Proof and Application of Cauchy's Residue Theorem

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Abstract. Complex analysis is a major subfield of mathematics that is concerned with investigating complex functions and their behaviors. The Cauchy's residue theorem plays an important role in complex analysis. It is also the main focus for this paper. The residue theorem connects complex integrals of functions with their residues at singular points. It is widely used to find integrals of complex functions that may be insolvable otherwise. This paper first looks at the definition of Cauchy's Residue Theorem. Next, it provides proof for the theorem from Laurent theorem and Cauchy-Goursat theorem. Finally, it looks at some applications of Cauchy's Residue theorem, including the theorem's use in evaluating integrals, in the proof of related theorems important to the field of complex analysis, and in its use in the formation of Fourier and Laplace transforms. Through these examples, the efficacy of the theorem, and the use of it in many areas, can be seen clearly.

Keywords: Cauchy-Goursat theorem, Cauchy's Residue Theorem, isolated singular points; residues, integrals.

1. Introduction

Complex function theory is a subdiscipline of mathematics that studies complex-valued functions, including the differentiation, integration, and analysis of functions in a complex domain. Complex analysis provides a method for solving many real-world problems that cannot be described by real-valued functions.

Cauchy's residue theorem is fundamental to complex analysis, the analysis of function with variables of complex numbers. It is used to calculate the complex variable integrals of analytic functions over closed contours. Cauchy's residue theorem was developed by Augustin-Louis Cauchy, a French mathematician, in the early nineteenth century, when complex analysis was emerging as a research field. In the 1700s, the works of mathematicians like Euler, Lagrange, Laplace, Poisson, and d'Alembert formed the basis of complex analysis and complex function theory [1]. In 1825, Cauchy first published the residue theorem and its proof, relating residues to the integrals of complex functions [1]. In subsequent years, the works of mathematicians such as Cauchy, Riemann, and Weierstrass greatly furthered the development of complex analysis. Cauchy's residue theorem was frequently used in the development of complex analysis and is utilized in the proof of many theorems such as the argument principle, Jordan normal form theorem, and fundamental theorem of algebra [2].

Cauchy was instrumental in the development of complex analysis, and the residue theorem also plays important roles in the development of complex function. Today, it remains the foundation of complex function theory [3]. Cauchy's residue theorem is used in many areas of mathematics, physics, and engineering [4], [5].

This paper will look at the concepts surrounding Cauchy's Residue Theorem, the theorem's proofs, and several applications of the theorem. It will begin with some background concepts of complex analysis and provide a definition of Cauchy's Residue Theorem and its related concepts. Then, it will include a proof of the theorem, and finally show some of its applications.

2. Cauchy's Residue Theorem

2.1. Definitions for Cauchy's residue theorem

Integrals of complex functions are defined in a way similar to the integral of real-valued functions.

$$\int_a^b f(t)dt = \int_a^b u(t)dt + i\int_a^b v(t)dt. \quad (1)$$

For a contour integral along a path γ, with the domain in the real-valued interval [a,b], the integral of a complex function f can be defined in the following way [6]:

$$\oint_\gamma f(z)dz = \int_a^b f\big(\gamma(t)\big)\gamma'(t)dt. \quad (2)$$

Cauchy's residue theorem provides one method to easily and systematically calculate the contour integrals of differentiable complex functions over closed curves.

Theorem 2.1 (Cauchy's Residue Theorem [7]) If a function f is analytic on and inside a simple closed and positively oriented contour, with exceptions for finite numbers of singular points, then the integral of that function can be found using the equation:

$$\oint f(z)dz = 2\pi i \sum_{k=1}^{n} Res_{z=z_k} f(z) \quad (3)$$

Where $\sum_{k=1}^{n} \mathrm{Res}_{z=z_k} f(z)$ is the summation of the residues found at the singular points.

As a note, simple closed contours are closed curves that does not cross over themselves, and singular points are points where the function f is not differentiable.

2.2. Proof

First, there are several theorems which are needed for proving Cauchy's residue theorem.

Theorem 2.2 (Laurent theorem [7]) Let f(z) be a function analytic inside the domain $R_1 < |z - z_0| < R_2$, with R_1>0 and suppose C is a simple closed contour that is around z_0 and located in the domain, then there is a Laurent expansion for f(z) with respect to z_0. In other words, f has a Laurent series, with the following form:

$$f(z) = a_1(z-z_0)^1 + a_2(z-z_0)^2 + a_3(z-z_0)^3 + \cdots + \frac{b_1}{(z-z_0)^1} + \frac{b_2}{(z-z_0)^2} + \frac{b_3}{(z-z_0)^3} + \cdots \quad (4)$$

In the Laurent expansion, the coefficients b_n, where n is positive, is also equal to $\frac{1}{2\pi\mathrm{i}}\oint \frac{f(z)dz}{(z-z_0)^{-n+1}}$. Thus, $b_1 = \frac{1}{2\pi\mathrm{i}}\oint \frac{f(z)dz}{(z-z_0)^{-1+1}} = \frac{1}{2\pi\mathrm{i}}\oint f(z)dz$, and $\oint f(z)dz = 2\pi\mathrm{i}b_1$.

Definition 2.3 (Residue) The coefficient $b_1 = \frac{1}{2\pi i}\oint f(z)dz$, (i.e., the coefficient when the power is -1), is said to be the residue belonging to the function f at the isolated singular point z_0.

Theorem 2.4 (Cauchy-Goursat theorem [8]) Suppose f is a complex analytic function for all points inside or on a contour C which is simply closed. Then

$$\oint_C f(z)dz = 0. \quad (5)$$

For a multiply connected domain as shown in the Fig. 1 below, applying Cauchy-Goursat theorem, it can be shown that

$$\oint_C f(z)dz - \sum_{k=1}^{n} \oint_{C_k} f(z)dz = 0. \quad (6)$$

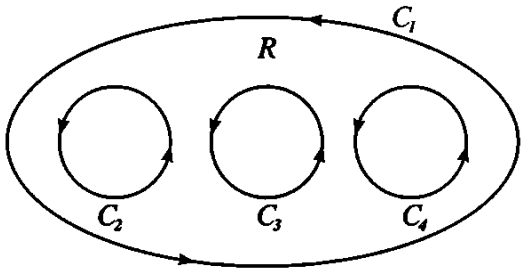


Figure 1. Contour used in proof for Cauchy-Goursat theorem for multiply connected domains [8].

Cauchy's Residue Theorem is derived from Cauchy-Goursat theorem. First, let there be a simple closed contour C. Suppose a function f is analytic on and inside the contour with exceptions for n singular points, z_1, z_2, …, z_n. Draw circles, C_1, C_2, …, C_n, centered at the corresponding points z_1, z_2, …, z_n, respectively, in C, in a way so that none of the circles have any points in common, as shown in Fig. 2:

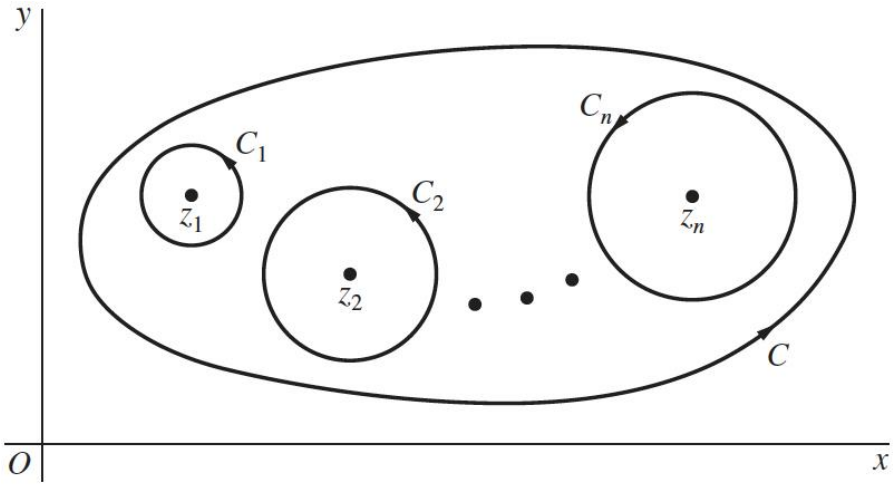


Figure 2. Contour used in proving Cauchy's Residue Theorem [7]

From Cauchy-Goursat theorem with adaption for multiply connected domains [7], the following result can be gotten:

$$\oint_C f(z)dz - \sum_{k=1}^{n} \oint_{C_k} f(z)dz = 0 \tag{7}$$

From the definition for residues, the result can be obtained that $\oint_{C_k} f(z)dz = 2\pi i Res_{z=z_k}$, fork from 1 to n.

Therefore,

$$\oint f(z)dz = 2\pi i \sum_{k=1}^{n} Res_{z=z_k} f(z) \tag{8}$$

And the proof is complete.

3. Applications

The Cauchy's residue theorem is used in many areas such as functional analysis, linear algebra, algebraic geometry, and quantum field theory. It has numerous practical applications, like solving integrals, proving related theorems, and constructing Fourier and Laplace transforms.

3.1. Solving integrals

One major use for Cauchy's residue theorem is in calculating complex variable integrals of analytic functions over closed contours.

Example 3.1 Let $f(z) = z^2 + z + \frac{1}{z}$ be a function with the domain of complex numbers. Take the contour integral off(z) with respect to the circle $\gamma(t) = 3$.

As illustrated in Fig. 3, there exists a singular point at z=0, with the residue equal to one. Therefore, utilizing the residue theorem, it is obtained that $\int_\gamma f(z) = 2\pi i$.

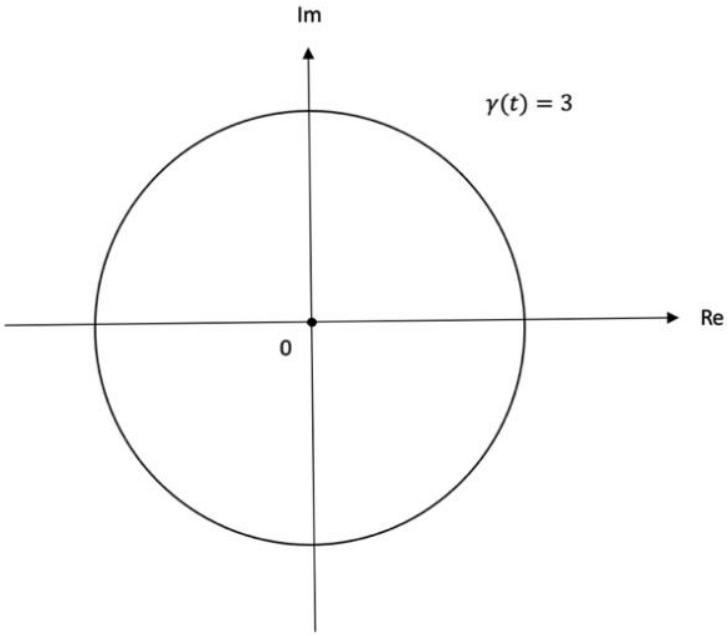


Figure 3. Contour for Example 1

Example 3.2 Let $f(z) = \frac{az^3+bz^2+cz+d}{z^4-1}$, with the coefficients a, b, c, and d having complex values. Discuss how to find contour integrals for this function.

Using partial fraction decomposition, f(z) is rewritten as $f(z) = \frac{A}{z-1} + \frac{B}{z-i} + \frac{C}{z+1} + \frac{D}{z+i}$, where $A = \frac{a+b+c+d}{4}$, $B = \frac{a-ib-c+id}{4}$, $C = \frac{a-b+c-d}{4}$, and $D = \frac{a+ib-c-id}{4}$.

The points z_0=1, z_0=i, z_0=-1, and z_0=-i are the singular points for f(z), since f(z) is not differentiable for these points.

Taking the Laurent expansion at z_0=1, the terms $\frac{B}{z-i}$, $\frac{C}{z+1}$, and $\frac{D}{z+i}$, according to Taylor's theorem, can all be expressed as a Taylor series: $f(z) = \sum_{n=0}^{\infty} a_n(z - z_0)^n$. Therefore, the residue of the function at z_0=1 is $A = \frac{a+b+c+d}{4}$. Similarly, the residues at z_0=i, z_0=-1, and z_0=-i are $B = \frac{a-ib-c+id}{4}$, $C = \frac{a-b+c-d}{4}$, and $D = \frac{a+ib-c-id}{4}$, respectively.

Thus, to find contour integrals for the function, when singular points z_0=1, z_0=i, z_0=-1, and z_0=-i are either inside or outside of the contour, the answer can be found by simply multiplying 2πi with the summation of the residues at the singular points located within the closed contour.

3.2. Use in proofs of related theorems

The residue theorem has widespread applications in the proof of many related theorems [6]. Here, this paper will present a proof for the argument principle, a theorem which relates the number of poles and zeros a function has to a contour integral of the function, as an example for a proof that utilizes Cauchy's residue theorem.

Definition 3.1 (Poles [7]) Poles are a type of isolated singularity. A point c that is located in to the domain of a complex function f is called a pole, if, when f is written as a Laurent series,

$$f(z) = a_1(z-c)^1 + a_2(z-c)^2 + a_3(z-c)^3 + \cdots + \frac{b_1}{(z-c)^1} + \frac{b_2}{(z-c)^2} + \frac{b_3}{(z-c)^3} + \cdots \quad (9)$$

There is a finite number of negative power terms of (z-c); i.e., there is a positive integer m for which $b_m \neq 0$, and, for every integer n that is greater than m, $b_n = 0$.

Theorem 3.2 (Argument principle [7]) Suppose f is a complex function meromorphic (analytic except for poles) on and in a simple closed and positively oriented contour C while also having no poles or zeros on C. Let P be the number of poles off(z) in the contour C counting multiplicities, Z be the number of zeros off(z) in the contour C counting multiplicities, and Δc argf(z) be the change in argf(z), the argument off(z), over C. Then,

$$\frac{1}{2\pi} \Delta c \, arg \, f(z) = Z - P. \quad (10)$$

To prove this theorem, first solve the integral $\oint_C \frac{f'(z)}{f(z)} dz$ using parametric representation. Letting z=z(t) and $f\big(z(t)\big) = \rho(t) e^{i \arg(f(z))}$, it can be found that $\oint_C \frac{f'(z)}{f(z)} dz = \mathrm{i} \Delta c \arg f(z)$ [7].

Then, apply Cauchy's residue theorem to solve the same integral. The result is that $\oint_C \frac{f'(z)}{f(z)} dz = 2\pi i(Z - P)$ [7].

From this, the result can be obtained that $\frac{1}{2\pi} \Delta c \arg f(z) = Z - P$, which completes the proof.

3.3. Fourier and Laplace transforms

Another major usage for Cauchy's residue theorem is in constructing Fourier transforms and Laplace transforms [9].

The Fourier transform expresses functions in a form that shows the frequencies of the functions [10]. Using the residue theorem to calculate, the following formula (Fourier transform formula) is obtained:

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{iwt} \tilde{f}(\omega) d\omega, \quad (11)$$

Were

$$\tilde{f}(\omega) = \int_{-\infty}^{+\infty} e^{-iwt} f(t) dt. \quad (12)$$

The Laplace transform converts a real function to a complex function. Utilizing Cauchy's residue theorem, the Laplace transform is found as:

$$f(t) = \frac{1}{2\pi i} \int_{a-i\infty}^{a+i\infty} e^{zt} \tilde{f}(z) dz, \quad (13)$$

Were

$$\tilde{f}(z) = \int_{0}^{+\infty} e^{-zt} f(t) dt. \quad (14)$$

These transformations have many real-word applications such as signal processing, system modeling, etc.,[11] showing applications of Cauchy's residue theorem in many areas.

4. Conclusion

This paper looks at the Cauchy's residue theorem, its proof, and its application in various areas. The residue theorem provides useful methods in finding complex variable integrals of analytic functions over closed contours. The theorem shows that the contour integral of a function analytic inside and on a simple closed contour, with exceptions for finite numbers of singular points, is equal to the summation of residues of the function at the singular points multiplied by $2\pi i$. Cauchy's residue theorem is proved using the Laurent Theorem and Cauchy-Goursat Theorem. The residue theorem has wide-ranging applications such as solving integrals, proving related theorems, constructing Fourier and Laplace transforms, and many other areas. This paper has just provided an overview of this widely used theorem and its applications, many of which could still develop significantly in the future.

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