Cauchy Theorem and Cauchy Residue Theorem

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Abstract. Cauchy theorem is widely used in solving analytic function problems in complex variables. It is an important theorem on path integrals of holomorphic functions in the complex plane. In this paper, the main work is about the application of the Cauchy theorem on the integrals which have singularities. The definition of the Cauchy theorem is given and proved by using the green theorem. By using the conclusion of Cauchy theorem, the integrals that have singularities can be solved, and the Cauchy residue theorem is proved. By using the Cauchy theorem, problems such as definite integrals, indefinite integrals, and differential equations can be solved. Some problems in other disciplines can also be solved such as the physics and computer. The Cauchy integral theorem states that if there are two distinct routes between two points and the function is holomorphic on both routes, then the integrals of the two pathways are equivalent.

Keywords: Cauchy Theorem, Cauchy Residue Theorem, integral.

1. Introduction

All Complex analysis is a branch of mathematics studying complex variable functions, including studying the properties and characteristics of complex variable functions. Complex number is the basic concept in complex analysis, and its introduction has solved many problems that cannot be solved by real variable functions. The problems involved in complex analysis include analytic function, holomorphic function, harmonic function, and conformal mapping. In complex variables, Cauchy theorem is a theorem about the path integral of holomorphic functions on a complex plane. The Cauchy theorem states that the integral of a holomorphic function on a simply connected closed region along any closed analytic curve is 0.

In this paper, the main work is about the Cauchy theorem. This theorem was proposed by Cauchy. From the founding of Cauchy's theorem in 1825 to the short proof of Cauchy's theorem in homology by Dickson in 1971, Cauchy's theorem has experienced a long history of nearly 150 years [1]. Cauchy theorem with singularities is studied, considering two cases where singularities are poles in the interior of a region and singularities are on the boundary of a region. Using residues to calculate Cauchy integral expressions, Cauchy theorem is obtained in different cases [2]. Cauchy's theorem and its extension theorem provide an important theoretical foundation for linear algebra and geometry. Cauchy's theorem is widely used in various disciplines. For example, by using Cauchy theorem of analytic function, a simple and straightforward proof method of the ampere loop theorem is given, which is suitable for the teaching and research of general physics and electromagnetism [3]. The Cauchy theorem can be used to establish the equality of two mathematical formulas for the anticipated number of individuals who are vulnerable to contracting an infection from a contaminated syringe in a group of individuals who share a syringe [4]. Cauchy integral formula for singular points on the contour is summarized and a conclusion is drawn that integral path C is a closed contour and the integral value is still zero when the self-intersection is finite or infinite [5]. For an extended class of holomorphic functions, an extension of the Cauchy integral theorem for functions valued in parameter-dependent elliptic algebras with structure polynomial is established [6]. By the main properties of homotopy of curves, the homotopy form of the Cauchy theorem in complex variables is proved [7]. The existence theorems under compactness-type constraints are established, together with the Cauchy theorem of fuzzy differential equations [8]. Based on the concept of diffraction of an ideal conducting cylinder, the problems in electromagnetic field can be solved by using Cauchy's integral theorem and residue theorem [9,10].

In the Section 2, the Cauchy Theorem is defined, and the Green Theorem is used to demonstrate its validity. Then, by using the Cauchy theorem, three examples of the complex integrals are solved. In the Section 3, the generalization of the Cauchy theorem is given, which includes the integral on the closed contours that have singularities. Then, the example of the integral which has singularities is given.

2. Cauchy Theorem

Theorem 2.1 (Cauchy theorem) If a continuous function g(z) is holomorphic everywhere in the domain D, then the integral in the closed contour C $\int_C g(z)dz$ is zero.

Proof: Let C represent the closed contour $z = z(m)$ $(x \le m \le b)$ that is outlined in the counterclockwise direction. g(z) is differentiable both within and outside of C.

$$\int_C g(z)dz = \int_a^b g[z(m)]z'(m)dm. \tag{1}$$

Suppose that $g(z) = h(x,y) + in(x,y)$, $z(m) = x(m) + iy(m)$.

Thus, $g[z(m)]z'(m) = h[x(m),y(m)] + in[x(m),y(m)] * (x'(m) + iy'(m))$. Since t is a real variable, then

$$\int_C g(z)dz = \int_a^b (hx' - ny')dm + i\int_a^b (nx' + hy')dm = \int_a^b (hdx - ndy) + i\int_a^b (ndx + hdy). \tag{2}$$

Use the green's theorem to calculate the right hand of the equation. The statement of the green theorem is that $\int_C (Qdy + Pdx) = \iint_R (-Py + Qx)dA$.

Since g is analytic and continuous anywhere on R, the function u and v are also continuous. Similarly, g, g', u', and v' are also continuous. Therefore, we can rewrite the equation by using the green's theorem

$$\int_C g(z)dz = \iint_R (-nx - hy)dA + i\iint_R (hx - ny)dA. \tag{3}$$

The Cauchy-Riemann equations' central assertion is that hx=ny and hy=-nx. The right hand of the formula is therefore equal to 0.

Example 2.1 Use Theorem 2.1 to demonstrate how $\int_C g(z)dz = 0$ with $g(z) = \frac{z^2}{z+3}$.

Proof: If C is the unit circle's contour $|z| = 1$, the equation $z = e^{i\theta}$ $(0 \le \theta \le 2\pi) = \cos\theta + i\sin\theta$ can be supposed. Then $\int_C g(z)dz = \int_0^2 \pi g(e^{i\theta})i\theta dt$. As g(z) is analytic anywhere in the contour, there is no singularities in it.

The real and imaginary halves of the function g(z) are represented by m(x,y) and n(x,y), respectively.

$$\begin{aligned} \int_C g(z)dz &= \int_0^2 \pi(mx' - ny')dt + i\int_0^2 \pi(nx' + my')dt \\ &= \int_0^2 \pi(mdx - ndy) + i\int_0^2 \pi(ndx + mdy). \end{aligned} \tag{4}$$

Since g(z) is an analytic function anywhere in the contour C, then mx=ny and my=-nx. By the green theorem.

$$\int_C g(z)dz = \iint_R (-nx - my)dA + i\iint_R (mx - ny)dA = 0. \tag{5}$$

Therefore, $\int_C g(z)dz = 0$.

Example 2.2 Prove $e^{-\pi b^2} = \int_{-\infty}^{+\infty} e^{-\pi x^2} e^{-2\pi i x b} dx,\ b \in R$.

Proof: $e^{-\pi x^2}$ is its own Fourier transform. If b=0, the function $g(z) = e^{-\pi x^2}$ would be the integral function in the close contour C.

So, the contour C would be a close rectangle whose vertex is R, $R + ib$, $-R$, and $-R + ib$, and the contour is along the counterclockwise. Therefore, according to the theorem 2.1, the integral $\int_C g(z)dz = 0$. Along the horizontal direction, the function is the integral of real numbers, the integral could be denoted as $\int_{-R}^{R} e^{-\pi x^2} dx$. The integral equals to 1when $R \to +\infty$.

Along the vertical direction, the integral on the right segment would be $\int_0^b g(R + iy)idy = \int_0^b e^{-\pi(R^2+2iRy-y^2)} idy$. When R→+∞, the integral equals to 0.

Similarly, the integral on the left segment would be 0. Consequently, the integral on the horizontal line segment would be

$$\int_{-\infty}^{+\infty} e^{-\pi(x+ib)^2} dx = -e^{\pi b^2} \int_{-\infty}^{+\infty} e^{-\pi x^2} dx. \tag{6}$$

Therefore, when $R \to +\infty$, by the Cauchy theorem, $0 = 1 - e^{\pi b^2} \int_{-\infty}^{+\infty} e^{-\pi x^2} dx$.

Example 2.3 Prove $\int_0^{+\infty} \frac{1-\cos x}{x^2} = \frac{\pi}{2}$.

Proof: Suppose that $g(z) = \frac{1-e^{iz}}{z^2}$, and calculate the integral of the function above the x-axis. The r is used to represent the radius of the smaller circle. R is used to represent the radius of bigger circle. C_r and C_R are used to represent the small and big circular contour. Suppose that C_r represents the clockwise of the contour, while C_R represents the counterclockwise of the contour. Therefore, according to the Cauchy theorem, the equation can be written as

$$\int_{-R}^{-r} \frac{1-e^{ix}}{x^2} dx + \int_{C_r} \frac{1-e^{iz}}{z^2} dz + \int_r^R \frac{1-e^{ix}}{x^2} dx + \int_{C_R} \frac{1-e^{iz}}{z^2} dz = 0. \tag{7}$$

If $R \to +\infty$, $|\frac{1-e^{iz}}{z^2}| \leq \frac{2}{|z|^2}$. Therefore, the integral on the contour C_R equals to 0. Besides,

$$\int_{|x|\geq r} \frac{1-e^{ix}}{x^2} dx = -\int_{C_r} \frac{1-e^{iz}}{z^2} dz. \tag{8}$$

Then, we can suppose that $g(z) = \frac{-iz}{z^2} + E(z)$.

When $z \to 0$, E(z) has a close contour, and on the contour C_r, we have $z = re^{i\theta}$ and $dz = ire^{i\theta}d\theta$. So, when $r \to 0$, $\int_{C_r} \frac{1-e^{iz}}{z^2} dz \to \int_\pi^0 (-ii)d\theta = -$.

Hence, it follows the real part of the integral: $\int_{-\infty}^{+\infty} \frac{1-\cos x}{x^2} = \pi$. Therefore,

$$\int_0^{+\infty} \frac{1-\cos x}{x^2} = \frac{\pi}{2}. \tag{9}$$

3. Cauchy Residue Theorem

A location z_0 is referred to be a singularity if f is not differentiable at z_0 but is differentiable elsewhere in each of its surroundings. For example, the point z = 0 is an obvious singularity of $g(z) = \frac{1}{z}$. These singular points must all be isolated if f is differentiable inside of the contour C, which is straightforward closed.

Theorem 3.1 (Cauchy Residue Theorem) The value of the integral of g around C is equal to $2\pi i$ times the sum of the residues of g at the singular points inside C if g is analytical on C and if C is positively orientated. Let C, as an illustration, be a straightforward closed contour that is positive characterized. g must be analytical both within and on C, with the exception of unique points $z_1, \ldots, z_n$ inside C, then

$$\int_C g(z)dz = 2\pi i \sum_{k=1}^{n} RES_{z=z_k} g(z). \tag{10}$$

Proof: In order to demonstrate the Theorem 3.1, assume the z_k are the middle points of positively orientated, internal to C_k, small enough circles so no two of them contain the same point. In a closed area where g is analytical and the interior of g is a multiply linked area made up of the points within C_k and external to each other, the circles and the simple closed contour C_k together define the region's border. Therefore, the integral would be 0 according to the Cauchy theorem's application to these domains.

$$\int_C g(z)dz - \sum_{k=1}^{n} \int_{C_k} g(z)dz = 0. \tag{11}$$

(11) proves the display $\int_C g(z)dz = 2\pi i \sum_{k=1}^{n} RES_{z=z_k} g(z)$, and the proof is complete.

Example 3.1 Let $g(z) = \frac{(4z-5)}{z(z-1)}$. Please apply the theorem to get $\int_C g(z)dz$. C stands for a circle, $|z| = 2$, that is described counterclockwise.

Proof: The two isolated singularities $z = 0$ and $z = 1$, and both of which are internal to C, are present in the integral. The Maclaurin series representation makes it simple to locate the equivalent residues h_1 when z = 0 and h_2 when z = 1.

$$\frac{1}{1-z} = \sum_{i=1}^{\infty} z^i \ (|\ z\ | < 1) \tag{12}$$

Therefore, when $|\ z\ |$ is smaller than 1, $\frac{4z-5}{z(z-1)} = (4 - \frac{5}{z})(-1 - z - z^2 - \cdots ...)$. It is evident that h1=5 by locating the 1/z coefficient.

Similarly, $\frac{4z-5}{z(z-1)} = (4 - \frac{1}{z-1})[1 - (z-1) + (z-1)^2 - \cdots ...]$. When $|\ z-1\ |$ is smaller than 1, it leads to that $h_2 = -1$.

Thus, $\int_C g(z)dz = 2\pi i(h_1 + h_2) = 8\pi i$.

The integral in this instance may be expressed as its partial fraction:

$$\frac{4z-5}{(z-1)z} = \frac{5}{z} - \frac{1}{z-1}. \tag{13}$$

It is possible to demonstrate the theorem because $\frac{5}{z}$ is a Laurent series if $|\ z\ |$ is smaller than 1, and $\frac{1}{z-1}$ is a Laurent series if $|\ z-1\ |$ is smaller than 1.

4. Conclusion

In other words, Cauchy theorem states that if an analytic function in a complex plane does not have any singularities in a certain domain, then the Cauchy integral of the function in that domain equals to 0. In this paper, the Cauchy theorem is given and proved. Then the generalization of the Cauchy theorem is applied to the complex integrals which have singularities. This is also equivalent to the Cauchy residue theorem. Then these theorems are applied on some problems such as some complex integrals which have singularities. Cauchy theorem is an important theorem in mathematics,

which is widely used in solving analytic function problems in complex variables. At the same time, Cauchy theorem also provides an important theoretical basis for the study of analytic functions in the complex plane. By solving or proving the real function integral through theorem 2.1, the problem of calculating real function integral can be simplified.

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