Smooth Uniform Approximation of Multivariate Spline Function

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Abstract. Multivariable spline approximation is of great significance both in mathematics and in practice. This involves the approximation of non-smooth functions. In this paper, the approximation of the non-smooth function $\min_2(g_1, g_2, \cdots, g_n)$, which is the most representative problem in robust statistics, is discussed. We propose a compact smooth uniform approximation algorithm using spline curves.

Keywords: Multivariate Spline; Smooth Uniform Approximation; Non-Smooth Function; Inductive; First-Order Smoothness.

1. Introduction

When a particular function is differentiable and its derivatives are continuous, we say that the function is smooth or differentiable. First-order smoothness means that the first-order derivative of the function exists and is continuous, second-order smoothness means that the second-order derivative of the function exists and is continuous, and so on, infinite smoothness means that all the first-order or partial derivatives of the function exist and are continuous in the usual sense. "Non-smooth" means that the differential and partial fractions of some order in an equation do not exist, or are not continuous.

In a mathematical programming problem, if the problem contains an objective function or a constraint function, and if one of the objective functions or a constraint function in the problem is not smooth, then the problem can be called a non-smooth problem [1]. At present, people have not studied the appropriate algorithm for the non-smooth optimal problem, which is limited to some specific cases. In recent years, non-smooth optimization has attracted much attention, including semi-smooth optimization, non-smooth optimization and matrix cone optimization. A non-smooth optimization problem often studied in the current research field is $\min_{t\in Y} g(t)$, where Y is the feasible domain, and this optimization problem has constraints [2]. According to penalty function theory, under certain conditions, we can treat the original non-smooth problem with constraints as an optimization problem without constraints, that is, $\min_{t\in R^n} g(t)$. A typical non-smooth function used for Rubon statistics is $\min_2(g_1, g_2, \cdots, g_n)$, and the typical methods to solve the non-smooth optimization problem include semi-smooth method, interior point method and smoothing method [3]. In this paper, we construct a smooth function $L\min_2(t_1, t_2, \cdots, t_n)(n \geq 3)$ uniformly approximates $\min_2(t_1, t_2, \cdots, t_n)(n \geq 3)$, and then compound function $L\min_2(g_1, g_2, \cdots, g_n)$ uniformly approximates $\min_2(g_1, g_2, \cdots, g_n)$ by using multivariate splines.

2. Non-smooth optimization algorithm

The commonly used iteration method is adopted. Each iteration uses $-h_i$ as the search direction, where $h_i \in \partial g(t_i)$ is used. If the step size ξ_i is determined in a linear manner rather than in a non-linear manner, the resulting row of points may not converge. Different step selection methods will lead to different algorithm convergence and function differentiability requirements [4]. Next, we will focus on how to choose the important step ξ_i :

(1) $\xi_i = \xi$. It is also relatively simple to use this method to obtain the step length algorithm, but the result is that the algorithm does not converge.

(2) Select such ξ_i, so that it meets condition $\xi_i > 0, \lim_{i\to\infty}\xi_i = 0, \sum_{i=1}^{\infty}\xi_i = \infty$, even if in this case, the algorithm generated by the selection method is convergent, the convergence speed will not be ideal, or even achieve the expected effect.

(3) $\xi_i \equiv \xi_0 m^i (0 < m < 1)$

The advantage of such step size selection is that it can meet the requirements of the algorithm, that is, ensure linear convergence of R and meet the effect of actual operation. However, whether the point column generated by the algorithm converges depends on the selection of ξ_0, m, which is an unsatisfactory result [5]. In the actual computing environment, people almost will not use this step size selection method, because it is not easy to choose the appropri ξ_0, m te B to ensure the convergence of the algorithm.

(4) $\xi_i \approx \mu\frac{g(t_i)-g^*}{\| h_i \|}, 0 < \mu < 2, g^*$ is the minimum of $g(t)$

The above step size selection method can achieve the expected effect, that is, ensure the linear convergence of the algorithm R, the only pity is that the algorithm must know g^* before running [6].

3. Semi-smooth study

The degree of semi-smoothness of a function can be measured by another quantity, first discussing the properties of the local Lipschitz function:

Definition 3.1: If the function $G:\phi^n \to \phi^n$ is locally Lipschitz continuous, then it is known by Rademacher's theorem that G is differentiable or smooth almost everywhere, let L_G represent the set of G differentiable, and let $\partial_C G(t)$ be defined as follows

$$\partial_C G(t) = \{J = \lim_{t_i \to t, t_i \in L_G} G'(t_i)\}$$

Then $\partial_C G(t)$ is called the Bouligand subderivative of G at point t. And $\partial_\xi G(t) = co\{\partial G(t)\}$ is called the generalized Jacobian of G at t.

Definition 3.2: The function $G:\phi^n \to \phi^n$ is a local Lipschitz function, for any direction $l \in \phi^n$, given a point t, if the following limit $\lim_{\substack{U\in\partial G(t+tl)\\ l\to l, t\to 0^*}} Ul$

We call G smooth at half t. If the function G is semi-smooth according to the above definition, we know that the directional derivative is equal to the limit defined above [7]. The familiar smooth functions, convex functions, and subsmooth functions are all semi-smooth functions, the above definition was originally proposed by Qi and Sun, and they also showed that if G is semi-smooth at t, then the directional derivative of G at t exists in every direction f, and

$$Uf - G'(t; f) = o(|f|), \ f \to 0$$

This is true for all $U \in \partial G(t+f)$. If for all $U \in \partial_C G(t)$ is non-singular, then given a point t, consider the problem in the neighborhood $N(t)$ of that point, given the constant ψ according to the specific case, such that for any $y \in N(t)$ and $U \in \partial_C G(y)$, we have $|U^{-1}| \le \psi$:

Assuming that t^* is a solution to the equation $G(t) = 0$, and that G is semi-smooth at t^*, and that all $U \in \partial_C G(t^*)$ is non-singular, then the iterative sequence $\{t^i\}$ generated by the algorithm is reasonably defined, and when $\left|t^* - t^0\right|$ is sufficiently small, the iterative sequence converges superlinearly to t^*.

We are given A region in a two-dimensional Euclid space ϕ^2, set as L, and for uniformity we denote the set of binary i real coefficient algebraic polynomials as Q_i, which is defined as follows:

$$Q_i = \left\{ q = \sum_{i=0}^{i} \sum_{j=0}^{i-i} \psi_{ij} x^i y^j \right\}$$

Definition 3.3: $q \in q_i$ binary polynomial a is called an irreducible polynomial if the polynomial is not divisible by any polynomial other than the constant and the polynomial itself [8]. If the polynomial $k(x,y)$ is an irreducible polynomial, then the equation of the algebraic curve is

$$\Xi : k(x,y) = 0, k(x,y) \in q_i$$

The space definition of a multivariate spline function is:

$$S_i^{\mu}(\Delta) := \left\{ s \in \psi^{\mu}(L) : s|_{L_i} \in Q_i, i = 1, \cdots, N \right\}$$

In order to achieve the overall smooth spline function to achieve the goal of smooth splicing, the following conditions must be met, which is embodied in the following theorem:

Theorem 3.1: Let $z = s(x,y)$ have different expressions on two adjacent cavities L_i and L_j on a given region L, defined as

$$z = q_i(x,y) \text{and} z = q_j(x,y)$$

Where $q_i(x,y), q_j(x,y) \in Q_i$ is such that condition $s(x,y) \in \psi^{\mu}\left(\overline{L_i \cup L_j}\right)$ is satisfied, if and only if the following conditions are satisfied, that is, there is a polynomial $w_{ij}(x,y) \in Q_{i-(\mu+1)d}$ that is

$$q_i(x,y) - q_j(x,y) = \left[k_{ij}(x,y) \right]^{\mu+1} \square w_{ij}(x,y)$$

The public internal network cable between $\overline{L_i}$ and $\overline{L_j}$ is

$$\Xi_{ij} : k_{ij}(x,y) = 0$$

And irreducible algebraic polynomial $k_{ij}(x,y) \in Q_l$. The polynomial factor $w_{ij}(x,y)$ defined above is called the smooth cofactor on the inner network line $\Xi_{ij} : \pm k_{ij}(x,y) = 0$. The existence of a smooth cofactor on the inner network Ξ_{ij} means that the equation as defined in the above theorem holds [9]. Let U be any given inner dot and the associated region of vertex U be $St(U)$. And $St(U)$ has N elements, that is, cell $L_1, L_2, \cdots, L_N$ is denoted as $\Xi_{ij} : k_{ij}(x,y) = 0$ by the common inner network line of any two adjacent cell cavities L_i and L_j, and we specify as follows

$$\Xi_{ij} = \Xi_{ji}; k_{ij}(x,y) = k_{ji}(x,y)$$

The smooth cofactor $w_{ij}(x,y)$ defined on the inner network line Ξ_{ij} and the smooth cofactor $w_{ji}(x,y)$ attached to the inner network line Ξ_{ji} satisfy the following relationship:

$$w_{ij}(x,y) = -w_{ji}(x,y)$$

Derive the definition of the coordination condition "at vertex U:

$$\sum\nolimits_U [k_{ij}(x,y)]^{\mu+1} \cdot w_{ij}(x,y) \equiv 0$$

For the sum of products of smooth cofactors and irreducible algebraic curves on an internal line with all internal nodes U being one endpoint, $w_{ji}(x,y)$ is the smooth cofactor on the corresponding irreducible curve Ξ_{ij}. Assuming a given partition Δ on region L, all its internal nodes are correspondingly specified as $U_1, \cdots, U_M$, then we give the concept of "global coordination condition", which is defined in the following way:

$$\sum\nolimits_{U_n} [k_{ij}(x,y)]^{\mu+1} \cdot w_{ij}(x,y) \equiv 0, n = 1, \cdots, M$$

All coordination conditions corresponding to the inner dot U_n and $w_{ij}(x,y)$ must satisfy the provisions or conditions specified in Theorem 3.1.

Theorem 3.2: In the given region L according to the specific problem, when the partition Δ is given according to the convention or the established way, the sufficient and necessary condition for the existence of the multivariate spline function $s(x,y) \in S_i^{\mu}(\Delta)$ is that the spline function $s(x,y)$ has a smooth cofactor on each internal line, and at the same time satisfies the global coordination condition specified by us. If the section Δ on region L appears to be formed in the following way: some straight lines through region L are used as internal lines to cut region L,

Let the region L be a simply connected region in the Euclidean space ϕ^n, now with a finite block hyperplane

$$\eta; a_1x_1 + a_2x_2 + \cdots + a_nx_n + b = 0$$

If region L is part Δ, region L is divided into a finite number of small cell cavities $L_1, \cdots, L_m$. The boundary surface of any cell cavity L_j can be called the partition surface of partition Δ. Polysplines are essentially polynomials, so here we define the class of polynomials of degree i with n elements:

$$Q_i := \left\{ \sum_{0 \le i_1 + \cdots i_n \le i} \psi_{i_1 \cdots i_n} x_1^{i_1} \cdots x_n^{i_n} \right\}$$

At the same time define the n-dimensional spline function as follows:

$$S_i^{\mu}(\Delta) := \{ s(x_1, \cdots, x_n) \in \psi^{\mu}(L) : s(x_1, \cdots, x_n)|_{L_j} \in Q_i, j = 1, \cdots, m \}$$

Similar to the binary spline function, also exists

Theorem 3.3: Let the expression of $s(x_1, \cdots, x_n)$ on two adjacent cell cavities L_i, L_j be $q_i(x_1, \cdots, x_n), q_j(x_1, \cdots, x_n)$ respectively to satisfy the following conditions $s(x_1, \cdots, x_n) \in \psi^{\mu}\left(\overline{L_i \cup L_j}\right)$, and its sufficient and necessary condition is that there is an equation $w_{ij}(x_1, \cdots, x_n) \in Q_{i-\mu-1}$ satisfying the relation

$$q_i(x_1, \cdots, x_n) - q_j(x_1, \cdots, x_n) = [k_{ij}(x_1, \cdots, x_n)]^{\mu+1} \cdot w_{ij}(x_1, \cdots, x_n)$$

Where $\eta_{ij} : k_{ij}(x_1, \cdots, x_n) = 0$ is the common inner profile of cell L_i and L_j.

The polynomial $w_{ij}(x_1, \cdots, x_n)$ in the above theorem is called a smooth cofactor of the inner section η_{ij}, from which the concept is derived. We specify $k_{ij}(x_1, \cdots, x_n) \equiv k_{ji}(x_1, \cdots, x_n)$ so according to the equation above, we get

$$w_{ij}(x_1, \cdots, x_n) \equiv -w_{ji}(x_1, \cdots, x_n)$$

The coordination condition on the inner edge AB is

$$\sum\nolimits_{AB} [k_{ij}(x,y,z)]^{\mu+1} \cdot w_{ij}(x,y,z) \equiv 0$$

Theorem 3.3: For L given region A, according to the specific research method, a partition method Δ is determined if the following conditions are met: The sufficient and necessary condition of polynomial $s(x,y,z) \in S_i^{\mu}(\Delta)$ is that the polynomial $s(x,y,z)$ must be accompanied by a smooth cofactor in each inner section, and all the associated regions or star regions of the inner edges meet the corresponding coordination conditions specified by us, that is, smooth cofactor coordination conditions.

Suppose $\eta_{ij} : k_{ij}(x,y,z) = 0$ is any intrinsic internal profile of flow line $\vec{\psi}$. When we start from a source cell that we have identified and flow through the profile in the direction or manner determined by the flow diagram $\vec{\psi}$, we call the union of all cells that can only be entered after crossing η_{ij}

$\cup(\eta_{ij}^{+})$. The set of all cell cavities that flow line $\vec{\psi}$ passes through before passing through inner profile η_{ij} is denoted $\cup(\eta_{ij}^{-})$ as a whole.

Define the front of η_{ij}:

$$g_r(\eta_{ij}) = \cup(\eta_{ij}^{+}) \setminus \cup(\eta_{ij}^{-})$$

Given any intrinsic internal profile of $\vec{\psi}$ hypothetical flow line A, whose equation is $\eta_{ij} : k_{ij}(x,y,z) = 0$ for the convenience of future algorithm design, a generalized truncation polynomial is defined below:

Theorem 3.4: In any given region and given partition of Euclidean space, any high-dimensional spline function $s(x,y,z)$ in $S_i^{\mu}(\Delta)$ can be uniquely represented as

$$s(x,y,z) = q(x,y,z) + \sum_{\vec{\psi}} [k_{ij}(x,y,z)]_{*}^{\mu+1} \square w_{ij}(x,y,z)$$

The polynomial $q(x,y,z) \in Q_i$ in the above equation is the expression of the polynomial $s(x,y,z)$ in the source cell cavity, and $\sum_{\vec{\psi}}$ represents the sum of the smooth cofactors on all the intrinsic inner sections of the streamline $\vec{\psi}$ and the product of the inner section equation.

4. Basic knowledge of simple complex

If for A finite set of points in a Dean Euclidean space or in a field of real numbers, denoted $A = \{a_0, a_1, \cdots, a_n\}$, the set of real numbers $\mu_0, \mu_1, \cdots, \mu_n$ must all be 0 if the following two conditions (1) $\sum_{i=0}^{n} \mu_i = 0$ (2) $\sum_{i=0}^{n} \mu_i a_i = 0$ are met, then they are said to be in a general position (or geometrically independent).

Definition 4.1: If there is A set of $n+1$ points $\{a_0, a_1, \cdots, a_n\}(n \geq 0)$ in a general position in a Euclidean space, and this set of points forms a convex hull, then the convex hull is called a n dimensional simplex [10]. It is also marked $(a_0, a_1, \cdots, a_n)$, and each point $a_i(i = 0,1,\cdots,n)$ is defined as the corresponding vertex of the simplex.

For any nonempty subset of A given Euclidean space, let's call it A. Then we represent the convex hull of the nonempty subset a as follows:

There are only $\left\{\sum_{a \in A} \mu_a a \mid \mu_a \geq 0\right\}$ finite number of them that are not 0 and satisfy $\left\{\sum_{a \in A} \mu_a = 1\right\}$ so let's take the point set $(a_0, a_1, \cdots, a_n) = \left\{\sum_{i=1}^{n} \mu_i a_i \mid \mu_i \geq 0, \sum_{i=1}^{n} \mu_i = 1\right\}$ $\forall x \in (a_0, a_1, \cdots, a_n)$. If there is such a case as a non-negative set of real numbers $\{\mu_0, \mu_1, \cdots, \mu_n\}$, such that the point satisfies $x = \sum_{i=0}^{n} \mu_i a_i$, and the following case $\sum_{i=0}^{n} \mu_i = 1$ is satisfied. Such a set of real numbers is uniquely determined by x, proving that if $\{\mu_0', \mu_1', \cdots, \mu_n'\}$ also fits the requirements, there will be the following result:

(1) $\sum_{i=0}^{n} (\mu_i - \mu_i') = \sum_{i=0}^{n} \mu_i - \sum_{i=0}^{n} \mu_i' = 0;$

(2) $\sum_{i=0}^{n} (\mu_i - \mu_i') a_i = \sum_{i=0}^{n} \mu_i a_i - \sum_{i=0}^{n} \mu_i' a_i = x - x = 0$

So we can derive $\mu_i - \mu_i^{'} = 0(i = 0,1,\cdots,n)$ from the general position of $\{a_0, a_1, \cdots, a_n\}$. We can now call the following set $\{\mu_0, \mu_1, \cdots, \mu_n\}$ full point x the barycentric coordinates of the simplex formed by the set of vertices $\{a_0, a_1, \cdots, a_n\}$.

The n dimensional subspace spanned by vector set $\{a_1 - a_0, \cdots, a_n - a_0\}$ is denoted L, and the hyperplane $L + a_0$ can be obtained by translating L as the translation vector of a_0. And that gives us $L + a_0 = \{\sum_{i=0}^{n} \mu_i a_i \mid \sum_{i=0}^{n} \mu_i = 1\}$. So $L + a_0 = L + a_i (i = 0,1,\cdots, a_n)$ is called the hyperplane spanned by $\{a_0, a_1, \cdots, a_n\}$.

Since $\{a_0, a_1, \cdots, a_n\}$ is in a general position, then we can deduce that every point x on $L + a_0$ determines an array $\{\mu_0, \mu_1, \cdots, \mu_n\}$ such that $x = \sum_{i=0}^{n} \mu_i a_i$, and $\sum_{i=0}^{n} \mu_i = 1$, call it the barycentric coordinates of x with respect to $\{a_0, a_1, \cdots, a_n\}$. Finally, the nonempty set $(a_0, a_1, \cdots, a_n)$ is a nonempty subset of points on the hyperplane $L + a_0$ that have nonnegative barycentric coordinates.

If A point in Euclidean space is called point $e_i = (0, \cdots, 0, 1, 0, \cdots)$, then $\{e_1, e_2, \cdots, e_{n+1}\}$ satisfies the condition of being in a general position, and it can be concluded that $(e_1, \cdots, e_{n+1})$ is a n dimensional natural simplex.

If the vertices of the simplex σ_1 are all vertices of the simplex σ_2, that is, the simplex σ_1 is all contained in the simplex σ_2, then σ_1 is said to be a face of σ_2, denoted $\sigma_1 < \sigma_2$. For example, obviously every vertex $\sigma_1 < \sigma_1, \sigma_1$ is a face of σ_1. If $\sigma_1 < \sigma_1$ and the dimension of the simplex σ_1 is less than the dimension of the simplex σ_2, then σ_1 can be said to be the true face of σ_2.

For example, a simplex $\sigma = (a_0, a_1, a_2)$, then its true face is: 0 dimensional face a_0, a_1, a_2;1 dimensional face $(a_0, a_1), (a_0, a_2), (a_1, a_2)$.

If $\sigma_1 < \sigma_1$ is considered as a set of points, the inclusion relation $\sigma_1 \subset \sigma_2$ is obtained. If is the true face of σ_2, then $\sigma_1 \subset \partial\sigma_2$. Conversely, every edge point of the simplex σ_2 must be on one of the true faces of the simplex σ_2. For example, if $x \in (a_0, a_1, \cdots, a_n)$, then its center of gravity coordinate $\mu_n = 0$, can also be on the true plane $(a_0, \cdots, a_{n-1})$. In summary, the edge of a simplex can be considered to be the union of all its true faces.

We know that we can define the corresponding polynomial on the simplex, so the first step is to first discuss how to find a suitable network representation of the defined polynomial on the simplex, which brings convenience to the study [11]. The symbol ϕ^n represents the n dimensional vector space of real numbers, and the symbol Z_+^n represents the set of all n repositive integer indicators. Consider $\xi = (\xi_1, \cdots, \xi_n) \in Z_+^n$, whose length is defined as $|\xi| = \sum_{i=1}^{n} \xi_i$, and the factorial is defined as follows: $\xi! = \xi_1! \cdots \xi_n!$

Let the symbol M be a non-negative integer, for the convenience of research and discussion in this paper, we stipulate the following symbol definition

$$\binom{r}{\xi} := \frac{r!}{\xi!}$$

It is now specified that the symbol e_i represents a vector in the n dimensional real space ϕ^n where almost all components are 0, but only the i component is 1.

Let the symbol σ represent a non-degenerate simplex in the real space ϕ^n, and the vertices of the simplex are respectively $v_0,\cdots,v_n$ then the simplex σ can be expressed as not $\sigma=\{v_0,\cdots,v_n\}$. For any $x\in\phi^n$, there is a unique $(\xi_0,\cdots,\xi_n)\in\phi^{n+1}$ that makes

$$\xi_i v_i,\sum_{i=0}^{n}\xi_i=1$$

Such $\xi=(\xi_0,\cdots,\xi_n)$ is called the barycentric coordinates of x with respect to the simplex σ. $u_i=(u_i^1,\cdots,u_i^n)\in\phi^n, i=0,\cdots,n$ notation

$$U(u_0,u_1,\cdots,u_n)=\begin{vmatrix}1 & 1 & \cdots & 1\\ u_0^1 & u_1^1 & \cdots & u_n^1\\ \vdots & \vdots & & \vdots\\ u_0^n & u_1^n & \cdots & u_n^n\end{vmatrix}/n!$$

Then $U(u_0,u_1,\cdots,u_n)$ is the directed volume of a simplex $\{u_0,u_1,\cdots,u_n\}$. Thus, we can conclude that the barycentric coordinate $(\xi_0,\cdots,\xi_n)\in\phi^{n+1}$ of a simplex $\sigma=\{v_0,\cdots,v_n\}$ for any point $x\in\phi^n$ can be expressed as follows:

$$\xi_i=V(v_0,\cdots,v_{i-1},x,v_{i+1},\cdots,v_n)/U(v_0,\cdots,v_n)$$

For $x\in\phi^n,\xi\in Z_+^n$ we specify the definition of monomial x^ξ:

$$x_1^{\xi_1}\cdots x_n^{\xi_n}$$

At the same time, it is stipulated that the space composed of all polynomial functions defined on the real number space of ϕ^n is denoted $\eta=\eta(\phi^n)$, where the subspace composed of all polynomials whose degree is less than or equal to k is denoted $\eta_k=\eta_k(\phi^n)$, and it is easy to know that this space is linear, then it conforms to the property of linear space.

Suppose that in the n dimensional real space B, for A given non-degenerate simplex $\sigma=\{v_0,\cdots,v_n\}$, which is non-empty, the set of indices $\xi=(\xi_1,\cdots,\xi_n)\in Z_+^n$, order

$$C_\xi(x)=\binom{|\xi|}{\xi}\xi^\xi$$

Here ξ is the barycentric coordinates of x with respect to the simplex σ. C_ξ is called the Bernstein polynomial corresponding to the index ξ. Any $q\in\eta_k$ can be uniquely represented as:

$$q=\sum_{|\xi|=k}b(\xi)C_\xi$$

Let $(b(\xi))_{|\xi|=k}$ be the barycentric coordinates of the polynomial $q\in\eta_k$ with respect to σ. make

$$x_{\xi,\sigma}:=\sum_{i=0}^{n}\xi_i v_i/k$$

Consider the following correspondence

$$x_{\xi,\sigma}\mapsto b(\xi),|\xi|=k$$

For any given simplex σ, the mapping defined above is called the Bernsted-Bezier net of some polynomial q about the simplex, or simply the C net $(B-net)$. Given this mapping, a one-to-one correspondence is established between the polynomial q and its C net representation F (that is, the mapping), similar to a single function.

5. Conclusion

In this paper, we discuss the approximation problem of A typical non-smooth function $\min_2(g_1, g_2, \cdots, g_n)$ in the field of robust statistics. The study of smooth uniform approximation of function $\min_2(g_1, g_2, \cdots, g_n)$ can help to solve the complementarity problem.

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