The Derivation and Some Explanations of the Uncertainty Principle

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Abstract. This paper is a small discussion about the uncertainty principle. With the great discoveries and researches made by predecessors, I try to answer some questions within the uncertainty principle. Using the definition of momentum, the momentum in de Broglie wave, the property of cosine function, the Euler's formula, the basic property of plane wave, the law of derivative, the energy of quantum discovered by Max Planck, the photon energy discovered by Albert Einstein, the energy equation of a system and the Hamiltonian operator, I figure out the Schrodinger equation successfully. With the help of Schrodinger equation, the possibility density of a wave function, the concept of normalization, the average value of position and momentum, the Fourier transform, the integration by parts and the limit of normalized wave functions, I find the position operator and the momentum operator. Then using some knowledge of the standard deviation and the variance, the concept of inner product of vectors, some property of Dirac symbol, the law of calculation of complex number and the law of operator calculation, I obtain the general uncertainty relationship of abstract mathematical quantities. According to the general relationship, I get the inequality of uncertainty principle, and discuss the specific relationship between the position and the momentum. After getting the inequality, I use Heisenberg's experiment of measuring an electron by using the γ ray to hit the electron to explain why we cannot measure the position and the momentum simultaneously. Based on Heisenberg's experiment, I give my own explanation by discussing the earth and a spacecraft. Transforming the perspective, we can easily see how the accuracies of position and momentum change. In addition, I attempt to figure out why the uncertainty exists. However, I can only explain it by discussing the time and frequency in mathematical method and have no idea to work it out in a pure physics approach.

Keywords: Quantum physics, classical physics, particle, uncertainty, position, momentum, Fourier transform.

1. Introduction

In the past centuries, physicists always tried to explain everything that happened in the universe, from the beginning of the cosmos, the big bang, to the things happening today, from the smallest quark to the whole universe. As the understanding of the world went deeper, with the technological development, scientists usually were allowed to gain some new information about the world, which often accompany holes in the old theory. Before the 20th century, the skyscraper of physics had already been built completely, and what remains was some decoration work. However, the beautiful sky of physics was covered by two black clouds [1, 2]. With the discovery of wave-particle duality, people realized that the building of physics system had an enormous gap between the microsystem and classical physics. As a result, with the effort of many famous physicists of that time, including Albert Einstein, Erwin Schrodinger and many other people, a new theoretical system was founded: quantum physics.

From the experience of physics through the whole period of the 20th century, we can easily know that the world and the universe still contain a huge number of things that we do not know. In addition, the old theory may not be completely perfect, and in the old theories, the holes and the unsolved questions are very likely to exist. In that case, we should think about the unsolved problems and make the skyscraper of physics more elaborate. In quantum fields, despite that quantum physics has tended to be matured throughout the 20th century and the first two decades of the 21st century, there are still many mysteries in the system. For example, why the particle has the property of uncertainty, and why

can people not measure the position and momentum of a particle at same time? No obvious evidence can interpret these questions completely. Suppose we can accurately see the state of the microparticle. In that case, we can make fewer mistakes when using the quantum computer, and eliminate many unnecessary steps when making new materials made by quantum substance, so it is very important to figure out the origin of uncertainty. In this paper, I will discuss the uncertainty principle, using the known knowledge concluded by predecessors, to try to explain these questions.

2. Mechanism

In quantum physics, many things are totally different from the things in classical field, as the momentum. In classical physics, the expression of momentum is

$$p = mv \tag{1}$$

However, in quantum field, due to the wave-particle duality raised by de Broglie, the momentum of the microparticle is [3]

$$p = \frac{h}{\lambda} \tag{2}$$

For the easiest wave, the cosine wave, we have

$$\psi(x) = \cos kx \tag{3}$$

We know the cosine function is periodic. One period corresponds to one wavelength, so based on the basic concept of cosine function, the wavelength is $\lambda = \frac{2\pi}{k}$

Then putting this relation into equation (2), we can get

$$\begin{aligned} p &= \frac{h}{\frac{2\pi}{k}} \\ &= \frac{hk}{2\pi} \\ &= \hbar k \left(\hbar = \frac{h}{2\pi}\right) \end{aligned} \tag{4}$$

With the help of Euler's formula [4]:

$$e^{ix} = \cos x + i \sin x \tag{5}$$

We can expand the wave function from the cosine wave to plane wave:

$$\psi(x) = \exp(ikx) \tag{6}$$

This is the time-independent form. If we add time into the function, we can get this:

$$\psi(x,t) = \exp[i(kx - \omega t)] \tag{7}$$

And make twice partial derivative to x, and then we will have these equations:

$$\begin{aligned} \frac{\partial}{\partial x}\psi(x,t) &= ik \exp[i(kx - \omega t)] \\ &= ik\psi(x,t) \end{aligned} \tag{8}$$

$$\begin{aligned} \frac{\partial^2}{\partial x^2}\psi(x,t) &= (ik)^2\psi(x,t) \\ &= -k^2\psi(x,t) \end{aligned} \tag{9}$$

Although not very rigorous, we can conclude this:

$$\frac{\partial^2}{\partial x^2} = -k^2 \tag{10}$$

And for the time t, we can also make once partial derivative, which looks like this:

$$\begin{aligned} \frac{\partial}{\partial t}\psi(x,t) &= -i\omega \exp[i(kx-\omega t)] \\ &= -i\omega\psi(x,t) \end{aligned} \tag{11}$$

Because of the energy equation discovered by Max Planck and Albert Einstein, which is [5][6]

$$E = h\nu \tag{12}$$

If we combine these relationships:

$$\hbar = \frac{h}{2\pi} \tag{13}$$

$$\omega = \frac{2\pi}{T} = 2\pi\nu \tag{14}$$

So, we can get another form used to describe the energy of a particle:

$$E = \hbar\omega \tag{15}$$

It means

$$\frac{\partial}{\partial t}\psi(x,t) = -i\frac{E}{\hbar}\psi(x,t) \tag{16}$$

$$\begin{aligned} -\frac{\hbar}{i}\frac{\partial}{\partial t}\psi(x,t) &= i\hbar\frac{\partial}{\partial t}\psi(x,t) \\ &= E\psi(x,t) \end{aligned} \tag{17}$$

In classical physics, we know that the expression of kinetic energy is

$$\begin{aligned} E_k &= \frac{1}{2}mv^2 \\ &= \frac{m^2v^2}{2m} \\ &= \frac{p^2}{2m} \end{aligned} \tag{18}$$

So, the kinetic energy in quantum field is

$$\begin{aligned} E_k &= \frac{p^2}{2m} \\ &= \frac{(\hbar k)^2}{2m} \\ &= -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} \end{aligned} \tag{19}$$

For the total energy of a particle, we have this formula:

$$E = E_k + V \tag{20}$$

And if we know the potential energy $V(x,t)$, we can write the Schrodinger equation in one dimension [7]:

$$i\hbar\frac{\partial}{\partial t}\psi(x,t) = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\psi(x,t) + V(x,t)\psi(x,t) \tag{21}$$

One thing necessary for Schrodinger equation is that the wave function must be normalized (equation (29)). For higher dimensions, the steps are similar to those in one dimension. The difference is that we need to make twice partial derivative of $x,\ y,\ z$ and so on. However, the twice derivative of each axis is too time-consuming and troublesome to write. Luckily, scientists use an operator called Hamiltonian operator to replace the complex form of partial derivative, which is

$$\nabla = \frac{\partial}{\partial x}\vec{\imath} + \frac{\partial}{\partial y}\vec{\jmath} + \frac{\partial}{\partial z}\vec{k} \tag{22}$$

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \tag{23}$$

So, we can write high dimension Schrodinger equation in this way [7]:

$$i\hbar\frac{\partial}{\partial t}\psi = \left(-\frac{\hbar^2}{2m}\nabla^2 + V\right)\psi \tag{24}$$

For one dimension Schrodinger equation (equation (21)), we can make some changes on it:

$$\frac{\partial}{\partial t}\psi(x,t) = \frac{i\hbar}{2m}\frac{\partial^2}{\partial x^2}\psi(x,t) - \frac{i}{\hbar}V(x,t)\psi(x,t) \tag{25}$$

And we also find its conjugate from:

$$\frac{\partial}{\partial t}\psi^*(x,t) = -\frac{i\hbar}{2m}\frac{\partial^2}{\partial x^2}\psi^*(x,t) + \frac{i}{\hbar}V(x,t)\psi^*(x,t) \tag{26}$$

Combine equation (25) and equation (26):

$$\begin{aligned}\frac{\partial}{\partial t}\|\psi\|^2 &= \frac{\partial}{\partial t}(\psi\psi^*)\\ &= \frac{i\hbar}{2m}\left(\psi^*\frac{\partial^2}{\partial x^2}\psi - \psi\frac{\partial^2}{\partial x^2}\psi^*\right)\\ &= \frac{i\hbar}{2m}\left[\frac{\partial}{\partial x}\left(\psi^*\frac{\partial}{\partial x}\psi - \psi\frac{\partial}{\partial x}\psi^*\right)\right]\end{aligned} \tag{27}$$

We already know that for any wave function $\psi(x,t)$, the possibility of the position $P(x,t)$ satisfies this equation [8]:

$$P(x,t) = \|\psi(x,t)\|^2 \tag{28}$$

If we integrate $\psi(x,t)$ over the whole real number domain, we will get this [8]:

$$\int_{-\infty}^{+\infty} P(x,t)\,dx = \int_{-\infty}^{+\infty}\|\psi(x,t)\|^2\,dx = 1 \tag{29}$$

We can see that the result of possibility is equal to 1, which means we can certainly find the particle in the whole position space. Of course, the momentum has the same property, which is the concept of normalization. It is very important to judge whether a wave function can be normalized or not. In more situations, the coefficient before the item $e^{i(kx-\omega t)}$ is unknown, and the normalization can help us get the exact value of the coefficient. However, some wave functions, such as the spherical wave, which cannot satisfy equation (29), since they have the discontinuous part over the whole real numbers, which means we cannot integrate them over the whole real number domain. In other words, they cannot be normalized, and we cannot find the value of coefficient. Until this step, we cannot solve this difficult problem with wave functions that cannot be normalized, so we only discuss those can be normalized. According to equation (29), for the possibility of the particle at the position x, we use a special operator $\langle x\rangle$ to represent it, which is the average value of x, and $\langle x\rangle$ should satisfy [8]

$$\begin{aligned}\langle x\rangle &= \int_{-\infty}^{+\infty} xP(x,t)\,dx \\ &= \int_{-\infty}^{+\infty} x\|\psi(x,t)\|^2\,dx\end{aligned} \tag{30}$$

So, we can make a partial derivative of time t, and we can get

$$\begin{aligned}\frac{d\langle x\rangle}{dt} &= \int_{-\infty}^{+\infty} x\frac{\partial}{\partial t}\|\psi\|^2\,dx \\ &= \frac{i\hbar}{2m}\int_{-\infty}^{+\infty} x\frac{\partial}{\partial x}\left(\psi^*\frac{\partial}{\partial x}\psi - \psi\frac{\partial}{\partial x}\psi^*\right)dx\end{aligned} \tag{31}$$

From the basic law of derivatives of two functions multiplied together, we have

$$[f(x)g(x)]' = f'(x)g(x) + f(x)g'(x) \tag{32}$$

Making indefinite integral for both two sides, we obtain this:

$$\int [f(x)g(x)]'\,dx = \int f'(x)g(x)\,dx + \int f(x)g'(x)\,dx \tag{33}$$

It can also be written in this form:

$$\int \frac{d[f(x)g(x)]}{dx}\,dx = \int \frac{df(x)}{dx}g(x)\,dx + \int \frac{dg(x)}{dx}f(x)\,dx \tag{34}$$

Writing it in a simple way, we get

$$\int f(x)\,dg(x) = f(x)g(x) - \int g(x)\,df(x) \tag{35}$$

This is known as integration by parts, which can be inferred from the Newton-Leibniz formula [9]. We can use this equation to deal with the integral of two functions multiplied together, so if we use it to calculate equation (31), we will obtain this result:

$$\begin{aligned}\frac{d\langle x\rangle}{dt} &= \frac{i\hbar}{2m}\int_{-\infty}^{+\infty} x\frac{\partial}{\partial x}\left(\psi^*\frac{\partial}{\partial x}\psi - \psi\frac{\partial}{\partial x}\psi^*\right)dx \\ &= \frac{i\hbar}{2m}\left[x\left(\psi^*\frac{\partial}{\partial x}\psi - \psi\frac{\partial}{\partial x}\psi^*\right)\Big|_{-\infty}^{+\infty} - \int_{-\infty}^{+\infty}\left(\psi^*\frac{\partial}{\partial x}\psi - \psi\frac{\partial}{\partial x}\psi^*\right)dx\right]\end{aligned} \tag{36}$$

For those wave functions which can be normalized, there is a property of them [10]:

$$\lim_{x\to\pm\infty}\psi = 0 \tag{37}$$

Of course, for the conjugate wave function ψ^*, the property is also true, so we are able to get this:

$$\begin{aligned}\frac{d\langle x\rangle}{dt} &= \langle v\rangle \\ &= -\frac{i\hbar}{2m}\int_{-\infty}^{+\infty}\left(\psi^*\frac{\partial}{\partial x}\psi - \psi\frac{\partial}{\partial x}\psi^*\right)dx\end{aligned} \tag{38}$$

Using the integration by parts (equation (35)) to the later item again, we will get

$$\begin{aligned}\langle v\rangle &= -\frac{i\hbar}{2m}\left(\int_{-\infty}^{+\infty}\psi^*\frac{\partial}{\partial x}\psi\,dx + \int_{-\infty}^{+\infty}\psi^*\frac{\partial}{\partial x}\psi\;dx\right) \\ &= -\frac{i\hbar}{m}\int_{-\infty}^{+\infty}\psi^*\frac{\partial}{\partial x}\psi\,dx\end{aligned} \tag{39}$$

So, if we use the formula of momentum in classical physics (equation (1)), we have

$$\begin{aligned}\langle p\rangle &= m\langle v\rangle \\ &= -i\hbar\int_{-\infty}^{+\infty}\psi^*\frac{\partial}{\partial x}\psi\,dx \\ &= \int_{-\infty}^{+\infty}\psi\left(-i\hbar\frac{\partial}{\partial x}\right)\psi^*\,dx\end{aligned} \tag{40}$$

We all know that the $\langle p\rangle$ can be expressed in this way (similar with equation (30)) [8]:

$$\langle p\rangle = \int_{-\infty}^{+\infty}\tilde{\psi}(p)\,p\tilde{\psi}^*(p)\,dx \tag{41}$$

However, this relation is founded in the momentum space, which is not convenient for us to deal with it, so we need to transform the space from momentum to position. Luckily, we can do this job since the momentum and the position of the particle can be transformed to each other by using the Fourier transform [11]:

$$\tilde{\psi}(p) = \frac{1}{\sqrt{2\pi\hbar}}\int_{-\infty}^{+\infty}\psi(x)\exp(-ikx)\;dx \tag{42}$$

$$\psi(x) = \frac{1}{\sqrt{2\pi\hbar}}\int_{-\infty}^{+\infty}\tilde{\psi}(p)\exp(ikx)\;dp \tag{43}$$

So, we can get

$$\langle p\rangle = \int_{-\infty}^{+\infty}\psi\,\hat{p}\psi^*\,dx \tag{44}$$

It means that we can find the momentum operator $\hat{p}$:

$$\hat{p} \equiv -i\hbar\frac{\partial}{\partial x} \tag{45}$$

And obviously, for the position operator $\hat{x}$, we have

$$\begin{aligned}\langle x\rangle &= \int_{-\infty}^{+\infty}\psi\,\hat{x}\psi^*\,dx \\ &= \int_{-\infty}^{+\infty}\psi\,x\psi^*\,dx\end{aligned} \tag{46}$$

$$\hat{x} \equiv x \tag{47}$$

For any observable quantity A, the uncertainty, or the standard deviation is

$$\Delta A = \sqrt{\langle A^2\rangle - \langle A\rangle^2} \tag{48}$$

And the variance is the square of standard deviation, so we have

$$\begin{aligned}s_A^2 &= (\Delta A)^2 \\ &= \langle A^2\rangle - \langle A\rangle^2\end{aligned} \tag{49}$$

In addition, the variance can also be written in this way:

$$\begin{aligned}s_A^2 &= \frac{1}{n}\sum_{i=1}^{n}(A_i - \langle A\rangle)^2 \\ &= \langle(\hat{A} - \langle A\rangle)^2\rangle\end{aligned} \tag{50}$$

Based on the definition of the inner product of any two vectors can be written by the mods of them:

$$\vec{a}\cdot\vec{b} = |\vec{a}||\vec{b}|\cos\langle\vec{a},\vec{b}\rangle \tag{51}$$

$\cos\langle\vec{a},\vec{b}\rangle$ is the cosine value of the angle between $\vec{a}$ and $\vec{b}$. Because the value domain of cosine function is always between 1 and -1, which is

$$\cos\langle \vec{a}, \vec{b} \rangle \in [-1, 1] \tag{52}$$

So, we can get

$$\vec{a} \cdot \vec{b} \leq |\vec{a}||\vec{b}| \tag{53}$$

$$(\vec{a} \cdot \vec{b})^2 = |\vec{a} \cdot \vec{b}|^2 \leq \left(|\vec{a}||\vec{b}|\right)^2 = |\vec{a}|^2|\vec{b}|^2 \tag{54}$$

With the help of the Dirac symbol, the inner product of two vectors can be expressed in this way:

$$\vec{a} \cdot \vec{b} = \langle \vec{a}|\vec{b} \rangle \tag{55}$$

Similarly, we have

$$\begin{aligned} |\vec{a}| &= \sqrt{\vec{a} \cdot \vec{a}} \\ &= \sqrt{\langle \vec{a}|\vec{a} \rangle} \end{aligned} \tag{56}$$

$$|\vec{a}|^2 = \langle \vec{a}|\vec{a} \rangle \tag{57}$$

It means that we can get this:

$$|\langle \vec{a}|\vec{b} \rangle|^2 \leq \langle \vec{a}|\vec{a} \rangle \langle \vec{b}|\vec{b} \rangle \tag{58}$$

And in quantum physics, the definition of the inner product of two wave functions is [12]

$$\langle \psi|\varphi \rangle = \int d\tau \, \psi^* \varphi \tag{59}$$

The meaning of $\int d\tau$ is to integrate the function in the whole space. In one dimension, we have this [12]:

$$\int d\tau = \int_{-\infty}^{+\infty} dx \tag{60}$$

It indicates

$$\langle \psi|\varphi \rangle = \int_{-\infty}^{+\infty} \psi^* \varphi \, dx \tag{61}$$

When these two functions are the same, the equation will become

$$\begin{aligned} \langle \psi|\psi \rangle &= \int_{-\infty}^{+\infty} \psi^* \psi \, dx \\ &= \int_{-\infty}^{+\infty} \|\psi\|^2 \, dx \end{aligned} \tag{62}$$

The result is obviously equal to 1 (equation (29)), so with some knowledge of vectors, we can deal with the item $(\hat{A} - \langle A \rangle)^2$, and we will obtain

$$\begin{aligned} \left(\hat{A} - \langle A \rangle\right)^2 &= \left\langle \psi \middle| \left(\hat{A} - \langle A \rangle\right)^2 \psi \right\rangle \\ &= \left\langle \psi(\hat{A} - \langle A \rangle) \middle| (\hat{A} - \langle A \rangle)\psi \right\rangle \end{aligned} \tag{63}$$

It means

$$s_A^2 = \left\langle \psi(\hat{A} - \langle A \rangle) \middle| (\hat{A} - \langle A \rangle)\psi \right\rangle \tag{64}$$

If we use a to represent $\psi(\hat{A} - \langle A \rangle)$, we will get

$$s_A^2 = \langle a|a \rangle \tag{65}$$

In the same way, we have

$$s_B^2 = \langle b|b \rangle \tag{66}$$

Multiplying equation (65) and equation (66) together, we can gain this equation:

$$s_A^2 s_B^2 = \langle a|a\rangle\langle b|b\rangle \geq |\langle a|b\rangle|^2 \tag{67}$$

The next step is to find a complex number to replace the $\langle a|b\rangle$, so we can define a complex number u:

$$u = c + di \tag{68}$$

The mod of u is

$$|u|^2 = c^2 + d^2 \tag{69}$$

Because u is a complex number, so $d \neq 0$, but for c, it occasionally can be 0, so we have

$$|u|^2 \geq d^2 \tag{70}$$

Through some complicated mathematical reasoning, the d^2 can be written is this way:

$$\begin{aligned} \left[\frac{1}{2i}(u-u^*)\right]^2 &= -\frac{1}{4}[(c+di)-(c-di)]^2 \\ &= -\frac{1}{4}(2di)^2 \\ &= d^2 \end{aligned} \tag{71}$$

Combining equations (70) and (71), we have

$$|u|^2 \geq [\tfrac{1}{2i}(u-u^*)]^2 \tag{72}$$

Using u to represent $\langle a|b\rangle$, we will get

$$u^* = \langle b|a\rangle \tag{73}$$

$$s_A^2 s_B^2 \geq \left[\tfrac{1}{2i}(\langle a|b\rangle - \langle b|a\rangle)\right]^2 \tag{74}$$

However, $\langle a|b\rangle$ is still not very convenient for us to calculate, so we have to simplify it. To do this, we need the formula mentioned in equations (63) and (64), and as a result, we obtain this:

$$\begin{aligned} \langle a|b\rangle &= \langle\psi(\hat{A}-\langle A\rangle)|(\hat{B}-\langle B\rangle)\psi\rangle \\ &= \langle\psi|(\hat{A}-\langle A\rangle)(\hat{B}-\langle B\rangle)\psi\rangle \\ &= \langle\psi|(\hat{A}\hat{B}-\hat{A}\langle B\rangle-\langle A\rangle\hat{B}+\langle A\rangle\langle B\rangle)\psi\rangle \\ &= \langle\hat{A}\hat{B}\rangle - \langle\psi|\hat{A}\langle B\rangle\psi\rangle - \langle\psi|\langle A\rangle\hat{B}\psi\rangle + \langle A\rangle\langle B\rangle\langle\psi|\psi\rangle \\ &= \langle\hat{A}\hat{B}\rangle - \langle A\rangle\langle B\rangle - \langle A\rangle\langle B\rangle + \langle A\rangle\langle B\rangle \\ &= \langle\hat{A}\hat{B}\rangle - \langle A\rangle\langle B\rangle \end{aligned} \tag{75}$$

In the same way, $\langle b|a\rangle$ is

$$\langle b|a\rangle = \langle\hat{B}\hat{A}\rangle - \langle B\rangle\langle A\rangle \tag{76}$$

So, we have

$$\langle a|b\rangle - \langle b|a\rangle = \langle\hat{A}\hat{B}\rangle - \langle\hat{B}\hat{A}\rangle \tag{77}$$

Due to the concept of commutation, the commutation relationship between $\hat{A}$ and $\hat{B}$ is

$$[\hat{A},\hat{B}] = \langle\hat{A}\hat{B}\rangle - \langle\hat{B}\hat{A}\rangle \tag{78}$$

It means

$$s_A^2 s_B^2 \geq \left[\frac{1}{2i}\langle[\hat{A},\hat{B}]\rangle\right]^2 \tag{79}$$

At this point, we gain a very important inequality, that is, the general uncertainty relationship. However, we still do not actually know why this relationship connects with those almost the most important physics quantities, since $\hat{A}$ and $\hat{B}$ have no meaning in our life. These operators are only abstract mathematical concepts. It seems like that we cannot make any effective progress when reaching this step. Still, if we remember the position operator $\hat{x}$ in equation (47) and the momentum operator $\hat{p}$ in equation (45), we can easily build a bridge between this mathematical equality and quantum physics. The first step is to find a method to connect position and momentum operators with $\hat{A}$ and $\hat{B}$. As a result, we can do this in a clever way, adding a function $f(x)$ without actual meaning but help us to calculate:

$$[\hat{x},\hat{p}]f(x) = \hat{x}\hat{p}f(x) - \hat{p}\hat{x}f(x) \tag{80}$$

Because $f(x)$ only has one independent variable x, the momentum operator in equation (45) has to be transformed into this form:

$$\hat{p} = -i\hbar\frac{d}{dx} \tag{81}$$

According to the law of operator calculation, we have

$$\begin{aligned}\hat{x}\hat{p}f(x) - \hat{p}\hat{x}f(x) &= x\left(-i\hbar\frac{d}{dx}f(x)\right) - \left[-i\hbar\frac{d}{dx}(xf(x))\right] \\ &= -i\hbar\left[x\frac{df(x)}{dx} - \left(f(x) + x\frac{df(x)}{dx}\right)\right] \\ &= -i\hbar f(x)\end{aligned} \tag{82}$$

It means

$$[\hat{x},\hat{p}]f(x) = -i\hbar f(x) \tag{83}$$

And if we want to make equation has meaning, it has to satisfy $f(x) \neq 0$, so we can eliminate the $f(x)$ [13]:

$$[\hat{x},\hat{p}] = -i\hbar \tag{84}$$

Since the operators mentioned in equation (79) are only the mathematical concept, we can use anything to replace them. In that case, we can make $\hat{x} = \hat{A}$ and $\hat{p} = \hat{B}$, and then, we have

$$s_A^2 s_B^2 \geq \left[\frac{1}{2i}\langle[\hat{x},\hat{p}]\rangle\right]^2 = \left[\frac{1}{2i}\langle -i\hbar\rangle\right]^2 = \left[\frac{1}{2i}(-i\hbar)\right]^2 = \left(\frac{\hbar}{2}\right)^2 \tag{85}$$

Because the standard deviations of momentum and position must be positive, we can take the quadratic roots of them, and we will gain a very important inequality. All of the microparticles have to follow this formula, the property of uncertainty, which means we cannot measure the position and the momentum of the particle at the same time. The uncertainty of position and of momentum satisfies this famous formula known as Heisenberg's uncertainty principle [14]:

$$\Delta x \Delta p \geq \frac{\hbar}{2} \tag{86}$$

The formula is really simple, but it contains information on the relationship between the position and the momentum. In the formula, the symbol Δx is the uncertainty of the particle position, which refers to each possible position of the micro substance. The value of Δx represents the point on which the particle focuses. The higher value means the particle may appear everywhere in the space on average. That is to say, the range of the possible position is very wide, so we cannot find an accurate location of particle. Compared with the high value, the low value indicates that the particle

almost focuses on one point in the given space, and we can find the particle is in that certain place. Very similar to the Δx, the value of Δp has the same meaning. The only different thing is that it refers to the uncertainty in momentum space. From the inequality, we can know that if the value of Δx is low, the value of Δp must be high to make the inequality true, which means we cannot make two values low enough to locate the state of the particle at the same time.

3. Result

We successfully obtain Heisenberg's uncertainty principle (equation (86)), and its important meaning for quantum physics. However, why does a quantum have the property of uncertainty? What is the exact connection between the uncertainty of position and of momentum? Why in classical physics, can everything be certain? From my perspective, the key to solve these questions are still Heisenberg's uncertainty principle though it only provides extremely simple information.

In quantum field, the particle is too small to be observed by normal ways, such as using the *microscope*. Due to the irregular motion of small particles all the time, it is very hard to find the position and the momentum of the particle. So, to find a particle, scientists need other useful ways. The most common way is to use other micro particles to hit the particle we want to find. For example, Heisenberg used γ ray to observe an electron. At that time, people already knew that the sharpness of images observed by γ ray, also known as the resolution, was directly linked to the wavelength and frequency of the ray. The shorter wavelength would lead to a sharper image, so that Heisenberg can define the position of the electron more clearly. However, the shorter wavelength would bring the higher frequency since the speed of electromagnetic wave, including γ ray, is a constant:

$$v = \frac{\lambda}{T} = \lambda f \tag{87}$$

Of couse, in quantum field, the frequency is ν, so we have

$$v = \lambda \nu \tag{88}$$

Combining equation (2), the momentum of γ ray would be higher with higher frequency. And we know that the collision between two substances contains a very large force between these objects, which is much more powerful than other forces coming from outside the system. So, the collision follows the law of conservation of momentum:

$$\vec{p}_1 = \vec{p}_2 \tag{89}$$

As a result, the γ ray with higher momentum would change the more momentum of the electron. It would add more difficulties for Heisenberg to measure the origin momentum of the electron.

This experiment done by Heisenberg well explained why we could not measure the accurate values of position and momentum at the same time. Based on this, I come up with a similar way to explain the uncertainty principle.

Since all of the small particles follow the rule of irregular movement, we can only observe one particle. Also take an electron as an example. We can enter the "perspective" of the electron, standing on the electron, just like standing on the earth, or imagining that the electron is a spacecraft, and we are staying in it. In these two cases, we can pretty clearly know what the momentum of the earth or the spacecraft is as for our perspective, so that we can easily know the momentum of electron. However, very similar to such situation that the people on the earth cannot feel the position in the universe of the earth, we cannot measure the place of the electron. Nevertheless, things are a little bit different from the condition on the earth, since the motion of the earth is certain. That is, the motion has a steady orbit and almost invariable speed. In other words, we can predict what the earth will do next. But when we come to quantum field, the electron, we must consider the irregular movement of small particles. Due to this problem, we have no idea about the position. We do not know the orbit, the direction since we do not know the original position of the particle, so we cannot predict what will happen on the electron next. Putting this conclusion into Heisenberg's uncertainty principle

(equation (86)), we will find that it matches the situation that $\Delta p \to 0$. To make the inequality true, we have to make $\Delta x \to +\infty$, which means the electron can be anywhere in space.

In other term, we can still observe only one *electron*, but this time we need to put ourselves out of the electron but stare at the electron. Similar to the method mentioned before, we can also imagine staying in a spacecraft or *standing* on *the* earth. However, we need to not to observe the object we stand on or stay in, but to look at the earth when staying in the spacecraft or observing the spacecraft when standing on the earth. Then we will find that we can figure out the place of the earth or spacecraft, since we can stay at a relatively settled point to observe the position. But for quantum particles, also due to the irregular movement, we have no method to measure its momentum, although we can actually see *where* the electron is. When using the uncertainty principle to describe this *phenomenon*, we have $\Delta x \to 0$ *and* $\Delta p \to +\infty$.

However, both the experiment held by Heisenberg and the explanation made by myself can only interpret the question: why we cannot measure the position and the momentum simultaneously? The most central problem is why microparticles have the property of uncertainty. This issue has not been solved yet. From mathematical angle, the Fourier transform transforms between the time t and the frequency f. If we discuss these two quantities in physics field, it is very easy to understand that there is also an uncertain relationship between them, and it is obviously understandable. Think about a scene: we are required to tear a song through listening to several notes. We all know that the pitch of each note is related to the frequency. In that case, the longer time we spend listening the song, the more note, representing the song's information, will be obtained, so that we are more likely to identify the song. Now, let us treat this event oppositely. If the test machine only cast one note, it will spend almost no time, but it is almost impossible for us to find an accurate song by only one note. As a result of this experiment, the length of time we spend, in fact, is the uncertainty of time. Whether we can figure out the song or not is, of course, the uncertainty of frequency. So obviously, the higher uncertainty of time brings the lower uncertainty of frequency and the lower uncertainty of time leads to the higher uncertainty of frequency. Even there is the most precise measuring machine, and it cannot still get the song by only one note. When we put this explanation into the field of quantum physics, we can discover that the uncertainty of position and momentum comes from Fourier transform, which means this property is the basic property of a microparticle without the connection with the observation method. Nevertheless, I find it is hard to prove this principle (equation (86)) physically: only describe the position and the momentum.

4. Conclusion

In this paper, with the help of predecessors, we have successfully solved some complex calculations and some thorny issues. I have discussed how Heisenberg's uncertainty principle comes from using some basic mathematical and physics knowledge. Based on the inequality of the uncertainty principle and the experiment of Heisenberg, I gave my own opinion to explain the question: why cannot we measure the position and the momentum at the same time? In addition, with the question of why the uncertainty exists, I tried to give my explanation mathematically by discussing the original Fourier transform about time and frequency, and gained a result that the uncertainty is not from the measuring method, but from the Fourier transform, which means if a quantity satisfies Fourier transform, it will have the uncertainty, including the position and the momentum. However, I have tried my best to clearly explain it in pure physics, and brought no result.

It is obvious that we have no clear way to explain why the uncertainty exists by only discussing the position and the momentum. Still, I believe this paper will have some little positive influence when researching the existence of uncertainty in the future.

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