# Fuzzy Julia Sets and Fuzzy Superior Julia Sets

Beining Mu *

College of Letters & Science, University of Wisconsin-Madison, Wisconsin, 53706, United States

* Corresponding author: bmu3@wisc.edu

**Abstract.** This article examine the past study of fuzzy Mandelbrot set and fuzzy superior Mandelbrot set, then give the definition of fuzzy Julia sets and fuzzy superior Julia sets with the idea of utilizing membership functions to represent the escape velocity of Julia sets or superior Julia sets of each complex number in the definition of fuzzy Mandelbrot set and fuzzy superior Mandelbrot set inherited while the membership functions are selected to distinguish complex numbers with different escape velocity and same orbit. Then some examples of fuzzy Julia sets and fuzzy superior Julia sets are presented. With some observation of the examples, some analytical and topological properties of fuzzy Julia sets are demonstrated and sketches of the proofs are also presented. In the part of fuzzy superior Julia sets, some trivial conclusions as well as an example is presented where more complicated properties which involves the study into very chaotic behaviors are left open.

**Keywords:** Fuzzy Julia sets, fuzzy superior Julia sets, escape criterion.

## 1. Introduction

Fractal geometry is a newly developed area in mathematics in the recent decades. It focuses on the geometric shapes where similar patterns appear at arbitrarily small scale, which is known as the self-similarity. The delicate structures of fractals, however, are usually generated by iterations of simple functions, which allows fractals to be connected with dynamic system and chaos theory. In the field of fractal geometry, there are two important objects, Julia sets and Mandelbrot set, which are generated by quadratic functions of complex numbers [1, 2]. Despite the fact that Julia sets and Mandelbrot set both have topological and analytical properties which are important in the theory of fractal figures, it should be noticed that Mandelbrot set share similar properties with Julia sets [3].

After Mann [4] introduced the superior sequence, Rani and Kumar in 2004 defined the superior Julia sets and superior Mandelbrot set [5, 6]. The introduction of these two sets raised a new approach to examine the convergence of superior sequence and constructed new fractal figures with higher complexity.

Another important mathematical object is fuzzy set. Zadeh in 1965 developed the theory of fuzzy sets so that mathematically description of the concept of fuzziness is achievable [7]. Different from classical set theory where one object is either in or not in the given set, the theory of fuzzy set, by assigning every element in a universal set a grade of membership, makes it possible to construct sets without precisely defined criteria. This idea promotes the development of Artificial Intelligence by mimicking the ambiguous thinking process of humans.

In 2021, İnce and Ersoy introduced the concept of fuzzy Mandelbrot set to describe the difference of the velocity of escaping the Mandelbrot set among points [8]. And in 2022, Mahmood and Ali introduced the idea of fuzzy superior Mandelbrot set [9]. Their studies raised a new approach of viewing the Mandelbrot set. Considering that the Mandelbrot set and Julia sets share many similarities, it is reasonable to extend their work and develop a theory on fuzzy Julia sets and fuzzy superior Julia sets.

In this paper, the theory of fuzzy Julia sets and fuzzy superior Julia sets will be developed by providing a criterion for assigning each complex number from a grade of membership according to the velocities of escaping of each complex number from Julia sets and superior Julia sets respectively. The topological and analytical properties of fuzzy Julia sets and superior Julia sets will also be investigated.

## 2. Preliminaries

### 2.1. Fuzzy Sets

Despite the deference between classical set theory and fuzzy set theory, fuzzy sets expand the concepts in classical set theory when taking the idea of partially inclusion into consideration.

**Definition 1.** Given a universal set U, a fuzzy set defined on U is a collection of ordered pairs $F = \{(u, \mu(u)) | u \in U\}$ where $\mu: U \to [0,1]$ is a membership function. For all $u \in U$, i. u is not included in F if $\mu(u) = 0$, x is partially included in F if $\mu(u) \in (0,1)$, and u is fully included F if $\mu(u) = 1$ [7].

**Definition 2.** Given a universal set U, fuzzy sets F and G defined on U with membership function $\mu_F$ and $\mu_G$ respectively, then the equality is defined as $F = G$ if $\mu_F(u) = \mu_G(u)$ for all $u \in u$; the inclusion is defined as $F \subset G$ if $\mu_F(u) \leq \mu_G(u)$ for all $u \in U$; and the union and intersection are defined as $F \cup G = \{(u, \max\{\mu_F(u), \mu_G(u)\}) | u \in U\}$, $F \cap G = \{u, \min\{\mu_F(u), \mu_G(u)\}) | u \in U\}$, the complement of F is $\neg F = \{(u, \sigma(u)) | \sigma(u) = 1 - \mu(u)\}$ [7].

The definition of fuzzy sets utilizes traditional set theory where every object has clear criterion to describe the fuzziness, which makes the production of an algorithm for computer in decision making possible. In most applications of fuzzy sets, given a criterion, a way to describe those objects which might be coincide with the criterion to a certain degree is demanded. Therefore, the support sets and $\delta$-cuts of a fuzzy set are defined.

**Definition 3.** Given a universal set U, fuzzy set $F = \{(u, \mu(u)) | u \in U\}$, the support set of F is $\text{supp}(F) = \{u \in U | \mu(u) \neq 0\}$ [7, 10].

**Definition 4.** Given a universal set U, fuzzy set $F = \{(u, \mu(u)) | u \in U\}$, and $\delta \in [0,1]$, the $\delta$-cut of F is $F^{\delta} = \{u \in U | \mu(u) \geq \delta\}$, and the strong $\delta$-cut of F is $F^{\delta+} = \{u \in U | \mu(u) > \delta\}$ [7, 10].

### 2.2. Julia Sets and Mandelbrot Set

**Definition 5.** For a function $f_c: \mathbb{C} \to \mathbb{C}, x \mapsto x^2 + c$ where $c \in \mathbb{C}$, the filled Julia set is $K_c = \{x \in \mathbb{C} | \{f_c^n(x)\}_{n \in \mathbb{N}} \text{ is bounded}\}$. The Julia sets are the boundary of filled Julia sets [1].

**Definition 6.** The Mandelbrot set is $\mathbb{M} = \{c \in \mathbb{C} | 0 \in K_c\}$ [2].

The following theorem gives the escape criterion of Mandelbrot set, which is essential in the definition of fuzzy Mandelbrot set.

**Theorem 1.** For all $c \in \mathbb{C}$, $c \in \mathbb{M}$ if and only if $|f_c^n(0)| \leq 2$ for all $n \in \mathbb{N}$ [11].

With this escaping criterion of Mandelbrot set, İnce and Ersoy introduced the idea of fuzzy Mandelbrot set by defining the escaping velocity by the times of iterations when $|f_c^n(0)|$ exceeds the escaping criterion for the first time.

**Definition 7.** A fuzzy Mandelbrot set is a fuzzy set $\overline{\mathbb{M}}$ defined on $\mathbb{C}$ such that $\overline{\mathbb{M}} = \{(c, \mu_{\mathbb{M}}) | c \in \mathbb{C}\}$ where:

$$\mu_{\mathbb{M}} = \begin{cases} 1 & \text{if } |f_c^n(0)| \leq 2 \text{ for all } n \in \mathbb{N} \\ \left| \frac{f_c^{n-1}(0)}{f_c^n(0)} \right| & \text{if } |f_c^n(0)| > 2 \text{ and } |f_c^n(0)| \leq 2 \text{ for an } n \in \mathbb{N} \end{cases} \quad (1)$$

In order to explore the theory on fuzzy Julia sets, it is reasonable that the escaping criterion of Julia sets plays an important role.

**Theorem 2.** For all $c \in \mathbb{C}$, for all $z \in \mathbb{C}$, $z \in K_c$ if and only if $|f_c^n(z)| \leq \max\{|c|, 2\}$ for all $n \in \mathbb{N}$.

### 2.3. Superior Julia Sets and Superior Mandelbrot Set

It should be noticed that the superior sequence is a significant part in the definition of the Superior Julia sets and superior Mandelbrot set.

**Definition 8.** The superior sequence is $x_{z_n} = s_n f_c(x_{n-1}) + (1 - s_n)x_{n-1}$ for $n \geq 1$ and $x_{z_0} = z$, where $\{s_n\}_{n\in\mathbb{N}}$ is convergent to a non-zero number and $s_n \in (0,1)$ for all $n \in \mathbb{N}$ [4].

**Definition 9.** For a $c \in \mathbb{C}$, the filled superior Julia set is $SK_c = \{z \in \mathbb{C} | \{x_{z_n}\}_{n\in\mathbb{N}} \text{ is bounded}\}$. The superior Jullia set is the boundary of filled superior Julia set [5].

**Definition 10.** The superior Mandelbrot set is $\mathbb{SM} = \{c \in \mathbb{C} | 0 \in SK_c\}$ [6].

Under the same philosophy of fuzzy Mandelbrot set, Mahmood and Ali defined the fuzzy superior Mandelbrot set by assigning each complex number the escaping velocity of superior Mandelbrot set. However, it should be noticed that they define superior Mandelbrot set by assuming $\{s_n\}_{n\in\mathbb{N}}$ as a constant sequence in [0,1].

**Definition 11.** A fuzzy superior Mandelbrot set is a fuzzy set $\overline{\mathbb{SM}}$ defined on $\mathbb{C}$ such that $\overline{\mathbb{SM}} = \{(c, \mu_{\mathbb{SM}}) | c \in \mathbb{C}\}$ where:

$$\mu_{\mathbb{SM}} = \begin{cases} 1 & \text{if } |x_{0_n}| \leq 2 \text{ for all } n \in \mathbb{N} \\ \left|\frac{x_{0_{n-1}}}{x_{0_n}}\right| & \text{if } |x_{0_n}| > 2 \text{ and } |x_{0_{n-1}}| \leq 2 \text{ for an } n \in \mathbb{N} \end{cases} \tag{2}$$

## 3. Fuzzy Julia Sets

From the observation of escaping criteria of Julia sets, an algorithm of finding filled Julia sets can be generated [12]. For a given c, let $A_0 = \mathbb{C}$. Then for every $n \in \mathbb{N}$, $A_{n+1} = \{z \in A_n | |f_c(z)| \leq \max\{|c|, 2\}\}$. As n goes to infinity, $A_n$ is gradually assessing to the filled Julia set at c. From this algorithm, one intrusive idea of the escaping velocity can be generated that the complex numbers that are ruled out for smaller n tend to have higher escaping velocity. In order to precisely describe this intrusive idea, the concept of fuzzy sets can be utilized.

However, it should be noticed that one observation which can be made in the construction of fuzzy Mandelbrot set and fuzzy superior Mandelbrot set is that $\{f_{c_1}^n(0)\}_{n\in\mathbb{N}} \cap \{f_{c_2}^n(0)\}_{n\in\mathbb{N}} = \{0\}$ if $c_1 \neq c_2$. Therefore, the membership function $\mu_{\mathbb{M}}$ defined in fuzzy Mandelbrot set coincides well with the idea that complex numbers that can reach the escape criterion with fewer iterations tends to have higher escaping velocity. However, in the case of filled Julia sets, for a given complex number c, $f_c^n(x) = f_c^m(y)$ for some distinct natural numbers m and n, and for distinct complex numbers x and y. This observation leads to the problem that the membership function fails to assign complex numbers with different escaping velocity different grades of membership if the membership function is defined as $\left|\frac{f_c^{n-1}(z)}{f_c^n(z)}\right|$ if $|f_c^n(z)| > \max\{|c|, 2\}$ and $|f_c^{n-1}(z)| \leq \max\{|c|, 2\}$ for an $n \in \mathbb{N}$. Hence, an adjustment to the membership function should be made to avoid this problem.

**Definition 12.** For a given c, a fuzzy Julia set is a fuzzy set $\overline{J_c}$ defined on $\mathbb{C}$ such that $\overline{J_c} = \{(z, \mu_{J_c}) | z \in \mathbb{C}\}$ where:

$$\mu_{J_c} = \begin{cases} 1 & \text{if } |f_c^n(z)| \leq \max\{|c|, 2\} \text{ for all } n \in \mathbb{N} \\ \left|\frac{f_c^n(z)}{f_c^{n+1}(z)}\right|^{\frac{1}{2^n}} & \text{if } |f_c^{n+1}(z)| > \max\{|c|, 2\} \text{ and } |f_c^n(z)| \leq \max\{|c|, 2\} \text{ for an } n \in \mathbb{N} \\ 0 & \text{if } |z| > \max\{|c|, 2\} \end{cases} \tag{3}$$

From the definition of fuzzy Julia sets, two trivial lemmas can be observed.

**Lemma 1.** For all $c, z \in \mathbb{C}$, $\mu_{J_c}(z) = \mu_{J_c}(-z)$. Therefore, fuzzy Julia sets are symmetrical about $0 + 0i$.

**Lemma 2.** For all $c \in \mathbb{C}$, $\text{supp}(\overline{J_c}) = \{z \in \mathbb{C} | |z| \leq \max\{|c|, 2\}\}$.

**Example 1.** For $c = 0$, $\max\{|c|, 2\} = 2$, and the corresponding filled Julia set is the closed disk $K_0 = \{z \in \mathbb{C} | |z| \leq 1\}$, therefore, $\mu_{J_0}(z) = 1$ for all $z \in K_0$. Also, it is trivial that $\mu_{J_0}(z) = 0$ for all z such that $|z| > 2$.

For $z \in \mathbb{C}$, $|f_0(z)| = |z^2| = |z|^2$. Therefore, for z such that $1 < |z| \leq 2$, suppose $|f_0^{n+1}(z)| > \max\{|c|, 2\}$ and $|f_0^n(z)| \leq \max\{|c|, 2\}$ for an $n \in \mathbb{N}$, then $2^{\frac{1}{2^{n+1}}} < |z| \leq 2^{\frac{1}{2^n}}$. Then, $\mu_{J_0} = \left|\frac{f_0^n(z)}{f_0^{n+1}(z)}\right|^{\frac{1}{2^n}} = \left|\frac{1}{f_0^n(z)}\right|^{\frac{1}{2^n}} = \frac{1}{|f_0^n(z)|^{\frac{1}{2^n}}} = \left(\frac{1}{(|z|^{2^n})}\right)^{\frac{1}{2^n}} = \frac{1}{|z|}$. This implies the following lemma.

**Lemma 3.** When c=0, $\mu_{J_0}$ is continuous on $\text{supp}(\overline{J_0})$. Moreover, $\mu_{J_0}$ has minimum $\frac{1}{2}$ on $\text{supp}(\overline{J_0})$.

**Theorem 3.** For $c \in \mathbb{C}$ with $|c| < 1$, the membership function $\mu_{J_c}$ is upper semi-continuous.

**Lemma 4.** If $|c| < 1$, then $\frac{|c|}{|f_c^k(z)|} < 1$ for $n \in \mathbb{N}$ such that $|f_c^{n+1}(z)| > 2$ and $|f_c^n(z)| \leq 2$ and for $z \in \mathbb{C}$ such that $z \notin K_c$ and $|z| < 2$.

**Proof:** If $|f_c^n(z)| \leq |c|$, $|f_c^{n+1}(z)| = |(f_c^n(z))^2 + c| \leq |f_c^n(z)|^2 + |c| \leq |c|^2 + |c| < 2$, which is contradictory with the assumption that $|f_c^{n+1}(z)| > 2$. Therefore, $|f_c^n(z)| > |c|$ and $\frac{|c|}{|f_c^n(z)|} < 1$.

**Proof of theorem 3:** First, in order to prove that $\mu_{J_c}$ is upper semi-continuous, it is necessary to show that $\mu_{J_c}^{-1}([0, \lambda))$ is open for all $\lambda \in (0,1]$. Note that:

$$\mu_{J_c}^{-1}([0,\lambda)) = \mu_{J_c}^{-1}(\{0\} \cup (0,\lambda)) = \{z \in \mathbb{C} | |z| > 2\} \cup \cup_k \{z | 0 < \left|\frac{f_c^n(z)}{f_c^{n+1}(z)}\right|^{\frac{1}{2^n}} < \lambda\} \tag{4}$$

Also note that $0 < \left|\frac{f_c^n(z)}{f_c^{n+1}(z)}\right|^{\frac{1}{2^n}} < \lambda$ implies that $\frac{1}{\lambda^{2^n}} < \frac{|f_c^{n+1}(z)|}{|f_c^n(z)|}$. Therefore, by triangle inequality and lemma 4, it can be derived that:

$$\frac{|f_c^{n+1}(z)|}{|f_c^n(z)|} = \frac{\left|(f_c^n(z))^2 + c\right|}{|f_c^n(z)|} \leq |f_c^n(z)| + \frac{|c|}{|f_c^n(z)|} < |f_c^n(z)| + 1 \tag{5}$$

Consequently, $\frac{1}{\lambda^{2^n}} - 1 < |f_c^n(z)|$. Then,

$$\mu_{J_c}^{-1}([0,\lambda)) = \{z \in \mathbb{C} | |z| > 2\} \cup \cup_n \{z | \frac{1}{\lambda^{2^n}} - 1 < |f_c^n(z)|\} \tag{6}$$

Given that for all $n \in \mathbb{N}$, $f_c^n(z)$ is a polynomial about z, it can be implied that $f_c^n: \mathbb{C} \to \mathbb{C}$ is continuous. As a result, $\mu_{J_c}^{-1}([0,\lambda))$ is open. This is followed directly by the conclusion that $\mu_{J_c}$ is upper semi-continuous.

**Theorem 4.** For $c \in \mathbb{C}$ such that $|c| < 1$, $\mu_{J_c}$ is continuous on $\text{supp}(\overline{J_c})$.

**Proof:** Consider $\mu_{J_c}^{-1}((\lambda, 1))$ for $\lambda \in (0,1)$, note that:

$$\mu_{J_c}^{-1}((\lambda,1]) = \cup_n \{ z | \lambda < \left|\frac{f_c^n(z)}{f_c^{n+1}(z)}\right|^{\frac{1}{2^n}} < 1\} \cup K_c \tag{7}$$

Since $\lambda < \left|\frac{f_c^n(z)}{f_c^{n+1}(z)}\right|^{\frac{1}{2^n}} < 1$ implies that $1 < \frac{|f_c^{n+1}(z)|}{|f_c^n(z)|} < \frac{1}{\lambda^{2^n}}$. Again, by the triangle inequality and lemma 4, it can be derived that:

$$\frac{|f_c^{n+1}(z)|}{|f_c^n(z)|} = \frac{\left|(f_c^n(z))^2+c\right|}{|f_c^n(z)|} \geq |f_c^n(z)| - \frac{|c|}{|f_c^n(z)|} > |f_c^n(z)| - 1 \tag{8}$$

Therefore, $|f_c^n(z)| < \frac{1}{\lambda^{2^n}} + 1$. Then,

$$\mu_{J_c}^{-1}((\lambda, 1]) = \cup_n \{z \,|\, |f_c^n(z)| < \tfrac{1}{\lambda^{2^n}} + 1.\} \cup K_c \tag{9}$$

By theorem 2, $K_c = \{z \in \mathbb{C} || f_c^n(z)| \leq 2 \text{ for all } n \in \mathbb{N}\}$, therefore,

$$K_c \subset \cup_n \{z \,|\, |f_c^n(z)| < \tfrac{1}{\lambda^{2^n}} + 1\} \tag{10}$$

Since $\frac{1}{\lambda^{2^n}} + 1 > 2$ for all $\lambda \in (0,1)$. By the continuity of $f_c^n: \mathbb{C} \to \mathbb{C}$, $\mu_{J_c}^{-1}((\lambda, 1])$ is open. Because $B = \{(0, \lambda) | \lambda \in (0,1)\} \cup \{(\lambda, 1] | \lambda \in (0,1)\}$ is a basis for the standard topology on $(0,1]$, $\mu_{J_c}$ is continuous on $\text{supp}(\overline{J_c})$.

**Corollary 1.** The strong $\delta$-cut of fuzzy Julia sets $\overline{J_c}^{\delta+}$ are open for all $c \in \mathbb{C}$ such that $|c| < 1$ and for all $\delta \in (0,1)$.

**Lemma 5.** The strong $\delta$-cut of fuzzy Julia sets $\overline{J_c}^{\delta+}$ are bounded for all $c \in \mathbb{C}$ and for all $\delta \in (0,1)$.

**Proof:** $\overline{J_c}^{\delta+} \subset \text{supp}(\overline{J_c}) = \{z \in \mathbb{C} || z| \leq \max\{|c|, 2\}\}$.

## 4. Fuzzy Superior Julia Sets

Similar to the idea in the construction of fuzzy Julia sets, in order to construct a membership function that assigns each complex numbers a grade of membership according to their escaping velocity of the superior Julia sets, the escape criterion of a superior Julia sets should be demonstrated. It should be noticed that the superior sequence is also treated as a real number in (0,1] in order to make the study of the basic properties of fuzzy superior Julia sets convenient.

**Theorem 5.** For $s \in (0,1]$, for $c \in \mathbb{C}$, for any $z \in \mathbb{C}$, $z \in SK_c$ if and only if $|x_{z_n}| \leq \max\{|c|, \frac{2}{s}\}$ for all $n \in \mathbb{N}$ [5].

With this escape criterion in mind, the fuzzy superior Julia sets can be defined as the following.

**Definition 13.** For a given c, and an $s \in (0.1)$, a fuzzy superior Julia set is a fuzzy set $\overline{SK_c}$ defined on $\mathbb{C}$ such that $\overline{SK_c} = \{(z, \mu_{SK_c}) | z \in \mathbb{C}\}$ where:

$$\mu_{SK_c} = \begin{cases} 1 & \text{if } |x_{z_n}| \leq \max\{|c|, \frac{2}{s}\} \text{ for all } n \in \mathbb{N} \\ \left|\frac{x_{z_n}}{x_{z_{n+1}}}\right|^{\frac{1}{2n}} & \text{if } |x_{z_{n+1}}| > \max\{|c|, \frac{2}{s}\} \text{ and } |x_{z_n}| \leq \max\{|c|, \frac{2}{s}\} \text{ for an } n \in \mathbb{N} \\ 0 & \text{if } |z| > \max\{|c|, \frac{2}{s}\} \end{cases} \tag{11}$$

**Example 2.** If $s = 1$, $x_{z_n} = f_c(x_{n-1})$. Therefore, $\overline{SK_c} = \overline{J_c}$ for all c.

The two trivial lemmas for fuzzy Julia sets can be adjusted into two trivial lemmas for fuzzy superior Julia sets.

**Lemma 6.** For all $c$, $z \in \mathbb{C}$, $\mu_{SK_c}(z) = \mu_{SK_c}(-z)$. Therefore, fuzzy superior Julia sets are symmetrical about $0 + 0i$.

**Lemma 7.** For all $c \in \mathbb{C}$, $\text{supp}(\overline{SK_c}) = \{z \in \mathbb{C} || z| \leq \max\left\{|c|, \frac{2}{s}\right\}\}$.

However, the fuzzy superior Julia sets tends to behave more chaotic than fuzzy Julia sets, therefore, more properties of fuzzy superior Julia sets are left open in this article.

## 5. Conclusion

In this paper, the definition of fuzzy Julia sets is given. Comparison of the definition of fuzzy Julia sets and fuzzy Mandelbrot set is made and the reasons for changes in the definition of fuzzy Juia sets are explained. And for cases of $|c| < 1$, the theorem that the membership function is upper semi-continuous and is continuous on the support set of fuzzy Julia sets is proved. Also, the definition of fuzzy superior Julia sets is given with some examples and trivial lemma. However, more thorough study on the properties of fuzzy Julia sets, especially those cases which involves with the observation of more chaotic behavior which may happen under a more general cases, are demanded. In addition, the connectedness of fuzzy Julia sets may vary significantly as c varies, which is another field that is meaningful to look into. As for the cases of fuzzy superior Julia sets, more complicated behavior can be observed given the fact that the variables include the superior sequences, which are usually viewed as a constant, and the constant term in the function of z. Therefore, the properties of fuzzy superior Julia sets are left open as well.

## References

[1] Julia G. Mémoire sur l'itération des fonctions rationnelles. Journal de mathématiques pures et appliquées, 1918, 147 - 245.

[2] Mandelbrot B B. Fractal aspects of the iteration of z→ Λz (1-z) for complex Λ and z. Annals of the New York Academy of Sciences, 1980, 357 (1): 249 - 259.

[3] Lei, T. Similarity between the Mandelbrot set and Julia sets. Communications in mathematical physics, 1990, 134: 587-617.

[4] Mann W R. Mean value methods ni iteration. Proc. Amer: Math. Soc. 1953, 4: 506 - 510.

[5] Rani M, Kumar V. Superior julia set. Research in Mathematical Education, 2004, 8 (4): 261 - 277.

[6] Zadeh L A. Fuzzy sets. Information and control, 1965, 8 (3): 338 - 353.

[7] İnce İ, Ersoy S. Fuzzy mandelbrot sets. Fuzzy sets and systems, 2022, 435: 78 - 88.

[8] Mahmood T, Ali Z. Fuzzy superior mandelbrot sets. Soft Computing, 2022, 26 (18): 9011 - 9020.

[9] Pao-Ming P, Ying-Ming, L. Fuzzy topology. I. Neighborhood structure of a fuzzy point and Moore-Smith convergence. Journal of mathematical analysis and applications, 1980, 76 (2): 571 - 599.

[10] Carleson L, Gamelin T W. Complex dynamics. Springer Science & Business Media, 2013.

[11] Devaney R L. A first course in chaotic dynamical systems: theory and experiment. CRC Press, 2018.