Unveiling Definite and Indefinite Integrals Through Flexibly Using Different Formulas

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Abstract. One of the most powerful branches of mathematics is calculus. It is closely related with the study of changing rate and the quantity accumulation, which provides a framework for understanding and analyzing complex, continuous processes. Integral calculus and differential calculus are two main branches in calculus. Differential calculus focuses on the concept of derivatives, which represent the rate at which a quantity changes related to another variable. Derivatives are fundamental in understanding and describing processes that involve instantaneous rates of change, such as velocity, acceleration, and growth rates. By calculating derivatives, one can determine critical points, identify trends, and solve optimization problems. In contrast, integral calculus focuses on the notion of integrals, which signify the gathering of quantities within a given interval. Integrals allow people to find areas, volumes, and total amounts by summing infinitesimally small contributions. They are especially useful for solving problems involving accumulation, such as finding the area under a curve, determining total distance traveled, and calculating the net change in a quantity over a given interval.

Keywords: Calculus, Definite integral, Indefinite integral, Derivatives.

1. Introduction

Integration, an essential concept in calculus, involves the determination of the area beneath a curve, the calculation of accumulated quantities, and the resolution of diverse mathematical problems. It serves as the inverse operation of differentiation and holds significant importance in numerous disciplines such as physics, engineering, economics, and statistics [1]. Integration allows people to calculate the total change or the sum of infinitesimally small changes over a given interval. At its core, integration involves finding the antiderivative of a function. An antiderivative, also known as an integral, represents the original function before it was differentiated.$\int f(x)dx$, and the process is often referred to as taking the integral. It signifies the summation of infinitely small increments of the function. There are different methods and techniques for performing integration, including definite and indefinite integration [2]. Definite integration is used to find the exact numerical value of the accumulated quantity within a specific interval. Indefinite integration, on the other hand, yields a general form of the integral without specifying the bounds of integration. Several important formulas are essential tools in integration [3].

These formulas, such as the power rule, the constant rule, and the linearity property, enable people simplify complex integrals and solve problems efficiently. The power rule has demonstrated that the x^n's integral in regard with x is $\left(\frac{1}{(n+1)}\right) \cdot x^{(n+1)}$, where n can be arbitrary real number exclude -1. The constant rule allows people to bring a constant factor outside the integral sign [4]. According to the linearity property, when integrating the sum or difference of two functions, the result is equal to the sum or difference of their individual integrals. Other widely used techniques in integration include substitution, integration by parts, and trigonometric substitutions. These techniques provide ways to simplify integrals by transforming them into more manageable forms or expressing them in terms of known functions [5].

Improper integrals refer to integrals where either one or both integration limits are infinite, and the function being integrated has an infinite discontinuity within the interval of integration [6]. Improper Integrals with Infinite Limits involve one or both limits of integration being infinite. Which mainly needs to consider convergence and divergence [7]. Convergence, in the context of improper integrals,

is used to describe an integral that is considered to converge if the limit of the integral exists and has a finite value. In other words, the area under the curve approaches a finite value as the upper limit tends to infinity. If the limit of the integral does not exist or is infinite, an improper integral is classified as divergent [8]. In this case, the area under the curve grows without bound as the upper limit tends to infinity. By contrast, $\int [a, \infty] f(x)dx$ $\int [a, \infty] f(x)dx$ converges, then

$$\int [a, \infty] f(x)dx = \lim_{b\to\infty}\left[\int [a, b] f(x)dx\right] \tag{1}$$

If $\int [a, \infty] f(x)dx$ diverges, then $\lim_{b\to\infty}\left[\int [a, b] f(x)dx\right] = \pm\infty$.

2. Different Methods to Calculate Integration

2.1. Definite Integration

Definite integration, a fundamental concept in calculus, finds extensive application in mathematics, physics, engineering, and various other disciplines. To compute definite integration, which entails determining the area beneath a curve or the overall accumulated change of a function within a specific interval, the subsequent steps can be followed [9].

In the process of carrying out definite integration, it begins by dividing the integration interval $[a, b]$ into smaller subintervals. A commonly used method involves dividing the interval into n equal subintervals, each with a width of $\Delta x = \frac{(b-a)}{n}$. The value of n determines the level of accuracy in the approximation. Within each subinterval, sample points need to be chosen at which the integrand is evaluated. The left endpoint, right endpoint, or midpoints of the subintervals are commonly chosen. These points are denoted as xi, where i ranges from 0 to n. For the left endpoint, $xi = a + i\Delta x$ for the right endpoint, $xi = a + (1 + i)\Delta x$; and for midpoints, $xi = a + (0.5 + i)\Delta x$. The definite integral is then approximated by summing the areas of rectangles or trapezoids formed below the curve.

Based on the selection of sample points, there are mainly three different approaches can be used. The left Riemann sum approximates the integral by evaluating the function at the left endpoints of the subintervals. The approximation is shown as:

$$\int_a^b f(x)\,dx \approx \Delta x * \left(f(x^0) + f(x^1) + \cdots + f(x_{\mathrm{n}}{}^{-1})\right). \tag{2}$$

The right Riemann sum estimates the integral by utilizing the function values at the right endpoints of the subintervals. The approximation is expressed as follows:

$$\int_a^b f(x)\,dx \approx \Delta x * \left(f(x^1) + f(x^2) + \cdots + f(x_{\mathrm{n}})\right) \tag{3}$$

The midpoint rule approximates the integral using the function values at the midpoints of the subintervals. The approximation shown as:

$$\int_a^b f(x)\,dx \approx \Delta x * \left(f(x^0 + \Delta x/2) + f(x^1 + \Delta x/2) + \cdots + f(x_{\mathrm{n}}{}^{-1} + \Delta x/2)\right). \tag{4}$$

As the number of subintervals, denoted as n, increases infinitely (in the limit as n approaches infinity), the approximation gradually converges to the precise value of the integral. This concept is referred to as a Riemann sum, and the definite integral is derived by taking the limit of the Riemann sum as the partition becomes progressively finer. The standard notation for the definite integral is: $\int_a^b f(x)\,dx$. Once the definite integral is determined, you can evaluate it using various methods like direct computation, substitution, or integration by parts.

2.2. Indefinite Integration

Antiderivative, also called indefinite integration, is a fundamental principle involved in calculus, which entails determining a function that has a derivative equivalent to the provided function. The action of discovering the antiderivative is recognized as integration. The function $f(x)$ has an indefinite integral regard as $\int f(x)\,dx$, where $f(x)$ denotes the function being integrated, and dx signifies the variable of integration. The outcome of the indefinite integral is a collection of functions, each of which possesses the same derivative as the original function $f(x)$, but with an added constant of integration [10].

To find the antiderivative of a function, the following techniques can be employed. There are some standard rules for integrating common functions. For instance, the power rule states that if $f(x) = x^n$ where n can be arbitrary real number except -1, then the antiderivative of $f(x)$ is $F(x) = \left(\frac{1}{n+1}\right) \cdot x^{n+1} + c$. The integration of a sum or difference of functions is equivalent to the sum or difference of their respective integrals. This property allows you to split the integral of a complex function into simpler parts. In some cases, a substitution can be made to simplify the integrand. This involves substituting a new variable or expression to transform the integral into a simpler form. This method relies on the product rule for differentiation, enabling the rewriting of the integral of a product of two functions as the addition or subtraction of two distinct integrals.

Trigonometric functions often require specific techniques for integration. Trigonometric identities, such as those involving sine, cosine, tangent, or secant, can be used to simplify the integral. When dealing with rational functions (quotients of polynomials), the method of partial fractions can be used to break them down into simpler fractions. This technique is useful for integrating complex rational functions.

3. Example with Solutions

3.1. Example I

Compute the integral

$$I = \int_0^\infty \frac{\ln x \ln(1+x)}{1+x^2} dx, \tag{5}$$

Where the integral interval is from 0 to infinity. By letting $x = \frac{1}{t}$, it can get

$$I = \int_0^\infty \frac{\ln\left(\frac{1}{t}\right)\ln\left(1+\frac{1}{t}\right)}{1+t^2} dt = \int_0^\infty \frac{\ln^2(t)}{1+t^2} - \int_0^\infty \frac{\ln t \ln(1+t)}{1+t^2} dt \tag{6}$$

Therefore, the equation can be simplified as

$$I = \frac{1}{2}\int_0^\infty \frac{\ln^2 x}{1+x^2} dx = \frac{1}{2}\int_0^{\frac{\pi}{2}} \ln^2 \tan x\, dx \tag{7}$$

By using the identity that $\tan x = \frac{\sin x}{\cos x}$, the integral can be written as

$$I = \frac{1}{2}\int_0^{\frac{\pi}{2}} (\ln \sin x - \ln \cos x)^2\, dx \tag{8}$$

By expanding the square, then it can get

$$I = \frac{1}{2}\int_0^{\frac{\pi}{2}} (\ln^2 \sin x + \ln^2 \cos x - 2 \ln \sin x \ln \cos x) dx, \tag{9}$$

Which is equivalent to the following expression

$$I = \frac{1}{2}\int_0^{\frac{\pi}{2}}(2\ln^2\sin x - 2\ln\sin x\ln\cos x)dx. \tag{10}$$

Here, the integrand $f(x)$ can be cast into the series form, namely,

$$f(x) = \left(-\ln 2 - \sum_{n=1}^{\infty}\frac{\cos 2nx}{n}\right)^2 - \left(-\ln 2 - \sum_{n=1}^{\infty}\frac{\cos 2nx}{n}\right)\left(-\ln 2 - \sum_{n=1}^{\infty}\frac{(-1)^n\cos 2nx}{n}\right) \tag{11}$$

After canceling out the same part, then

$$I = \int_0^{\frac{\pi}{2}}\sum_{n=1}^{\infty}\left(\frac{\cos^2 2nx}{n^2} + (-1)^{n-1}\frac{\cos^2 2nx}{n^2}\right)dx \tag{12}$$

Expand the square of $\cos^2 2nx$ by using double angle formula

$$I = \int_0^{\frac{\pi}{2}}\sum_{n=1}^{\infty}\left(\frac{1+\cos 4nx}{2n^2} + (-1)^{n-1}\frac{1+\cos 4nx}{2n^2}\right)dx \tag{13}$$

Substitute the value of x into the equation, it is calculated that

$$I = \frac{\pi}{4}\left(\sum_{n=1}^{\infty}\frac{1}{n^2} + \sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^2}\right) = \frac{\pi}{4}\left(\frac{\pi^2}{6} + \frac{\pi^2}{12}\right) = \frac{\pi^3}{16} \tag{14}$$

3.2. Example II

Compute the integral:

$$I = \int_0^{\infty}\frac{1}{x^4-x^2+1}\ln^2\frac{x^2}{x^4-x^2+1}dx \tag{15}$$

By letting $x = \frac{1}{x}$, it can get

$$I = \int_0^{\infty}\frac{x^2}{x^4-x^2+1}\ln^2\frac{x^2}{x^4-x^2+1}dx \tag{16}$$

Consequently, by taking constant value $\frac{1}{2}$ out of the integral, the equation appeared as

$$I = \frac{1}{2}\int_0^{\infty}\frac{x^2+1}{x^4-x^2+1}\ln^2\frac{x^2}{x^4-x^2+1}dx \tag{17}$$

Substitute $x = \frac{1}{x}$ into the equation, and times x^2 to both numerator and denominator, the integral appeared as

$$I = \frac{1}{2}\int_0^{\infty}\frac{1+\frac{1}{x^2}}{x^2+\frac{1}{x^2}-1}\ln^2\left(x^2+\frac{1}{x^2}-1\right)dx \tag{18}$$

Then by letting $t = x - \frac{1}{x}$, the range of the integral change to positive infinity to negative infinity. The function be simplified as:

$$I = \frac{1}{2}\int_{-\infty}^{+\infty}\frac{\ln^2(t^2+1)}{t^2+1}dt \tag{19}$$

After some simple calculation, the equation changes to $I = \int_0^{\frac{\pi}{2}}\ln^2\cos^2 u\,du$. By using the trigonometry identity (the value of $\cos u$ is equal to $\sin u$ in the first quadrant), then it is found that

$$I = 4\int_0^{\frac{\pi}{2}}\ln^2\sin u\,du = \frac{\pi^3}{6} + 2\pi\ln^2 2. \tag{20}$$

4. Conclusion

In summary, there exist several methods such as substitution, integration by parts, trigonometric substitution, exchanging integral order, and tangent half angle substitution that can simplify the computation of definite integrals. Each method has its own specific application depending on the equation being solved. While the final answer for each equation may involve π, the approaches needed to solve these functions differ. Example I illustrate the utilization of all the mentioned formulas, involving extensive work and relying on experience to find the solution. On the other hand, Example II employs some of the formulas in a simpler and more direct manner, resulting in fewer steps. This brings up the inquiry of whether comparable procedures can be employed to solve related definite integrals. It is important to note that there is no perfect formula for these types of equations. However, by systematically solving them like typical problems, one can eventually obtain simple and direct answers. In conclusion, calculus empowers people to explore the fabric of the universe, unravel its intricacies, and unlock the secrets hidden within. It is an indispensable tool for those seeking to understand the world and push the boundaries of knowledge. As people continue to expand understanding and applications of calculus, its impact on science, technology, and society will undoubtedly continue to grow, paving the way for future discoveries and innovations.

References

[1] Chen L., Li X., Numerical calculation of regular and singular integrals in boundary integral equations using Clenshaw–Curtis Quadrature rules. Engineering Analysis with Boundary Elements, 2023, 155, 25–37.

[2] Franca W., Menegatto V. A. Positive definite functions on products of metric spaces by integral transforms. Journal of Mathematical Analysis and Applications, 2022, 514 (1), 126304.

[3] Ergene Ö., Özdemir A. Ş. Understanding the definite integral with the help of Riemann sums. Participatory Educational Research, 2022, 9 (3), 445 – 465.

[4] Jones S. R., Areas, anti-derivatives, and adding up pieces: Definite integrals in pure mathematics and Applied Science Contexts. The Journal of Mathematical Behavior, 2015, 38, 9 – 28.

[5] Stornaiuolo A. & Garg R., Cosmopolitan literacies, and the limits of understanding: Possible directions in global literacy research. International Encyclopedia of Education (Fourth Edition), 2023.

[6] Zheng H.-S., Ming W.-Y., Yuan D.-M., The construction of several new inequalities for definite integral. Science Asia, 2021, 47 (5), 651.

[7] Tikhomirov V. M., Axiomatic definition of the integral. Selected Works of A. N. Kolmogorov, 1991, 13–14.

[8] Mahmudov E., The definite integral. Single Variable Differential and Integral Calculus, 2013, 259 – 334.

[9] Pattaraintakorn P., Peters J. F., Ramanna S., Capacity-based definite rough integral, and its application. Man-Machine Interactions, 2009, 299 – 308.

[10] Serhan D., Students' understanding of the definite integral concept. International Journal of Research in Education and Science, 2014, 1 (1), 84.