Stark Effect of Electron-Proton System with Computer-generated Visualizations

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Abstract. The energy level and wave function obtained by solving the Schrödinger equation are extremely important in quantum mechanics. In this study, the solution to the Schrödinger equation of the electron-proton system by separation of variables was briefly described and the visualizations of the various solution were generated by Mathematica. The perturbation theory was discussed with the exclusion of the correction terms coming from the degenerate subspace. Applying the perturbation theory to the wavefunctions of the electron-proton system to study the Stark effect, the perturbation Hamiltonian is a constant electrostatic field. For the principal quantum numbers $n = 1$, $n = 2$ and $n = 3$, this paper obtains the energy corrections to the first and second degree, the corrections of the wavefunction to the first degree. The visualizations of the perturbed wavefunctions were obtained by setting the summation limits $k = 20$ and other parameters to 1. The visualizations of various unperturbed-states and perturbed-states were generated by Mathematica version 13.2.

Keywords: Stark effect; Schrödinger equation; Perturbation theory; Electron-proton system.

1. Introduction

Ever since the introduction of the Schrödinger equation (SE) in 1926 [1], understanding of the behavior of atomic system has been improved. However, the number of analytical solutions to the SE is very small, like quadratic potential, linear potential, step potential and the Coulomb potential. The utilities of the analytical solutions to the SE are limited, because the real world is much more complicated. For this reason, the study of approximation methods becomes very important. There are two primary approximation methods in quantum mechanics, one is perturbation theory while the other is variational method. With the understanding of the perturbation theory, researchers can investigate the line splitting of the electron-proton system in the presence of a strong electrostatic field [2] and the first observation of this effect was reported by Stark in 1914 [3].

Stark effect is the splitting of the line spectrum of the electron-proton system in the presence of a very strong electrostatic field. In the subsequent years, equipment capable of generating much higher strength electrostatic field were developed and the Stark effect was more visible and easier to observe [4]. The Stark effect has been observed to occur at nano-scale structures such as quantum wells [5] and quantum dots [6], [7]. The Stark effect is not exclusive to the electron-proton system, other species of atoms can also be made to exhibit the Stark effect [8]. Because the Stark effect is induced by a linear potential, by changing the electrostatic field to the gravitational field, astrophysicists could potentially investigate the gravity of distant stars by analyzing their emission spectra.

This paper is an attempt to discuss the Stark effect in the context of a simplified perturbation theory. Firstly, this paper will describe the solution to the SE without spin and relativistic effects and show some computer-generated figures of the probability density. Secondly, this paper will briefly describe the mathematical theory of the degenerate perturbation theory with the exclusion of the correction terms coming from the degenerate subspace. Finally, this paper will apply the perturbation theory to the electron-proton system, and give examples for the principal quantum number $n = 1$, $n = 2$ and $n = 3$.

2. The Electron-Proton System and Its Energy Spectrum

For the electron-proton system problem, there are two ways to solve it, one is factorization [9], the other is separation of variables. The SE in 3-D is

$$\left(-\frac{\hbar^2\nabla^2}{2\mu}-\frac{e^2}{4\pi\varepsilon_0 r}\right)\Psi = E\Psi \tag{1}$$

Where $\mu = \frac{m_{electron}\cdot m_{proton}}{m_{electron}+m_{proton}}$ is the effective mass, and ∇^2 is

$$\nabla^2 = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right)+\frac{1}{r^2\sin\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right)+\frac{1}{r^2\sin^2\theta}\left(\frac{\partial^2}{\partial\varphi^2}\right). \tag{2}$$

The standard way of solving this equation is the method of separation of variables. Writing the wavefunction in the form of $\Psi(x,y,z) = \frac{u(r)}{r}\Theta(\theta)\Phi(\varphi)$, and the normalization condition in spherical coordinates is

$$\int_0^{2\pi}\int_0^{\pi}\int_0^{\infty} r^2\sin\theta\Psi^*(r,\theta,\varphi)\Psi(r,\theta,\varphi)drd\theta d\varphi = 1. \tag{3}$$

The results of separation of variables are

$$\frac{d^2\Phi}{d\varphi^2} = -m^2\Phi, \tag{4}$$

$$-\frac{d^2\Theta}{d\theta^2}-\cot\theta\frac{d\Theta}{d\theta}+\frac{m^2\Theta}{\sin^2\theta} = l(l+1)\Theta, \tag{5}$$

$$-\frac{\hbar^2}{2\mu}\frac{d^2u}{dr^2}+\left(-\frac{e^2}{4\pi\varepsilon_0 r}+\frac{\hbar^2}{2\mu}\frac{l(l+1)}{r^2}\right)u = Eu. \tag{6}$$

Using series expansion, these three equations can be analytically solved

$$\Phi_m(\varphi) = A_m e^{im\varphi}, \tag{7}$$

$$\Theta_{lm}(\theta) = B_{lm}P_l^{|m|}(\cos\theta), \tag{8}$$

$$u_{nl}\left(\frac{r}{r_0 n}\right) = C_{nl}\left(\frac{r}{r_0 n}\right)^{l+1}e^{-\frac{r}{r_0 n}}L_{n-l-1}^{2l+1}\left(2\frac{r}{r_0 n}\right). \tag{9}$$

Here, $P_l^{|m|}$ and L_{n-l-1}^{2l+1} are associated Legendre polynomial and associated Laguerre polynomial respectively, and A_m, B_{lm}, C_{nl} are the normalization factors. The quantity r_0 is the Bohr radius $r_0 = \frac{4\pi\varepsilon_0\hbar^2}{\mu e^2}$. The quantization conditions are $l = 0, 1, 2, 3 \ldots$, $m = -l, -l+1, \ldots 0, \ldots l-1, l$, and $n = 1, 2, 3 \ldots$. The eigen-energies are:

$$E_n = -\frac{\mu}{2\hbar^2}\frac{e^4}{(4\pi\varepsilon_0)^2}\frac{1}{n^2} = -\frac{e^2}{2r_0}\frac{1}{n^2} = -\frac{E_0}{n^2}, \tag{10}$$

Where $E_0 = \frac{\mu}{2\hbar^2}\frac{e^4}{(4\pi\varepsilon_0)^2} \approx 13.6$ eV. The full expression of the electron-proton system wavefunction is

$$\Psi_{nlm}(r,\theta,\varphi) = \sqrt{\left(\frac{2}{nr_0}\right)^3\frac{(n-l-1)!}{2n[(n+l)!]^3}}\left(\frac{r}{r_0 n}\right)^l e^{-\frac{r}{r_0 n}}L_{n-l-1}^{2l+1}\left(2\frac{r}{r_0 n}\right)Y_l^m(\theta,\varphi) \tag{11}$$

Where$Y_l^m(\theta,\varphi) = P_l^{|m|}(\cos\theta)e^{im\varphi}$.

For the first few cases, some visualizations of the probability density are shown in Fig. 1 and Fig. 2.

3. Perturbation Theory

3.1. Corrections of Eigenenergy

According to the SE, the eigen-energies satisfy the following equation:

$$(H^{(0)} + \lambda\delta H)|n(\lambda)\rangle = E_n(\lambda)|n(\lambda)\rangle \tag{12}$$

Where λ is a small parameter. This equation can be solved by Taylor expansion of the eigen-state and eigen-energy, i.e.,

$$|n(\lambda)\rangle = \lambda^0|n^{(0)},l,m\rangle + \lambda^1|n^{(1)},l,m\rangle + \lambda^2|n^{(2)},l,m\rangle + \cdots \lambda^k|n^{(k)},l,m\rangle \tag{13}$$

Where the term $|n^{(k)},l,m\rangle$ is the k-th correction to the n-th state-vector, and

$$E_n(\lambda) = \lambda^0 E_{nlm}^{(0)} + \lambda^1 E_{nlm}^{(1)} + \lambda^2 E_{nlm}^{(2)} + \cdots \lambda^k E_{nlm}^{(k)} \tag{14}$$

Where the term $E_{nlm}^{(k)}$ is the k-th correction to the eigen-energy. By defining the matrix element of δH to be $\delta H_{k\beta\gamma,nlm} \equiv \langle k^{(0)},\beta,\gamma|\delta H|n^{(0)},l,m\rangle$, it is found that the first energy correction, the second energy correction and the perturbed wavefunction can be written as

$$E_{nlm}^{(1)} = \delta H_{nlm,nlm}. \tag{15}$$

The energy and the corresponding eigen-state are

$$E_{nlm}^{(2)} = \langle n^{(0)},l,m|\delta H|n^{(1)},l,m\rangle \approx \sum_{k,\beta,\gamma,k\neq n} \frac{\delta H_{k\beta\gamma,nlm}\cdot\left(\delta H_{k\beta\gamma,nlm}\right)^*}{E_{nlm}^{(0)} - E_{k\beta\gamma}^{(0)}} \tag{16}$$

And

$$|n,l,m\rangle \approx |n^{(0)},l,m\rangle + \sum_{k,\beta,\gamma,k\neq n} |k^{(0)},\beta,\gamma\rangle \cdot \frac{\delta H_{k\beta\gamma,nlm}}{E_{nlm}^{(0)} - E_{k\beta\gamma}^{(0)}}. \tag{17}$$

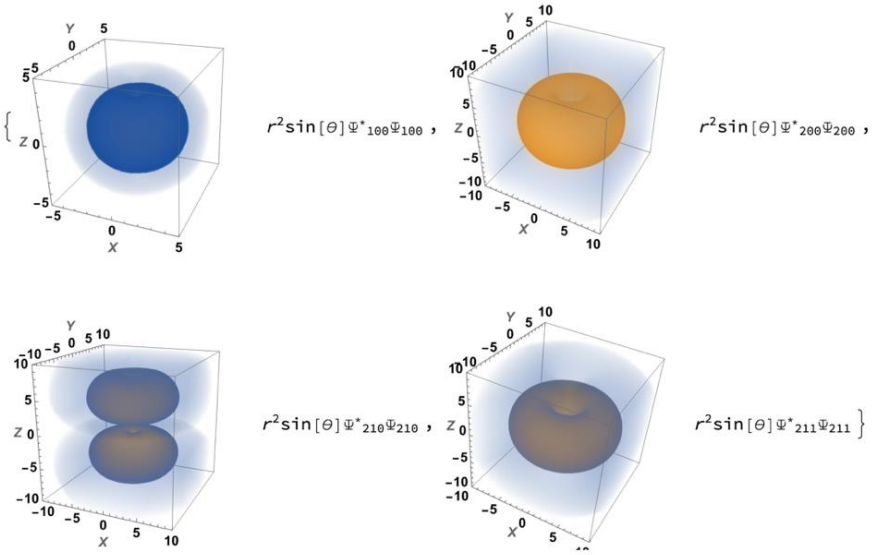


Fig. 1 Probability density of Ψ_{100} (upper left), Ψ_{200} (upper right), Ψ_{210} (lower left), and Ψ_{211} (lower right).

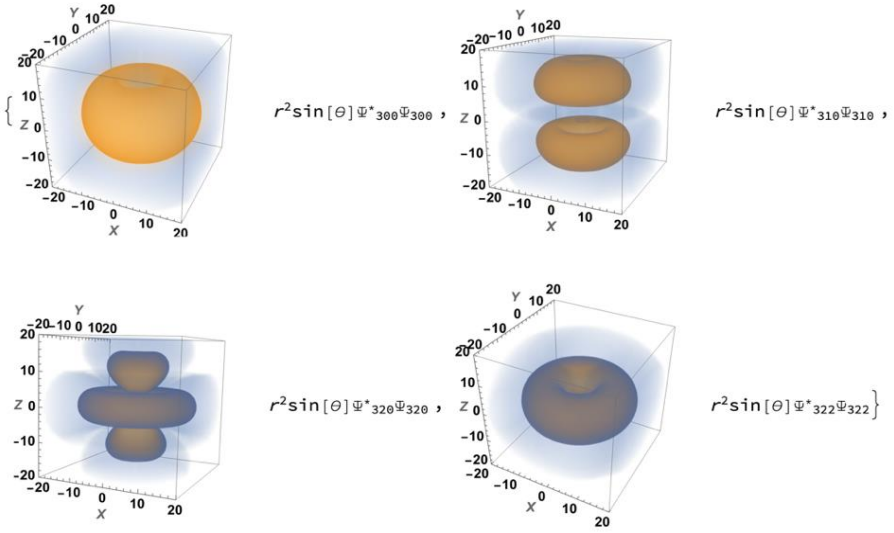


Fig. 2 Probability density of Ψ_{300} (upper left), Ψ_{310} (upper right), Ψ_{320} (lower left), and Ψ_{322} (lower right).

3.2. The Electron-Proton System Subjected to a Strong Electrostatic Field

The energy spectrum of the electron-proton system in the present of an intensive electrostatic field will change, and this had been observed by Stark in 1914 [3]. To calculate the energy corrections of the n-th state to the first and second degree, it is useful to choose the z direction to be the electrostatic field direction $\delta H = -e\mathcal{E}z = -e\mathcal{E}r\cos\theta$, where $\mathcal{E}$ is the strength of the electrostatic field. The result is:

$$E_{nlm}^{(1)} = -e\mathcal{E}\left\langle n^{(0)}\middle|r\middle|n^{(0)}\right\rangle \cdot \langle l,m|\cos\theta|l,m\rangle. \tag{18}$$

In order to compute the energy corrections, the first step is to calculate the matrix elements of $-e\mathcal{E}z$: $\delta H_{k\beta\gamma,nlm} = -\left\langle k^{(0)},\beta,\gamma\middle|e\mathcal{E}r\cos\theta\middle|n^{(0)},l,m\right\rangle$. By separating the r,θ,φ variables, the above matrix element can be evaluated as the angular term and the radial term. The evaluation of the angular term is

$$\langle\beta,\gamma|\cos\theta|l,m\rangle = \int_0^{2\pi}\int_{-1}^{1} N_{\beta\gamma}P_\beta^\gamma e^{-i\gamma\varphi}\cdot\cos\theta\cdot N_{lm}P_l^m e^{im\varphi}\cdot d(\cos\theta)d\varphi\,. \tag{19}$$

The recurrence relation of the associated Legendre polynomial is:

$$xP_l^m(x) = \sqrt{\frac{(l+1)^2-m^2}{(2l+3)(2l+1)}}P_{l+1}^m(x) + \sqrt{\frac{l^2-m^2}{(2l+1)(2l+1)}}P_{l-1}^m(x) \tag{20}$$

Written in the spherical harmonic form:

$$\cos\theta\cdot Y_l^m(\theta,\varphi) = \sqrt{\frac{(l+1)^2-m^2}{(2l+3)(2l+1)}}Y_{l+1}^m + \sqrt{\frac{l^2-m^2}{(2l+1)(2l+1)}}Y_{l-1}^m \tag{21}$$

Therefore, the angular integration reduces to:

$$\langle\beta,\gamma|\cos\theta|l,m\rangle = \sqrt{\frac{(l+1)^2-m^2}{(2l+3)(2l+1)}}\delta_{\beta,l+1}\delta_{\gamma,m} + \sqrt{\frac{l^2-m^2}{(2l+1)(2l+1)}}\delta_{\beta,l-1}\delta_{\gamma,m}. \tag{22}$$

Therefore, β and γ must be $\gamma = m$ and $\beta = l+1$ or $l-1$. For $\beta = l-1, \gamma = m$:

$$\langle l-1,m|\cos\theta|lm\rangle = \Pi_{lm}^{l-1,m} = \sqrt{\frac{l^2-m^2}{(2l+1)(2l+1)}}, \tag{23}$$

While for $\beta = l+1, \gamma = m$:

$$\langle l+1,m|\cos\theta|lm\rangle = \Pi_{lm}^{l+1,m} = \sqrt{\frac{(l+1)^2-m^2}{(2l+3)(2l+1)}}. \tag{24}$$

Thus, the non-zero matrix elements are $\delta H_{k(l+1)m,nlm} = -e\mathcal{E}\langle k^{(0)}, l+1, m|r\cos\theta|n^{(0)}lm\rangle$ and $\delta H_{k(l-1)m,nlm} = -e\mathcal{E}\langle k^{(0)}, l-1, m|r\cos\theta|n^{(0)}lm\rangle$.

The radial integrals were calculated by W. Gordon [10]:

$$\langle k,l-1|r|n,l\rangle = \int_0^\infty u_{k,l-1}\cdot r\cdot u_{n,l}\,dr = C_{n,l}^{k,l-1} = C_{k,l-1}^{n,l}$$

And

$$C_{n,l}^{k,l-1} = \frac{(-1)^{k-l}}{4(2l-1)!}\sqrt{\frac{(n+l)!\,(k+l-1)!}{(n-l-1)!\,(k-l)!}}\left(\frac{4nk}{(n-k)^2}\right)^{l+1}\left(\frac{n-k}{n+k}\right)^{n+k}\cdot g \tag{25}$$

Where

$$g = F\left(l+1-n, l-k, 2l, -\frac{4kn}{(k-n)^2}\right) - \left(\frac{n-k}{n+k}\right)^2 F\left(l-1-n, l-k, 2l, -\frac{4kn}{(k-n)^2}\right) \tag{26}$$

With $F(\alpha,\beta,\gamma;x) = \frac{\Gamma(\gamma)}{\Gamma(\alpha)\Gamma(\beta)}\sum_{k=0}^{k=\infty}\frac{\Gamma(\alpha+k)\Gamma(\beta+k)}{\Gamma(\gamma+k)}\frac{x^k}{k!}$. Hence, the relevant matrix elements are $\delta H_{k(l+1)m,nlm};\ \delta H_{k(l-1)m,nlm}$.

4. The Stark Effect in Electron-Proton System

4.1. The Ground State $n = 1$

For the ground state of the electron-proton system, the quantum numbers are $n = 1, l = 0, m = 0$. The relevant matrix elements are:

$$E_{100}^{(1)} = -e\mathcal{E}\langle 1,0,0|r\cos\theta|1,0,0\rangle = -e\mathcal{E}\Pi_{1,0}^{1,0}C_{1,0}^{1,0} = 0 \tag{27}$$

And

$$\delta H_{k10,100} = -e\mathcal{E}\Pi_{1,0}^{2,0}C_{1,0}^{k,1} = \frac{-e\mathcal{E}}{\sqrt{3}}\sqrt{\frac{256\cdot k^7(k-1)^{2k-5}}{(k+1)^{2k+5}}}\cdot r_0. \tag{28}$$

Therefore, the energy correction of the ground state to the second degree is:

$$E_{100}^{(2)} = -\frac{512\mathcal{E}^2r_0^3}{3}\sum_{k=2}^{k=\infty}\frac{k^2}{k^2-1}\cdot\frac{k^7(k-1)^{2k-5}}{(k+1)^{2k+5}} \approx -1.8314\mathcal{E}^2r_0^3 \tag{29}$$

The expression of the perturbed ground state wavefunction is:

$$|1,0,0\rangle \approx |1,0,0\rangle^{(0)} - \lambda\frac{r_0}{e^2}\sum_{k=2}^{k=\infty}\frac{2k^2\delta H_{k10,100}}{k^2-1}|k,1,0\rangle \tag{30}$$

Where $|1,0,0\rangle^{(0)}$ is the unperturbed ground state. More explicitly:

$$|1,0,0\rangle \approx |1,0,0\rangle^{(0)} + \lambda \frac{r_0^2 \mathcal{E}}{e} \sum_{k=2}^{k=\infty} \sqrt{\frac{256 \cdot k^7 (k-1)^{2k-5}}{(k+1)^{2k+5}}} \frac{2k^2}{k^2-1} |k,1,0\rangle. \quad (31)$$

In Fig. 3, it chooses the quantities $e = 1,\ \mathcal{E} = 1, r_0 = 1, \lambda = 1$ to visualize the perturbed ground state wavefunction.

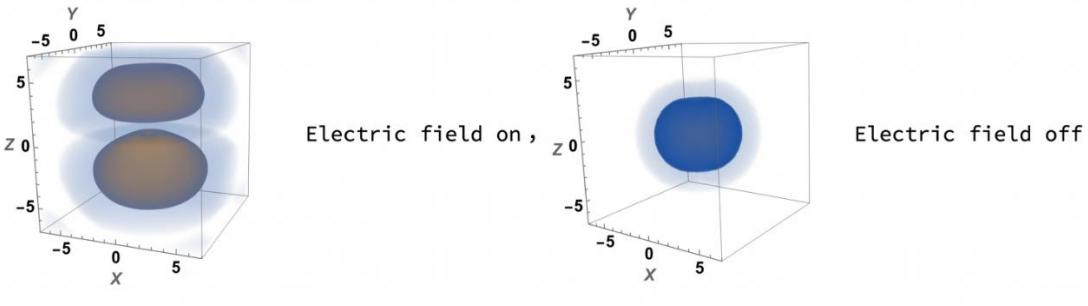


Fig. 3 Probability density of the perturbed state Ψ_{100} (left, k is from 2 to 20), and the unperturbed state (right).

4.2. The First Excited State $n = 2$

The electron-proton system has 4 states corresponding to $n = 2$. They are $|2,0,0\rangle$, $|2,1,0\rangle$, $|2,1,1\rangle$, and $|2,1,-1\rangle$. It is convenient to combine these states in such a way that the z components of the angular momentum are 0,1 and -1, i.e.,

$$|1\rangle = \frac{|2,1,0\rangle + |2,0,0\rangle}{\sqrt{2}}; |2\rangle = |2,1,1\rangle; |3\rangle = |2,1,-1\rangle; |4\rangle = \frac{|2,1,0\rangle - |2,0,0\rangle}{\sqrt{2}}. \quad (32)$$

The matrix elements of $-e\mathcal{E}z$ in the new basis are:

$$-e\mathcal{E}\begin{pmatrix} \langle 1|z|1\rangle & \langle 1|z|2\rangle & \langle 1|z|3\rangle & \langle 1|z|4\rangle \\ \langle 2|z|1\rangle & \langle 2|z|2\rangle & \langle 2|z|3\rangle & \langle 2|z|4\rangle \\ \langle 3|z|1\rangle & \langle 3|z|2\rangle & \langle 3|z|3\rangle & \langle 3|z|4\rangle \\ \langle 4|z|1\rangle & \langle 4|z|2\rangle & \langle 4|z|3\rangle & \langle 4|z|4\rangle \end{pmatrix} = -e\mathcal{E}r_0 \begin{pmatrix} -3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix}. \quad (33)$$

Therefore, according to Eq. 15, the first energy correction to these states are: $E_{|1\rangle}^{(1)} = 3e\mathcal{E}r_0$; $E_{|2\rangle}^{(1)} = E_{|3\rangle}^{(1)} = 0$; $E_{|4\rangle}^{(1)} = -3e\mathcal{E}r_0$. The relevant matrix elements to the perturbation of $|2,1,1\rangle$ are $\delta H_{k21,211}$:

$$\delta H_{k21,211} = -e\mathcal{E}r_0 \frac{1}{\sqrt{5}} \sqrt{\frac{2^{19} k^9 (k^2-1)(k-2)^{2k-7}}{3(k+2)^{2k+7}}} \quad (34)$$

The approximate wavefunction according to Eq. (17) is given by

$$|2\rangle = |2,1,1\rangle \approx |2,1,1\rangle^{(0)} - \lambda \frac{r_0}{e^2} \sum_{k=3}^{k=\infty} \delta H_{k21,211} \frac{8k^2}{k^2-4} |k,2,1\rangle, \quad (35)$$

Where $|2,1,1\rangle^{(0)}$ is the unperturbed state. The energy correction of the state $|2,1,1\rangle$ to the second degree is:

$$E_{211}^{(2)} = -\frac{8r_0}{e^2} \sum_{k=3}^{k=\infty} \left|\delta H_{k21,211}\right|^2 \frac{k^2}{k^2-4} \approx -75.2\mathcal{E}^2 r_0^3 \quad (36)$$

For simplicity the quantities $e, \mathcal{E}, r_0, \lambda$ are all set to 1. The visualization of the perturbed state and unperturbed state is shown in Fig. 4.

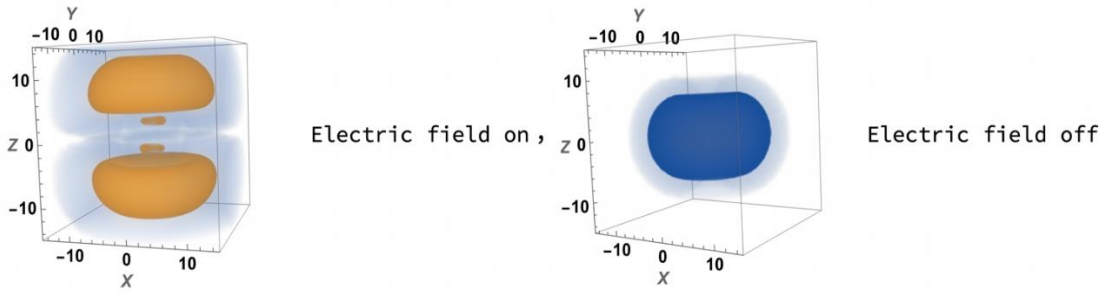


Fig. 4 Probability density of the perturbed state Ψ_{211} (left, k is from 3 to 20), and the unperturbed state (right).

4.1.3 The Second Excited State $n = 3$

There are nine states associated with $n = 3$, they can be arranged in such a way that the matrix-$e\mathcal{E}z$ is diagonalized. It is found that $|1\rangle = \frac{1}{\sqrt{3}}|3,0,0\rangle - \frac{1}{\sqrt{2}}|3,1,0\rangle + \frac{1}{\sqrt{6}}|3,2,0\rangle$; $|2\rangle = \frac{1}{\sqrt{2}}|3,2,-1\rangle + \frac{1}{\sqrt{2}}|3,1,-1\rangle$; $|3\rangle = \frac{1}{\sqrt{2}}|3,2,1\rangle + \frac{1}{\sqrt{2}}|3,1,1\rangle$; $|4\rangle = |3,2,2\rangle$; $|5\rangle = \sqrt{\frac{2}{3}}|3,2,0\rangle - \sqrt{\frac{1}{3}}|3,0,0\rangle$; $|6\rangle = |3,,2,-2\rangle$; $|7\rangle = \frac{1}{\sqrt{2}}|3,2,1\rangle - \frac{1}{\sqrt{2}}|3,1,1\rangle$; $|8\rangle = \frac{1}{\sqrt{2}}|3,2,-1\rangle - \frac{1}{\sqrt{2}}|3,1,-1\rangle$ and $|9\rangle = \frac{1}{\sqrt{3}}|3,0,0\rangle + \frac{1}{\sqrt{2}}|3,1,0\rangle + \frac{1}{\sqrt{6}}|3,2,0\rangle$. The matrix form of -$e\mathcal{E}z$ is:

$$-e\mathcal{E}r_0\begin{pmatrix} 9 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{9}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{9}{2} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -\frac{9}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{9}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 9 \end{pmatrix} \tag{37}$$

Therefore, the first energy corrections to these states are: $E^{(1)}_{|1\rangle} = -9e\mathcal{E}r_0$; $E^{(1)}_{|2\rangle} = E^{(1)}_{|3\rangle} = -\frac{9e\mathcal{E}r_0}{2}$; $E^{(1)}_{|4\rangle} = E^{(1)}_{|5\rangle} = E^{(1)}_{|6\rangle} = 0$; $E^{(1)}_{|7\rangle} = E^{(1)}_{|8\rangle} = \frac{9e\mathcal{E}r_0}{2}$; $E^{(1)}_{|9\rangle} = 9e\mathcal{E}r_0$. The relevant matrix elements to the perturbation of $|3,2,2\rangle$ are $\delta H_{k32,322}$:

$$\delta H_{k32,322} = -e\mathcal{E}r_0\frac{1}{\sqrt{7}}5184\sqrt{\frac{6k^{11}(k^2-1)(k^2-4)(k-3)^{2k-9}}{5(k+3)^{2k+9}}} \tag{38}$$

The energy correction of the state $|3,2,2\rangle$ to the second degree is:

$$E^{(2)}_{322} = -\frac{18r_0}{e^2}\sum_{k=4}^{k=\infty}\left|\delta H_{k32,322}\right|^2\frac{k^2}{k^2-9} \approx -682.2\mathcal{E}^2r_0^3 \tag{39}$$

The approximate wavefunction is:

$$|3,2,2\rangle \approx |3,2,2\rangle^{(0)} - \lambda \frac{r_0}{e^2} \sum_{k=4}^{k=\infty} \delta H_{k32,322} \frac{18k^2}{k^2-9} |k,3,2\rangle\,, \tag{40}$$

Where $|3,2,2\rangle^{(0)}$ is the unperturbed state. For simplicity the quantities $e, \mathcal{E}, r_0, \lambda$ are all set to 1. The visualization of the perturbed state and the unperturbed state is shown in Fig. 5.

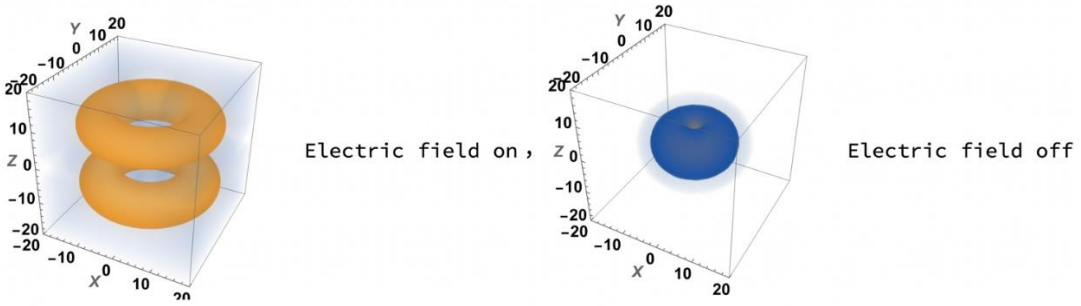


Fig. 5 Probability density of the perturbed state Ψ_{322} (left, k is from 4 to 20), and the unperturbed state (right).

5. Conclusion

In conclusion, this paper, firstly, describes the solution to the Schrödinger equation of the electron-proton system by separation of variables, and uses Mathematica to generate visualizations of the probability density of various states. Secondly, it discusses the degenerate-perturbation theory with the exclusion of the correction terms coming from the degenerate-subspace and obtains the formulas of the energy corrections to the first and second degree, the correction of the wavefunction to the first degree. Finally, it applies this simplified perturbation theory to the solutions of the electron-proton system and calculates the energy corrections to the first and second degree, first degree wavefunction corrections with some 3-D visualizations. The visualizations are only approximate since the summation limit is 20. Due to the complexity of the correction terms, the summation limits are not able to go high. The potential is that with more refined computer program in the future the visualizations will be more accurate. Energy levels of the electron-proton system in the presence of a very strong electrostatic field do split in accord with the perturbation theory. This splitting is now called the Stark effect and it has been experimental verified. The Stark effect is not confined to the electron-proton system, but it also can be observed at nano-scale structures such as quantum dots and quantum wells, and other species of atoms.

References

[1] Schrödinger E. An Undulatory Theory of the Mechanics of Atoms and Molecules. Physical Review, 1926, 28(6): 1049-1070.

[2] Silverstone H. J. Perturbation theory of the Stark effect in hydrogen to arbitrarily high order. Physical Review A, 1978, 18(5): 1853-1864.

[3] Stark J. Beobachtungen über den Effekt des elektrischen Feldes auf Spektrallinien. I. Quereffekt. Annalen der Physik, 1914, 348: p. 965-982.

[4] Jager H. Experimental Investigations of The Stark-Effect at High-Field Strengths. Physica Scripta, 1989, T26: 74-83.

[5] Diroll B. T. Circularly Polarized Optical Stark Effect in CdSe Colloidal Quantum Wells. Nano Letters, 2020, 20(11): 7889-7895.

[6] Zhang L., et al. Quantum-confined stark effect in the ensemble of phase-pure CdSe/CdS quantum dots. Nanoscale, 2019, 11(26): 12619-12625.

[7] Aghaeimeibodi S., et al. Large stark tuning of InAs/InP quantum dots. Applied Physics Letters, 2019, 114(7): 071105.

[8] Khadjavi A., Lurio A., Happer W. Stark Effect in the Excited States of Rb, Cs, Cd, and Hg. Physical Review, 1968, 167(1): 128-135.

[9] Schrödinger E. Further Studies on Solving Eigenvalue Problems by Factorization. Proceedings of the Royal Irish Academy. Section A: Mathematical and Physical Sciences, 1940, 46: 183-206.

[10] Gordon W. Zur Berechnung der Matrizen beim Wasserstoffatom. Annalen der Physik, 1929, 394(8): 1031-1056.