The Abel Theory of Power Series in Split-Complex Analysis

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Abstract. The Split-complex number $\mathbb{R}^{1,1}$ is a generalization of real number $\mathbb{R}$, which is a commutative ring with two zero divisors generated by two real numbers. In this paper, we first study the Abel theorem of power series and obtain root and ratio convergent criterion in split-complex analysis. Then this paper shows these regions of convergence by figures in split-complex space. Finally, from our perspective, the results presented in this work represent the extensions of the related ideas or findings in real analysis, which makes it possible to study them in a larger domain.

Keywords: Split Complex Number $\mathbb{R}^{1,1}$, Power Series, h-Module.

1. Introduction

The split complex number is a generalization of the real numbers that is closely related to Minkowski space, quantum physics, Möbius geometry , gravitational theory, and relativity. In 2007, Libine [1] explored analogous simulations of the Cauchy integral formulation on the split complex numbers and extended the Cauchy-Fueter and Poisson formulas to Minkowski spaces. In 2014, Deckelman and Robson [2] discussed the application of the split complex numbers in quantum physics, where they provide a mathematical definition of Hyperbolic Dirac Networks (HDN) using the split complex numbers. In 2015, Emanuello and Nolder [3] studied Möbius geometry on $\mathbb{R}^{1,1}$ and also considered conformal maps on compactifications and their fundamental properties. For more details, please see earlier studies research [4-8] and so on.

In recent years, more and more people have extended some conclusions from real analysis to split-complex analysis. In 2007, Gürlebeck et al. [9] extended the theory of harmonic functions in real analysis and the minimal property of harmonic functions to split-complex analysis. In 2012, Colombo et al. [10] studied the Dirac operator for split complex variables and drew some new conclusions, such as generalizing the Fourier transform from real analysis to the case of split complex variables by introducing appropriate generalized functions. Motivated by these developments, we study the Abel theorem of power series and obtain root and ratio convergent criterion in split-complex analysis. The first step is to show the convergence regions in the split-complex space using figures. From our perspective, the results presented in this work represent the extensions of the related ideas or findings in real analysis, which makes it possible to study them in a larger domain.

This paper is organized as follows. In Section 2, we briefly introduce the basic concepts of split complex numbers and the partial order of split complex numbers on $\mathbb{R}^{1,1}$ .In Section 3, we discuss the properties of power series on $\mathbb{R}^{1,1}$, including studying Abel theorem, root and ratio convergent criterion in split-complex analysis. Finally, we shall give the convergence of $\mathbb{R}^{1,1}$ is more complex than that of real number $\mathbb{R}$ and complicated number $\mathbb{C}$.

2. Preliminaries

2.1. Split Complex Number $\mathbb{R}^{1,1}$

The set which has the properties is called Split Complex Numbers $\mathbb{R}^{1,1}$ (as in [3]),

$$\mathbb{R}^{1,1}=\{\zeta=x+yj|x,y\in\mathbb{R},j^2=1\} \tag{1}$$

$\mathbb{R}^{1,1}$ has a different algebraic structure. For instance, $\mathbb{R}^{1,1}$ has zero divisors:

$$(1+j)(1-j)=1-j^2=0 \tag{2}$$

Two special zero factors are defined in [3].

$$j_+=\frac{1+j}{2},\quad j_-=\frac{1-j}{2} \tag{3}$$

which can be used as a basis for $\mathbb{R}^{1,1}$.

For any element ζ in $\mathbb{R}^{1,1}$, it can be represented as

$$\zeta=uj_++vj_- \tag{4}$$

where $u=x+y,v=x-y$ [3].

We can find that j_+,j_- has the following arithmetic properties:

$$j_++j_-=1;j_+-j_-=j \tag{5}$$

$$\begin{aligned}
j_+\cdot j_+&=\frac{1+j}{2}\cdot\frac{1+j}{2}=\frac{1+j^2+2j}{4}=\frac{2+2j}{4}=\frac{1+j}{2}=j_+\\
j_-\cdot j_-&=\frac{1-j}{2}\cdot\frac{1-j}{2}=\frac{1+j^2-2j}{4}=\frac{2-2j}{4}=\frac{1-j}{2}=j_-\\
j_+\cdot j_-&=\frac{1+j}{2}\cdot\frac{1-j}{2}=\frac{1-j^2}{4}=0
\end{aligned} \tag{6}$$

In [13], there are some important arithmetic properties, let $u=u_1j_++u_2j_-,v=v_1j_++v_2j_-$, we have

$$u^2=u_1^2j_++u_2^2j_- \tag{7}$$

and

$$uv=u_1v_1j_++u_2v_2j_- \tag{8}$$

since the orthogonal property of j_+,j_-. The strength properties can be concluded naturally.

$$\sqrt[n]{u}=\sqrt[n]{u_1}j_++\sqrt[n]{u_2}j_- \tag{9}$$

and

$$\frac{u}{v}=\frac{u_1}{v_1}j_++\frac{u_2}{v_2}j_- \tag{10}$$

if $v_1,v_2>0$

After we have these based knowledge, we can define the power series in $\mathbb{R}^{1,1}$.

2.2. Partial Order on $\mathbb{R}^{1,1}$

In real number field, we have the size order relationship. The concept of partial order is introduced in [15]. A partial order relationship $\prec,\preccurlyeq$ [15] is defined. In some special case, we can compare some numbers in split-complex number $\mathbb{R}^{1,1}$

Let $a,b\in\mathbb{R}^{1,1}$,

$$\begin{aligned} a &= u_1 j_+ + v_1 j_- \\ b &= u_2 j_+ + v_2 j_- \\ u_1, u_2, v_1, v_2 &\in \mathbb{R} \end{aligned} \tag{11}$$

We denote $a \preccurlyeq b$ if $u_1 \leqslant u_2$ and $v_1 \leqslant v_2$. And denote $a \prec b$ if $u_1 < u_2$ and $v_1 < v_2$. We can see the partial order relationship on Figure 1.

In the split-complex number case, we consider the generalized metric h-Module [14][15] (a map from $\mathbb{R}^{1,1}$ to $\mathbb{R}^{1,1}$) as follows:

let $\zeta = uj_+ + vj_-$,

$$|\zeta|_h = |u| j_+ + |v| j_- \tag{12}$$

We denote this map $|\cdot|_h$

After the definition of h-Module, we can define the split-complex circumference. The circumference with ζ_0 as the center and R as the radius on the split complex plane has a different geometric figure with case $\mathbb{R}$. It represents a set $\mathbb{S}(\zeta_0, R) = \{\zeta \,|\, |\zeta - \zeta_0|_h \prec R, R \in \mathbb{R}^{1,1}, R \succ 0\}$. This is shown in Figure 2.

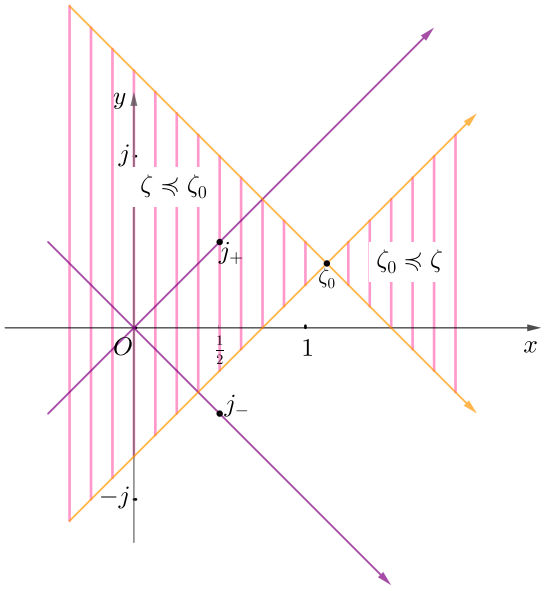
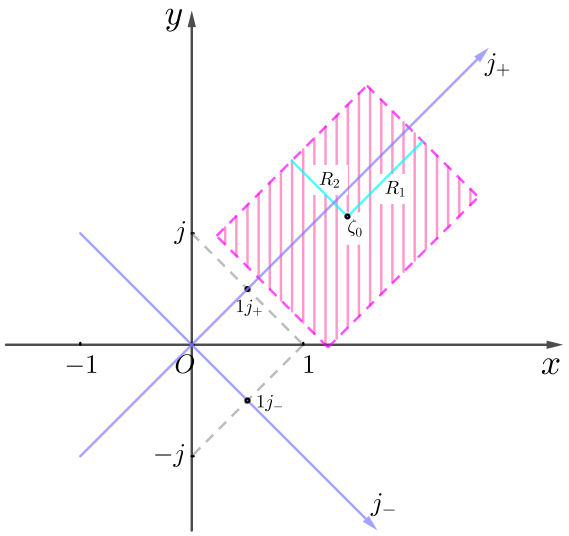


Figure 1. Partial Order on $\mathbb{R}^{1,1}$



$|\zeta - \zeta_0|_h \prec R$, $R = R_1 j_+ + R_2 j_-$

Figure 2. Split-complex circumference with ζ_0 as center and R as radius

After the definition of partial order and generalized metric, the split complex sequence limit on $\mathbb{R}^{1,1}$ can be defined Definition 5.1.1 from [15].

Definition(Sequence Limit on $\mathbb{R}^{1,1}$): A sequence of split complex numbers $\{\zeta_n\}_{n\in\mathbb{N}}$ is called convergent if there exists $\zeta_0 \in \mathbb{R}^{1,1}$ such that for any $\varepsilon \in \mathbb{R}^{1,1}$, where $\varepsilon \succ 0$, there exists $N \in \mathbb{N}$, such that for any $n > N$, there holds

$$|\zeta_n - \zeta_0|_h \prec \varepsilon \tag{13}$$

In this case, we say ζ_0 is the limit of the sequence that we write as

$$\lim_{n\to\infty} \zeta_n = \zeta_0 \tag{14}$$

When we have these preliminaries, we can develop our theme as follows.

3. The Properties of Power Series on $\mathbb{R}^{1,1}$

Here we consider the series on $\mathbb{R}^{1,1}$ of form (15)

$$f(\zeta)=\sum_{n=0}^{\infty}a_n(\zeta-\zeta_0)^n \tag{15}$$

where $a_n,\zeta_0\in\mathbb{R}^{1,1}$, then we have $a_n=a_{1,n}j_+ + a_{2,n}j_-$ and $\zeta-\zeta_0=(\zeta_1-\zeta_{1,0})j_+ + (\zeta_2-\zeta_{2,0})j_-$. The power series can be written as (16) shown:

$$f(\zeta)=\sum_{n=0}^{\infty}a_{1,n}(\zeta_1-\zeta_{1,0})^n j_+ + \sum_{n=0}^{\infty}a_{2,n}(\zeta_2-\zeta_{2,0})^n j_- \tag{16}$$

Approximate to case on $\mathbb{R}$, we have Abel split complex number theorem as follows:

Theorem 1 (*Split-Complex Abel theorem*). Consider a Split Complex $\sum_{n=0}^{\infty}a_n(\zeta-\zeta_0)^n$ and let $\mathbb{S}(\zeta_0,\rho)=\{\zeta\in\mathbb{R}^{1,1}||\zeta-\zeta_0|_h=\rho,\rho\in\mathbb{R}^{1,1},\rho\succ 0\}$. If there exists a point $\tilde{\zeta}\in\mathbb{S}(\zeta_0,\rho)$ such that the series $\sum_{n=0}^{\infty}a_n(\zeta-\zeta_0)^n$ converges at $\tilde{\zeta}$, then it is absolutely converges in the open split complex ball $\mathbb{B}(\zeta_0,\rho)=\{\zeta\in\mathbb{R}^{1,1}||\zeta-\zeta_0|_h\prec\rho\}$ of the same radius and it converges on any compact subsets of this ball.

Proof: In the case of $\zeta=\zeta_0$, the series is converge obviously.

Let $\zeta\in\mathbb{B}(\zeta_0,\rho)\setminus\{\zeta_0\}$. This series converges on $\tilde{\zeta}$. Here, we have $a_n=a_{1,n}j_+ + a_{2,n}j_-,\zeta=\zeta_1 j_+ + \zeta_2 j_-,\zeta_0=\zeta_{1,0}j_+ + \zeta_{2,0}j_-$ and $\tilde{\zeta}=\tilde{\zeta}_1+\tilde{\zeta}_2$, we have

$$\begin{aligned}|a_n(\zeta-\zeta_0)^n|_h &= |(a_{1,n}j_+ + a_{2,n}j_-)((\zeta_1 j_+ + \zeta_2 j_-)-(\zeta_{1,0}j_+ + \zeta_{2,0}j_-))|_h\\ &= |(a_{1,n}j_+ + a_{2,n}j_-)((\zeta_1-\zeta_{1,0})j_+ + (\zeta_2-\zeta_{2,0})j_-)|_h\\ &= |a_{1,n}(\zeta_1-\zeta_{1,0})j_+ + a_{2,n}(\zeta_2-\zeta_{2,0})j_-|_h\\ &= |a_{1,n}(\zeta_1-\zeta_{1,0})|j_+ + |a_{2,n}(\zeta_2-\zeta_{2,0})|j_-\end{aligned} \tag{17}$$

For the reason that the $\mathbb{R}^{1,1}$ series

$$\sum_{n=0}^{\infty}a_n(\zeta-\zeta_0)^n=\sum_{n=0}^{\infty}a_{1,n}(\zeta_1-\zeta_{1,0})^n j_+ + \sum_{n=0}^{\infty}a_{2,n}(\zeta_2-\zeta_{2,0})^n j_- \tag{18}$$

is converge. i.e. the $\mathbb{R}$ series

$$\sum_{n=0}^{\infty}a_{1,n}(\zeta_1-\zeta_{1,0})^n \tag{19}$$

and

$$\sum_{n=0}^{\infty}a_{2,n}(\zeta_2-\zeta_{2,0})^n \tag{20}$$

is both converge, thus $\{a_{1,n}(\tilde{\zeta}_1-\zeta_{1,0})^n\}_{n=0}^{\infty}$, $\{a_{2,n}(\tilde{\zeta}_2-\zeta_{2,0})^n\}_{n=0}^{\infty}$ is boundary , i.e. there exists $M_1,M_2\in\mathbb{R}$ such that $|a_{1,n}(\tilde{\zeta}_1-\zeta_{1,0})^n|\leqslant M_1$ and $|a_{2,n}(\tilde{\zeta}_2-\zeta_{2,0})^n|\leqslant M_2$, so

$$\begin{aligned}|a_n(\zeta-\zeta_0)^n|_h &= |a_{1,n}(\zeta_1-\zeta_{1,0})|j_+ + |a_{2,n}(\zeta_2-\zeta_{2,0})|j_- \\ &= \left|a_{1,n}(\tilde{\zeta}_1-\zeta_{1,0})^n\left(\frac{\zeta_1-\zeta_{1,0}}{\tilde{\zeta}_1-\zeta_{1,0}}\right)^n\right|j_+ + \left|a_{2,n}(\tilde{\zeta}_2-\zeta_{2,0})^n\left(\frac{\zeta_2-\zeta_{2,0}}{\tilde{\zeta}_2-\zeta_{2,0}}\right)^n\right|j_- \\ &= |a_{1,n}(\tilde{\zeta}_1-\zeta_{1,0})^n|\cdot\left|\frac{\zeta_1-\zeta_{1,0}}{\tilde{\zeta}_1-\zeta_{1,0}}\right|^n j_+ + |a_{2,n}(\tilde{\zeta}_2-\zeta_{2,0})^n|\cdot\left|\frac{\zeta_2-\zeta_{2,0}}{\tilde{\zeta}_2-\zeta_{2,0}}\right|^n j_- \\ &= |a_{1,n}(\tilde{\zeta}_1-\zeta_{1,0})^n|\cdot\left(\frac{|\zeta_1-\zeta_{1,0}|}{|\tilde{\zeta}_1-\zeta_{1,0}|}\right)^n j_+ + |a_{2,n}(\tilde{\zeta}_2-\zeta_{2,0})^n|\cdot\left(\frac{|\zeta_2-\zeta_{2,0}|}{|\tilde{\zeta}_2-\zeta_{2,0}|}\right)^n j_- \\ &\preccurlyeq M_1\left|\frac{r_1}{\rho_1}\right|^n j_+ + M_2\left|\frac{r_2}{\rho_2}\right|^n j_-\end{aligned} \tag{21}$$

where $|\zeta-\zeta_0|_h = |\zeta_1-\zeta_{1,0}|j_+ + |\zeta_2-\zeta_{2,0}|j_- = r_1 j_+ + r_2 j_- = r$, $\left|\frac{r}{\rho}\right|_h \prec 1$ since $\zeta\in\mathbb{B}(\zeta_0,\rho)$, we have $r = r_1 j_+ + r_2 j_-$ and $\rho = \rho_1 j_+ + \rho_2 j_-$, so we have

$$\left|\frac{r}{\rho}\right|_h = \left|\frac{r_1}{\rho_1}\right|j_+ + \left|\frac{r_2}{\rho_2}\right|j_- \prec 1j_+ + 1j_- = \frac{1+j}{2} + \frac{1-j}{2} = 1 \tag{22}$$

i.e.

$$\sum_{n=0}^{\infty}|a_n(\zeta-\zeta_0)|_h^n \preccurlyeq M_1\sum_{n=0}^{\infty}\left|\frac{r_1}{\rho_1}\right|^n j_+ + M_2\sum_{n=0}^{\infty}\left|\frac{r_2}{\rho_2}\right|^n j_- \tag{23}$$

let

$$A_n = M_1\left|\frac{r_1}{\rho_1}\right|^n j_+ + M_2\left|\frac{r_2}{\rho_2}\right|^n j_- \tag{24}$$

We have $|a_n(\zeta-\zeta_0)|_h \prec A_n$, from Weierstrass converge criterion(in ref [15]), the power series is absolutely converging in $\mathbb{B}(\zeta_0,\rho)$.

Approximate to case on $\mathbb{R}$, we have Root Criterion on split-complex numbers as follows:

Theorem 2 (*Converge Radius Theorem for Split-Complex Power Series***).** For a given split complex power series $f(\zeta) = \sum_{n=0}^{\infty}a_n(\zeta-\zeta_0)^n = \sum_{n=0}^{\infty}a_{1,n}(\zeta_1-\zeta_{1,0})^n j_+ + \sum_{n=0}^{\infty}a_{2,n}(\zeta_2-\zeta_{2,0})^n j_-$, we can define the converging radius of the two series as R_1 and R_2. From the Root Criterion, we can identify the converge radius R of the power series $f(\zeta) = \sum_{n=0}^{\infty}a_n(\zeta-\zeta_0)^n$ is as (25):

$$R = \frac{1}{\overline{\lim_{n\to\infty}\sqrt[n]{|a_n|}}} = \frac{1}{\overline{\lim_{n\to\infty}\sqrt[n]{|a_{1,n}|}}}j_+ + \frac{1}{\overline{\lim_{n\to\infty}\sqrt[n]{|a_{2,n}|}}}j_- = R_1 j_+ + R_2 j_- \tag{25}$$

If each a_n is invertible. i.e. $|a_n|_h \succ 0$ for each n, we can also use the Ratio Criterion as (26)

$$R = \overline{\lim_{n\to\infty}}\frac{|a_n|_h}{|a_{n+1}|_h} = \overline{\lim_{n\to\infty}}\frac{|a_{1,n}|_h}{|a_{1,n+1}|_h}j_+ + \overline{\lim_{n\to\infty}}\frac{|a_{2,n}|_h}{|a_{2,n+1}|_h}j_- \tag{26}$$

The power series $\sum_{n=0}^{\infty}a_n(\zeta-\zeta_0)^n = \sum_{n=0}^{\infty}a_{1,n}(\zeta_1-\zeta_{1,0})^n j_+ + \sum_{n=0}^{\infty}a_{2,n}(\zeta_2-\zeta_{2,0})^n j_-$ converges absolutely in split complex ball $\mathbb{B}(\zeta_0,R)$.

Proof: **Step1**, from the analysis knowledge, we know that the number series $\sum_{n=0}^{\infty} a_{1,n}$ on $\mathbb{R}$ converges if $A_1 = \overline{\lim_{n\to\infty}} \sqrt[n]{|a_{1,n}|} < 1$.

Step2, we prove that the power series $\sum_{n=0}^{\infty} a_{1,n}(\zeta_1 - \zeta_{1,0})^n$ converges if $|\zeta_1 - \zeta_{1,0}| < R_1 = \frac{1}{A_1}$.

Let $\overline{\lim_{n\to\infty}} \sqrt[n]{|a_{1,n}(\zeta_1 - \zeta_{1,0})|^n} < 1$, i.e. $\overline{\lim_{n\to\infty}} \sqrt[n]{|a_{1,n}|} \cdot |\zeta_1 - \zeta_{1,0}| < 1$, i.e. $|\zeta_1 - \zeta_{1,0}| < \frac{1}{\overline{\lim_{n\to\infty}} \sqrt[n]{|a_{1,n}|}} = \frac{1}{A_1} = R_1$. At this time, the power series converge.

Step3, we prove that the power series on split number $\mathbb{R}^{1,1}$ converges in $\mathbb{B}(\zeta_0, R)$.

We can write R as $R = R_1 j_+ + R_2 j_-$, write a_n as $a_n = a_{1,n} j_+ + a_{2,n} j_-$ and $\mathbb{B}(\zeta_0, R) = \{\zeta | \zeta - \zeta_0|_h < R\}$, $|\zeta - \zeta_0|_h < R$ i.e. $|\zeta_1 - \zeta_{1,0}| < R_1$ and $|\zeta_2 - \zeta_{2,0}| < R_2$ here $a_{1,n}, a_{2,n}, \zeta_1, \zeta_2, \zeta_{1,0}, \zeta_{2,0} \in \mathbb{R}$, $a_n, \zeta, \zeta_0 \in \mathbb{R}^{1,1}$, we have

$$\sum_{n=0}^{\infty} a_n(\zeta - \zeta_0)^n = \sum_{n=0}^{\infty} a_{1,n}(\zeta_1 - \zeta_{1,0})^n j_+ + \sum_{n=0}^{\infty} a_{2,n}(\zeta_2 - \zeta_{2,0})^n j_- \tag{27}$$

For the reason that the converge radius of power series $\sum_{n=0}^{\infty} a_{1,n}(\zeta_1 - \zeta_{1,0})^n$ is $R_1 = \frac{1}{\overline{\lim_{n\to\infty}} \sqrt[n]{|a_{1,n}|}}$, i.e. this power series converge when $\zeta_1 \in \{\zeta | \zeta - \zeta_{1,0}| < R_1\}$. And the converge radius of power series $\sum_{n=0}^{\infty} a_{2,n}(\zeta_2 - \zeta_{2,0})^n$ is $R_2 = \frac{1}{\overline{\lim_{n\to\infty}} \sqrt[n]{|a_{2,n}|}}$, i.e. this power series converge when $\zeta_2 \in \{\zeta | \zeta - \zeta_{2,0}| < R_2\}$. The power series (27) converges if and only if both two series converges, i.e. $|\zeta_1 - \zeta_{1,0}| < R_1$ and $|\zeta_2 - \zeta_{2,0}| < R_2$, from the definition of R, the power series converges in $\zeta \in \{\zeta | \zeta - \zeta_0|_h < R\}$.

Remark: The convergence of split-complex power series has nine situations, and it is shown as follows.

We can divide it into 9 situations as follows:

1. $0 < R_1, R_2 < \infty$, this situation is shown in Figure 3.
2. R_1 or R_2 is equal to infinity and the other value is finite, which includes

1). $R_1 = \infty, R_2 = 0$, this situation is shown in Figure 4.

2). $R_1 = \infty, 0 < R_2 < \infty$, this situation is shown in Figure 5.

3). $R_1 = 0, R_2 = \infty$, this situation is shown in Figure 6.

4). $0 < R_1 < \infty, R_2 = \infty$, this situation is shown in Figure 7.

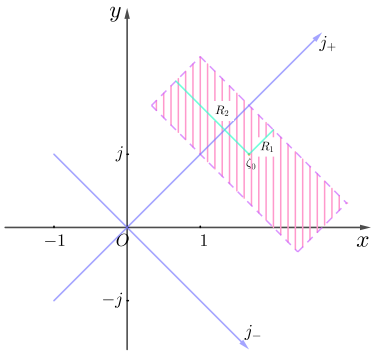
3. R_1 or R_2 is equal to 0 and the other value is nonzero finite.

1). $R_1 = 0, 0 < R_2 < \infty$, this situation is shown in Figure 8.

2). $0 < R_1 < \infty, R_2 = 0$, this situation is shown in Figure 9.

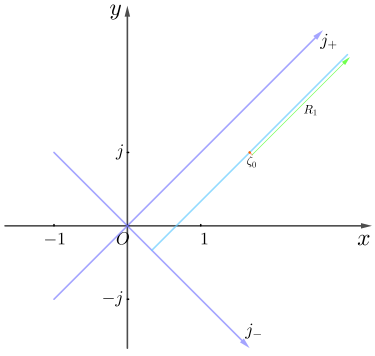
4. $R_1, R_2 = \infty$, this situation is shown in Figure 10.

5. $R_1 = R_2 = 0$, this situation is shown in Figure 11.



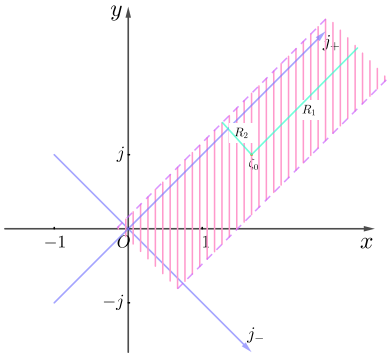
$|\zeta-\zeta_0|_h \prec R\,, R=R_1 j_+ + R_2 j_-$
$(R_1 \in (0,\infty), R_2 \in (0,\infty))$

Figure 3. $0<R_1,R_2<\infty$



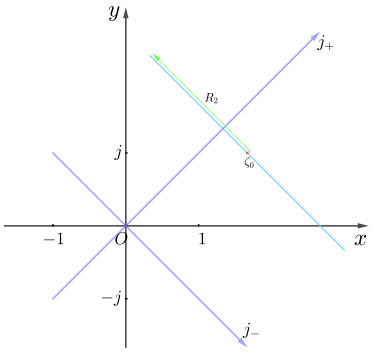
$|\zeta-\zeta_0|_h \prec R\,, R=R_1 j_+ + R_2 j_-$
$(R_1 = \infty, R_2 = 0)$

Figure 4. $R_1=\infty,\ R_2=0$



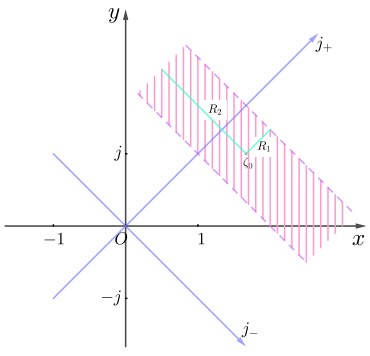
$|\zeta-\zeta_0|_h \prec R\,, R=R_1 j_+ + R_2 j_-$
$(R_1 = \infty, R_2 \in (0,\infty))$

Figure 5. $R_1=\infty,\ 0<R_2<\infty$



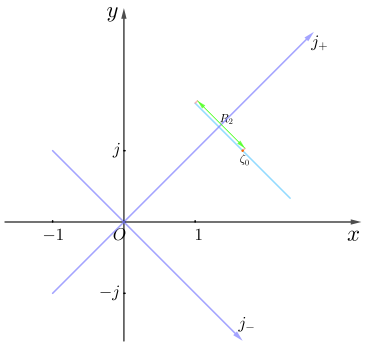
$|\zeta-\zeta_0|_h \prec R\,, R=R_1 j_+ + R_2 j_-$
$(R_1 = 0, R_2 = \infty)$

Figure 6. $R_1=0,\ R_2=\infty$



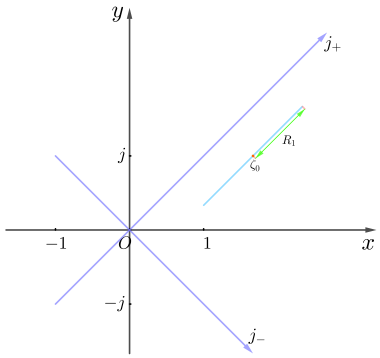
$|\zeta-\zeta_0|_h \prec R\,, R=R_1 j_+ + R_2 j_-$
$(R_1 \in (0,\infty), R_2 = \infty)$

Figure 7. $0<R_1<\infty,\ R_2=\infty$



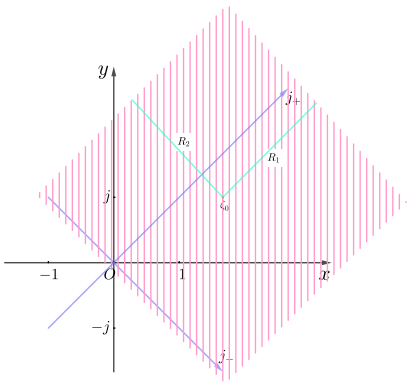
$|\zeta-\zeta_0|_h \prec R\,, R=R_1 j_+ + R_2 j_-$
$(R_1 = 0, R_2 \in (0,\infty))$

Figure 8. $R_1=0,\ 0<R_2<\infty$



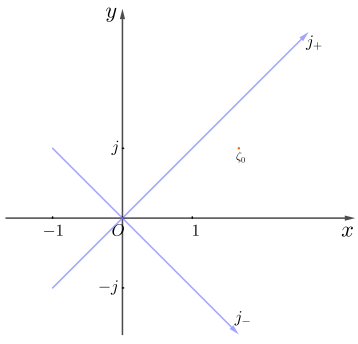
$|\zeta-\zeta_0|_h \prec R\,, R=R_1 j_+ + R_2 j_-$
$(R_1 \in (0,\infty), R_2 = 0)$

Figure 9. $0<R_1<\infty,\ R_2=0$



$|\zeta-\zeta_0|_h \prec R\,, R=R_1 j_+ + R_2 j_-$
$(R_1 = \infty, R_2 = \infty)$

Figure 10. $R_1,\ R_2=\infty$



$|\zeta-\zeta_0|_h \prec R\,, R=R_1 j_+ + R_2 j_-$
$(R_1 = 0, R_2 = 0)$

Figure 11. $R_1=0,\ R_2=0$

4. Conclusions

In this paper, we have studied the Abel theory of split-complex power series and attained root and ratio convergent criterion in split-complex analysis. The first step is to show the convergence regions in the split-complex space using figures. Finally, from our perspective, the results presented in this work represent the extensions of the related ideas or findings in real analysis, which makes it possible to study them in a larger domain.

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