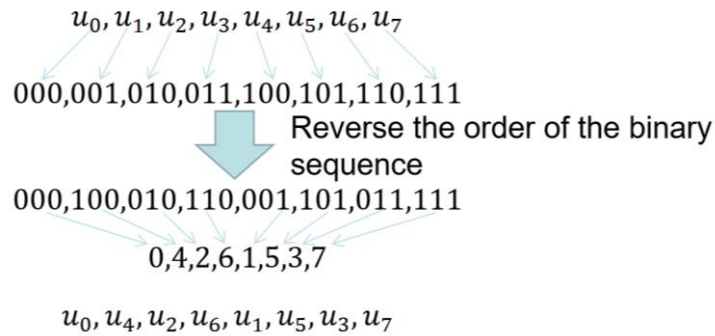


Figure 10. An practice method of change the parity order of 8-bits input.

The second method to obtain the generator matrix and furthermore the codewords is based on the first structure. In this method $G_N = \begin{bmatrix} G_{2z} \\ G_{2r} \end{bmatrix}$. Defining that $G_N = [\bar{c}_1, \bar{c}_2 \dots \bar{c}_N]$, $G_{2z} = [\bar{c}_1, 0, \bar{c}_2, 0 \dots, 0, \bar{c}_N]$. It means adding a zero-vector next to each column vector. $\bar{c}_N$ is the column vector of the generator matrix. $G_{2r} = [\bar{c}_1, \bar{c}_1, \bar{c}_2, \bar{c}_2 \dots, \bar{c}_N, \bar{c}_N]$. “r” means repetition, for it repeats each column.

3.2. Decoding

Decoding in the reverse process of channel encoding. This study have known the encoded results $[x_1^N]$, and this study need to obtain $[u_1^N]$. This study use the first encoding structure, $[u_1^N]G_N = [x_1^N]$. From the property of the generator matrix(6), this study can obtain that $[u_1^N = x_1^N G_N^{-1}]$. This study first use figure 3 as an example. It is a $N=2$ input polar encoding. This study have known the possibility $P(x_1 = 0), P(x_1 = 1), P(x_2 = 1), P(x_2 = 0)$. The $N=4$ example is shown in figure 11.

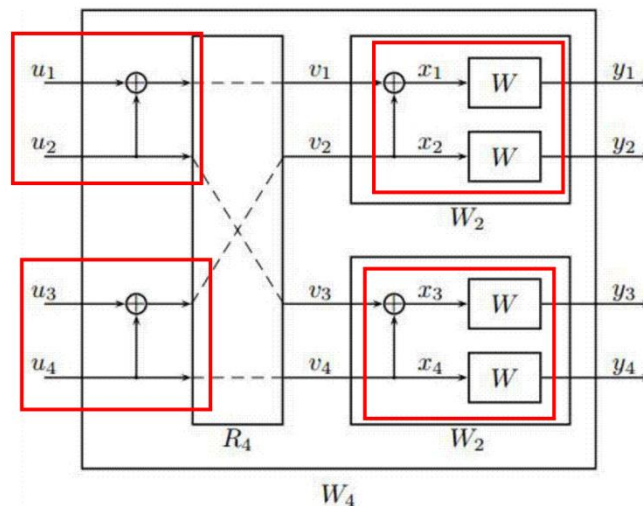


Figure 11. An example of 4-inputs polar encoding system.

The red frame can be taken as a W_2 subsystem. For convenience, $LR(u_i) = \frac{P(u_i=1)}{P(u_i=0)}$ in the following contents, while $LR(u_i) = \frac{P(u_i=0)}{P(u_i=1)}$ in some books. To understand the decoding process more thoroughly, this study can using a tree structure instead of seeing it directly. The tree structure can help us devise the algorithm clearly when experimenting. There are totally $1 + \log_2 N$ number of tree level, N is the number of total input. $N = 2^n$. $i = 0, 1, \dots, 1 + \log_2 N$ is the number of level. And the i^{th} level has 2^i nodes. In figure 12, it is an example that $N=8$ and this study are aiming to decode the u_8 since this study have known the $[u_1, u_2, u_3, \dots, u_7]$.

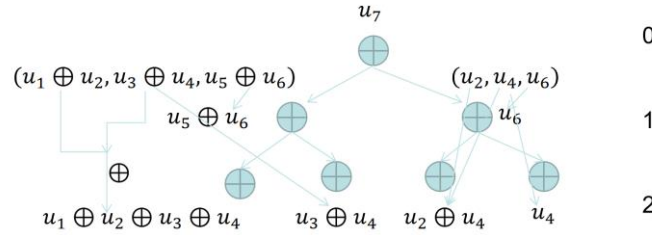


Figure 12. An example of 8-input tree structure.

In the tree structure, this study need to obtain u_i since this study have known the $[u_1, u_2, u_3, \dots, u_{i-1}]$. This study can summarize the principles as below.: If i is odd: Allocate the bit on the right on the head node as u_7 in figure 12. The odd decoding bit sequence $\bar{x}_o = [u_1, u_3, \dots, u_{i-2}]$, The even decoding bit sequence $\bar{x}_e = [u_2, u_4, \dots, u_{i-1}]$. The left-child node of each head node $\bar{x}_l = \bar{x}_e \oplus \bar{x}_o$. The right-child node $\bar{x}_r = \bar{x}_e$. If i is even, no bits will be allocated on each head node. To calculate the likelihood ratio, this study can find that the node with allocated bit (i is odd) is $LR(u_j) = LR_1^{1-2^{u_a}}(u_l)LR_r(u_r)$, the node without allocated bit (i is even) is $\frac{LR(u_l)+LR(u_r)}{LR(u_l)LR(u_r)+1}$. u_a is the allocated bit on the node, u_l and u_r is the left and right child node, i is the number of any node.

In the tree structure, this study can use the recursive method as above to calculate the likelihood ratio in any nodes and then obtain the information rate.

information data. While in the high-capacity channels, this study can transmit the information bits.

4. Conclusion

The article presented in this paper provides a comprehensive explanation of the polarization code process, including its principles and coding and decoding methods. The coding process is based on channel combination theory, while the decoding process relies on channel splitting theory, both forming the basis of channel polarization theory. In addition to discussing the standard coding structure, this paper also explores an alternative approach using two recursive methods from different perspectives. The decoding process uses a tree structure and a recursive method of calculating the likelihood ratio to give the reader a clear understanding of the decoding process of polar codes. The article also introduces a recursive method to calculate the channel capacity in binary erasure channels, which results from the input polarization structure with $N=2$. The Bhattacharyya parameters are then introduced to understand the split channel capacity better, enabling a recursive calculation of capacity attribution. These results can be used to decide which bits are sent through channels with low capacity, while those with high reliability can be used to transmit critical information bits. In addition, this paper provides a thorough analysis of the polar code process and its advantages, with multiple explanations of each step. Polar codes have been shown to reach the Shannon limit and have promising applications. Therefore, this paper is a valuable reference for those seeking to understand the basic process of polar codes.

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