Application of Cauchy's Residue Theorem in Several Improper Integrals

Hongyi Zhang *

Wuhan Britain-China School, Wuhan, Hubei, China

* Corresponding author: zunjunwan@hmjy12.wecom.work

Abstract. Calculating definite integrals in complex functions requires the Cauchy's residue theorem, which is a key concept in the complex variables. It is based on several ideas, including the isolated singular points theory, the Laurent theorem, and the Cauchy integral theorem. Nevertheless, it is highly challenging to calculate closed curves using the conventional integrals method since numerous formulas must be substituted. The singular points must all be isolated in the case of a holomorphic function f that belongs to an enclosing contour C and C is positively oriented, at which time the contour integral C can be used to resolve the issue. Additionally, the problem can be solved by using visual aids to help people understand them. By doing so, the notion of solving the problem will be made clearer and some calculating steps will be cut short. This study exhibits the residue theorem's potent efficiency in computing improper integrals and other difficult problems.

Keywords: Cauchy's residue theorem, Laurent series, Isolated singular points, Improper integral.

1. Introduction

The study of analytical functions and the solution of complex integrals are central to complex analysis in a whole. There are a variety of techniques for resolving complex integrals along closed curves, including the residue theorem, Laurent series, Cauchy integral formula, parametric form of definite integrals, and the logarithmic residue approach [1]. This study contrasts the benefits and drawbacks of various approaches by solving a typical situation. It concludes that the most adaptable and widely applicable method is the Cauchy residue theorem.

The residue theorem is a crucial and extensively used theorem in the mathematical physics approach taught in the prerequisite course for physics majors at universities. Using the topological phase transition problem of the one-dimensional special model as an illustration, it is shown how the residue theorem may be used to obtain the equation of the second-order displacement and how this physical quantity characterizes the topological phase transition [2]. Additionally, as a complex function approaches infinity, its integral value is equal to that of another complex function. The infinite summation of all the roots of transcendental equations can be calculated analytically using the residue theorem [3] and the governed residue theorem [4] further characterizes the residual field

The Cauchy residue theorem is briefly introduced to set the stage for this research. The discussion of various important ideas that follow includes the Laurent series and isolated solitary points. These ideas are used to solve real-world problems, and the study finishes by outlining its goals and potential directions.

2. Methods

2.1. Cauchy's Residue Theorem

The paper precisely states that the value of f 's integral in the neighbor of C is equal to the summation of f 's residues at singularities within C if and only if C is positively oriented [5]. Assume that C is a simple, positively specified closed contour (see Fig. 1). If the function f is analytic both within and on C and lacks any singular points inside of C ($k = 1, 2, \ldots, n$).

$$\int_c f(z)dz = 2\pi i \sum_{k=1}^{n} \operatorname*{Res}_{z=z_k} f(z) \tag{1}$$

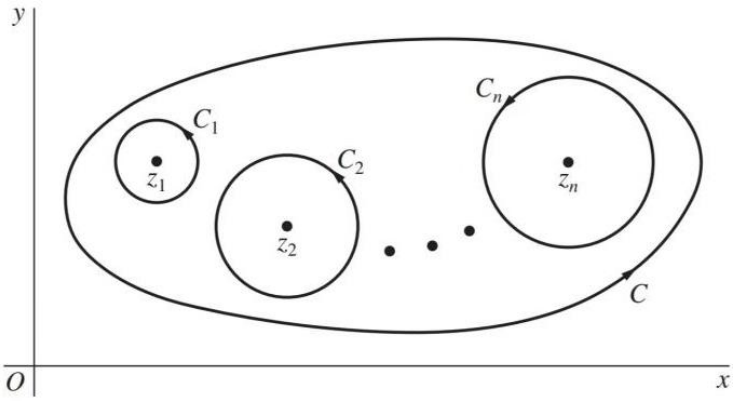


Figure 1. Illustration of the integration contour

Then, in order to demonstrate this theorem, it is supposed that all the points are in the positive-inclination circles that are situated inside C and are sufficiently tiny to guarantee that no overlapping data exist. These circles define the edge of a closed region when f is analytical, together with the simple closed contour C. This region's interior is made up of points inside C and outside of each circle and is a multiply connected domain. As a result, it follows that using the revised Cauchy-Goursat theorem that applies to these domains [6]

$$\int_c f(z)dz - \sum_{k=1}^{n} \int_{c_k} f(z)dz = 0. \tag{2}$$

In other words, it is said that

$$\int_{c_k} f(z)\, dz = 2\pi i \underset{z=z_k}{Res}\, f(z) \tag{3}$$

Where $k = 1, 2, \ldots, n$.

2.2. Laurent Series

The Laurent's theorem states that if C is surrounded and located in an annular region $R_1 < |z - z_0| < R_2$ that is centered at z_0 and f is regular (see Fig. 2), then the series representation of f (z) is given by [7]

$$f(z) = \sum_{n=-\infty}^{\infty} C_n (z - z_0)^n \tag{4}$$

Where $R_1 < |z - z_0| < R_2$, $n = 0, \pm 1, \pm 2, \ldots$

$$C_n = \frac{1}{2\pi i} \int_C \frac{f(z)dz}{(z-z_0)^{n+1}} \tag{5}$$

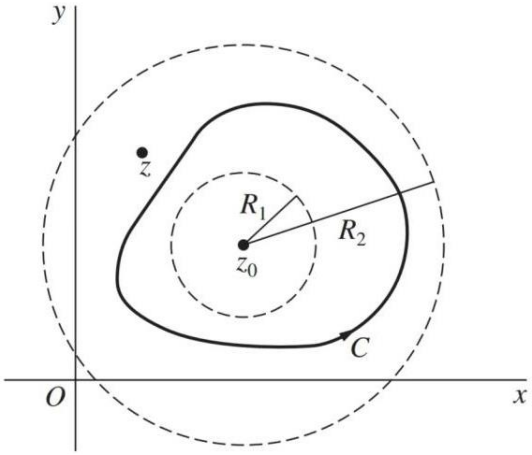


Figure 2. the integration contour of integral in 2.2

2.3. Three Types of Isolated Singular Points

The common Laurent series representation is [8]

$$f(z) = \sum_{n=0}^{\infty} a_n(z - z_0)^n + \frac{b_1}{z-z_0} + \frac{b_2}{(z-z_0)^2} + \cdots + \frac{b_n}{(z-z_0)^n}, \quad (6)$$

Which is defined in a hole in the disk $0 < |z - z_0| < R_2$. The primary part denotes the section of the series dealing with negative powers of $z - z_0$ is

$$\frac{b_1}{z-z_0} + \frac{b_2}{(z-z_0)^2} + \cdots + \frac{b_n}{(z-z_0)^n}, \quad (7)$$

And singular points that can be removed are the first kind. On the other hand,

$$f(z) = \sum_{n=0}^{\infty} a_n(z - z_0)^n = a_0 + a_1(z - z_0) + a_2(z - z_0)^2 + \cdots \quad (8)$$

Where $0 < |z - z_0| < R_2$ and $b_n = 0$, is referred to as a detachable singular point here. Keep in mind that a detachable singular point's residue is always zero. Equation (8) is true for the whole region if f is defined as $f(z_0) = a_0$. Essential singular points comprise the second category. This point of f is said to exist if many b_n in the primary part are finite. Poles of order make up the third category. If the Eq. (6) looks like

$$f(z) = \sum_{n=0}^{\infty} a_n(z - z_0)^n + \frac{b_1}{z-z_0} + \frac{b_2}{(z-z_0)^2} + \ldots + \frac{b_m}{(z-z_0)^m} \quad (9)$$

Where $0 < |z - z_0| < R_2$ and and $b_m \neq 0$. The isolated unique point in this instance is referred to as a pole of order m. A simple pole is typically for the case where $m = 1$.

3. Applications of Residue Theorem in Definite Integrals

3.1. Computing $\int_0^{2\pi} R(\cos\theta, \sin\theta)d\theta$ type Integral

Let $R(\cos\theta, \sin\theta)$ be a rational function with real variables and continuous on $[0, 2\pi]$. Let $z = e^{i\theta}(0 < \theta < 2\pi)$, according to Euler's formula $e^{i\theta} = \cos\theta + i\sin\theta$ to get $\cos\theta = \frac{z+z^{-1}}{2}$, $\sin\theta = \frac{z-z^{-1}}{2i}$, $d\theta = \frac{dz}{iz}$. Then when z goes around the path $|z|=1$ once in the anticlockwise direction, so that equation become the following form [9]

$$\int_0^{2\pi} R(\cos\theta, \sin\theta)\, d\theta = \oint_{|z|=1} R\left(\frac{z+z^{-1}}{2}, \frac{z-z^{-1}}{2i}\right)\frac{dz}{iz} \quad (10)$$

Then the integral can be calculated by applying Cauchy residue theorem,

$$\int_0^{2\pi} R(\cos\theta, \sin\theta)\, d\theta = 2\pi i \sum_{k=1}^{n} Res f(z) \quad (11)$$

Next, this part will elaborate the method for solving these types of Integral with some examples. The first example is

$$I = \int_0^{2\pi} \frac{\cos 2\theta}{1 - 2p\cos\theta + p^2}\, d\theta \quad (12)$$

Where $0 < |p| < 1$. Because $0 < |\mathrm{p}| < 1$, $1 - 2\mathrm{p}\cos\theta + p^2 = (1-p)^2 + 2p(1-\cos\theta) \neq 0$, so the Integral is existed. Then, due to $\cos 2\theta = \frac{1}{2}\left(e^{i2\theta} + e^{-i2\theta}\right) = \frac{1}{2}(z^2 + z^{-2})$, the Integral become $\mathrm{I} = \oint_{|z|=1} \frac{\left(z^2+z^{-2}\right)}{2} \cdot \frac{1}{1-2\mathrm{p}\cdot\frac{z+z^{-1}}{2}+\mathrm{p}^2} \cdot \frac{\mathrm{dz}}{iz}$, then multiply the numerator and denominator by z^2 and simplify the whole formula, get $\mathrm{I} = -\frac{1}{2pi}\oint_{|z|=1} \frac{1+z^4}{z^2\left(z-\frac{1}{p}\right)(z-p)}dz = -\frac{1}{2pi}\oint_{|z|=1} f(z)dz$. The singularity of Integral are the poles at the points $z_1 = 0$ of the order two and $z_2 = \frac{1}{p}$ and $z_3 = \mathrm{p}$, two simple poles. However, the point z_2 is not lying in the normalized circle $|z| = 1$.

Thus, it only has the requirement to compute the residues at z_1 and z_3. Due to residue theorem, $\operatorname{Res}_{\mathrm{z}=0} f(\mathrm{z}) = \left[\frac{1+z^4}{\left(z-\frac{1}{p}\right)(z-p)}\right]\Big|_{z=0} = \frac{1+p^2}{p}$ and $\operatorname{Res}_{\mathrm{z}=\mathrm{p}} f(\mathrm{z}) = \left[\frac{1+z^4}{z^2\left(z-\frac{1}{p}\right)}\right]\Big|_{z=p} = -\frac{1+p^4}{p(1-p^2)}$, thus

$$I = \int_0^{2\pi} \frac{cos\,2\theta}{1-2p\,cos\,\theta+p^2}d\theta = -\frac{1}{2pi}2\pi i\left[\frac{1+p^2}{p} - \frac{1+p^2}{p(1-p^2)}\right] = \frac{2\pi p^2}{1-p^2} \tag{13}$$

The second series of example is

$$I_1(n) = \int_0^{\pi} e^{\cos x}\cos(\sin x - nx)dx \tag{14}$$

Where n $\in$ R, and

$$I_2 = \int_0^{\pi} e^{cos\,x}\,sin(sin\,x)\,cot\frac{x}{2}dx \tag{15}$$

Let f (x) = $e^{\cos x}\cos(\sin x - nx)$, due to $f\,(2\pi - x) = f(x)$, so $f\,(x)$ is symmetric about $x = \pi$. Therefore, simplify the equation to get that

$$I(n) = \frac{1}{2}Re\int_0^{2\pi} e^{e^{ix}}\cdot e^{-inx}dx \tag{16}$$

Then, let $e^{ix} = z, \mathrm{dx} = \frac{dz}{iz}$, so $\int_0^{\pi} e^{\cos x}\cos(\sin x - nx)dx = \frac{1}{2}Re\left(\frac{2\pi}{n!}\lim_{z\to 0}\frac{d^n}{dz^n}e^z\right) = \frac{\pi}{n!}$. In addition, let $\mathrm{h(x)} = e^{\cos x}\sin(\sin x)\cot\frac{x}{2}$, according to $h(2\pi - x) = h(x)$, so $h(x)$ is symmetric about $x = \pi$. Therefore,

$$I = \frac{1}{2}\int_0^{2\pi} e^{cos\,x}\,sin(sin\,x)\frac{1+cos\,x}{sin\,x}dx = \frac{1}{2}Im\int_0^{2\pi} e^{cos\,x+i\,sin\,x}\cdot\frac{1+cos\,x}{sin\,x}dx \tag{17}$$

Let $e^{ix} = z$, so $\cos\theta = \frac{z+z^{-1}}{2}$, $\sin\theta = \frac{z-z^{-1}}{2i}$, $\mathrm{dx} = \frac{dz}{iz}$, and thus

$$I = \frac{1}{2}Im\int_{C_1} e^z\frac{1+\frac{1}{2}\left(z+z^{-1}\right)}{\frac{1}{2i}(z-z^{-1})\cdot iz}dz = \frac{1}{2}Im\int_{C_1} e^z\frac{z+1}{z(z-1)}dz \tag{18}$$

Due to the outliers have singularities on the circumference of the circle, as considered by digging out the integral path of the singularity (see Fig. 3). Because of residue theorem, $\int_{C_1} e^z\frac{z+1}{z(z-1)}dz + \int_{C_1} e^z\frac{z+1}{z(z-1)}dz = 2\pi i\,Res\left[e^z\frac{z+1}{z(z-1)};0\right]$. Because of small circle lemma, $\lim_{\delta\to 0}\int_{C_\delta} e^z\frac{z+1}{z(z-1)}dz = -\pi i\,Res\left[e^z\frac{z+1}{z(z-1)};1\right] = -2e\pi i$ and $\mathrm{Res}\left[e^z\frac{z+1}{z(z-1)};0\right] = -1$. so, $\int_{C_1} e^z\frac{z+1}{z(z-1)}dz = 2(e-1)\pi i$, and thus

$$\int_0^{\pi} e^{cos\,x}\,sin(sin\,x)\,cot\frac{x}{2}dx = \frac{1}{2}Im[2(e-1)\pi i] = (e-1)\pi. \tag{19}$$

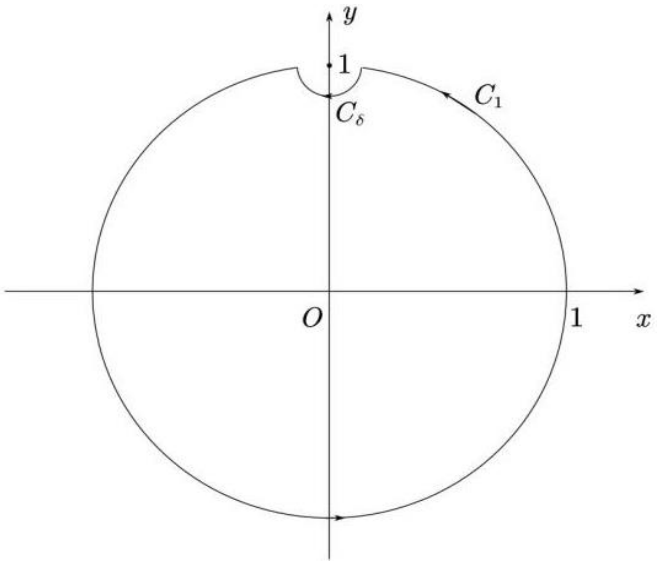


Figure 3. the integration contour of integral in example 2

3.2. Computing $\int_{-\infty}^{+\infty} f(x)$ type Integral

Theorem 1. let f is holomorphic except for $z_1,\ z_2,...,z_n$ on the upper half plane $\{z: Imz > 0\}$ (see Fig. 4), let f is continuous except for $z_1,\ z_2,...,z_n$ on the $\{z: Im\ z > 0\}$, if $\lim_{z\to\infty} zf(z) = 0$, then it can get that

$$\int_{-\infty}^{+\infty} f(x)dx = 2\pi i \sum_{k=1}^{n} Res[f; z_k] \tag{20}$$

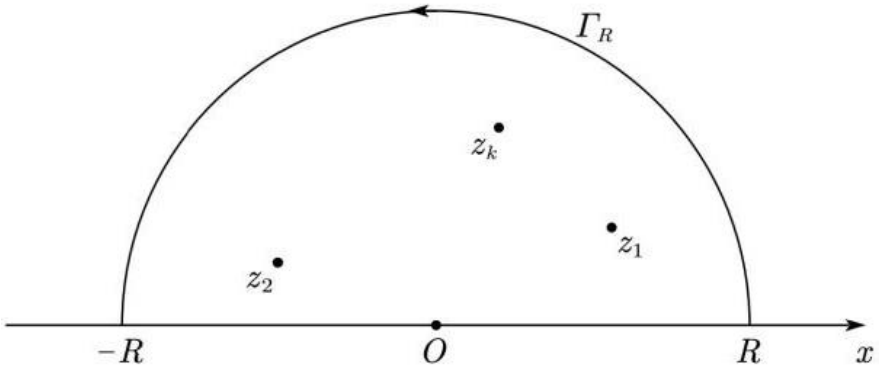


Figure 4. The general integration contour of integral

The third example is

$$\int_{-\infty}^{+\infty} \frac{sin(x+a)\, sin(x-a)}{x^2-a^2} dx, a > 0 \tag{21}$$

Let $\mathrm{I} = \frac{1}{2}\int_0^{+\infty} \frac{\cos 2a - \cos 2x}{x^2-a^2} dx = \frac{1}{4}\int_{-\infty}^{+\infty} \frac{\cos 2a - \cos 2x}{x^2-a^2} dx$ and let $f(\mathrm{z}) = \frac{\cos 2a - e^{2iz}}{x^2-a^2}$, they take the following integration path, see Fig. 5).

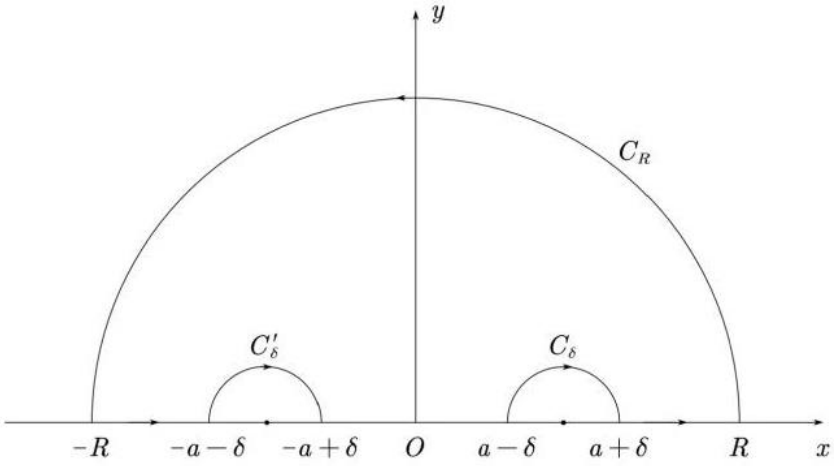


Figure 5. The integration contour of integral in example 3

Due to $f(z)$ have singular points -a and a on the real axis, therefore use small circle lemma to dig out these two singular points [10]. So $\lim_{\delta\to 0}\int_{c_\delta} f(z)dz = -\pi i \lim_{z\to a}(z-a)f(z) = \frac{\pi(\cos 2a - e^{2ai})}{2ai}$, and $\lim_{\delta\to 0}\int_{c_{\delta 1}} f(z)dz = -\pi i \lim_{z\to -a}(z+a)f(z) = -\frac{\pi(\cos 2a - e^{-2ai})}{2ai}$. Then according to $\lim_{R\to\infty}\int_{C_R} f(z)dz = 0$ and residue theorem, $\int_{-R}^{-a-\delta} f(x)dx + \int_{C_{\delta 1}} f(z)dz + \int_{-a+\delta}^{a-\delta} f(x)dx + \int_{C_\delta} f(z)dz + \int_{a+\delta}^{R} f(x)dx + \int_{C_R} f(z)dz = 0$, let $\delta \to 0$, R$\to\infty$ get V. P. $\int_{-\infty}^{+\infty} \frac{\cos 2a - e^{2ix}}{x^2-a^2}dx = \frac{\pi}{a}\sin 2a$, therefore,

$$\int_{-\infty}^{+\infty} \frac{sin(x+a)\,sin(x-a)}{x^2-a^2}dx = \frac{\pi}{4a}sin\,2a \tag{22}$$

3.3. Computing $\int_0^{+\infty} f(x)dx$ type Integral

The big circle lemma states that let f be continuous on the big circle $C_R = \{z; z = a + Re^{i\theta}, \theta_0 \le \theta \le \theta_0 + a\}$. If $\lim_{x\to+\infty}(z-a)f(z) = A$, then it is found that

$$\lim_{R\to+\infty}\int_{C_R} f(z)dz = iAa \tag{23}$$

Where, the direction of $C_R = \{z; z = a + Re^{i\theta}, \theta_0 \le \theta \le \theta_0 + a\}$ is along with the direction of the increasing value of argument. If the single-valued function is holomorphic except for z_1, z_2,...,z_n on the complex plane and these singular points all not on the positive real axis including the origin, z = ∞ is the zero of m order ($m \ge 1$) of $f(z)$, then

$$\int_0^{+\infty} x^p f(x)dx = \frac{2\pi i}{1-e^{2p\pi i}} Res[z^p f(z); z_k] \tag{24}$$

Where $-1 < p < 0$.

The fourth example is

$$\int_0^{+\infty} \frac{x^b}{1+x^2} cos\left(ax - \frac{b}{2}\pi\right)dx, a \ge 0, -1 < b < 1 \tag{25}$$

Let $f(z) = \frac{e^{iaz}z^b}{1+z^2}$, set $0 \le$ argz $\le 2\pi$, then Res $[f(z); i] = \lim_{z\to e^{\frac{\pi i}{2}}} \frac{e^{iaz}z^b}{z+i} = \frac{e^{-a}}{2i}e^{\frac{b\pi i}{2}}$, and that $Res[f(z); -i] = \lim_{z\to e^{\frac{3\pi i}{2}}} \frac{e^{iaz}z^b}{z-i} = \frac{e^a}{2i}e^{\frac{3b\pi i}{2}}$, according to the integration path shown in Fig. 6.

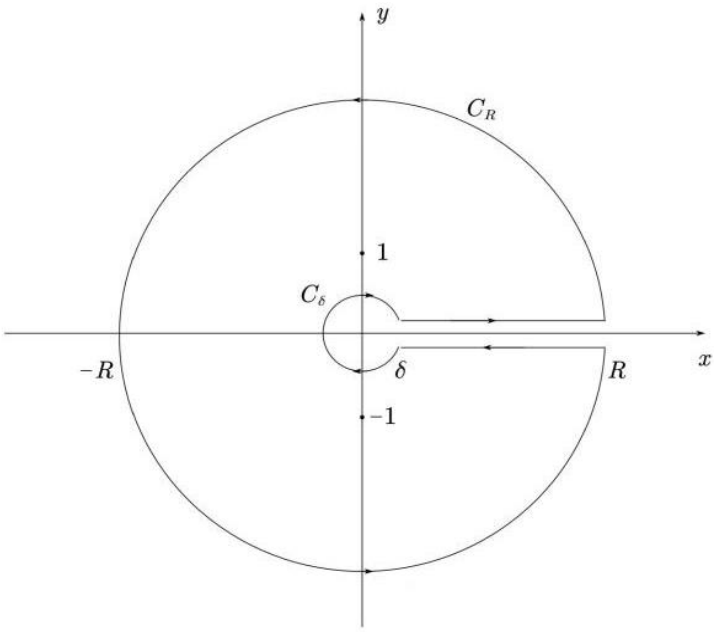


Figure 6. The integration contour of integral in example 4.

Moreover, since $\int_0^{+\infty} \frac{e^{iax}x^b}{1+x^2}dx = \frac{2\pi i}{1-e^{2b\pi i}}\{Res[f(z);i] + Res[f(z);-i]\} = -\frac{\pi}{2}\left[-\frac{\cosh(a)}{\cos\frac{b\pi}{2}}i\frac{\sinh(a)}{\sin\frac{b\pi}{2}}\right]$, it is direct to get $\int_0^{+\infty} \frac{x^b\cos ax}{1+x^2}dx = \frac{\pi\cosh(a)}{2\cos\frac{b\pi}{2}}$, $\int_0^{+\infty} \frac{x^b\sin ax}{1+x^2}dx = -\frac{\pi\sinh(a)}{2\sin\frac{b\pi}{2}}$, then take these expressions to original equation, it can get

$$\int_0^{+\infty} \frac{x^b}{1+x^2}cos\left(ax - \frac{b}{2}\pi\right)dx = \frac{\pi}{2}[cosh(a) - sinh(a)] = \frac{\pi}{2}e^{-a} \tag{26}$$

4. Conclusion

The Cauchy residue theorem and its practical application are briefly discussed in this paper. A generic formula for a real infinite integral may be obtained by using Cauchy-Goursat theorem, Cauchy's integral theorem, and the Laurent series. In some straightforward situations, all that is required is to determine the coefficient of the z^{-1} and compute the residue of these locations in accordance with the singular point. This simplifies the calculation procedure because the value of f is then equal to the sum of residues at singularities. For functions of complex variables, the residue theorem is very important and it is convenient to use it to calculate the integral of a closed curve and real infinite integral and it is directly related to some other theorems, such as Jordan's lemma. In addition, the residue theorem has important significance in various fields of research, such as solving complex waveform and radio wave problems through the residue theorem and solving complex formulas in physics. According to Cauchy's residue theorem which is a very important concept in the complex variable, it is possible to explore some more interesting contents in this region, such as Cauchy-Riemann equations. It uses partial derivative to solve the component function and two dimensions are urgent to think about, including vertical and horizontal approaches. This concept is also useful in complex variable, and thus many new and complicate concepts can be achieved.

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