Application and Extension of Residue Theorem

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Abstract. The Complex analysis is a branch of mathematics studying complex variable functions, mainly studying the properties and characteristics of complex variable functions. This paper first introduces the background and development of the residue theorem. Then this paper briefly explains the definition of the residue theorem. It also introduces the applications of residue theorem in different field of physics. The residue theorem is widely used in the integration of complex variable functions, and it is also applicable to the integration of real functions. Several examples of integral solved by the residue theorem in real variable functions are used to further understand the residue theorem and explore the application of the logarithmic residue theorem. The logarithmic residue is an important and effective method to discuss the number of poles and zeros of analytic functions. By proving and applying various inferences, the zeros and poles can be calculated efficiently and quickly, which greatly saves time for the study and applications of complex functions.

Keywords: Cauchy Residue Theorem, logarithm, integral.

1. Introduction

Complex number is the basic concept in complex analysis. It has solved many problems that cannot be described by real variable functions. Considering the early development of complex analysis theory, this concept is of great significance to the classification of isolated singularities and the relations among them. At the same time, it promotes the method of solving the value of the definite integral to a new stage. Through the selection of functions, the selection of integral routes, it solves many cases in which the original function of the integrable function could not be solved, which laid the foundation for the development of the integral theory [1]. In analogy with the case of calculating real integrals, the problems related to complex integrals are solved, and the definition of residues is given [2]. Subsequently, Cauchy further developed and refined the concept of residues, forming the following definition [3]. If the function $g(z)$ is holomorphic, in $D(a,t)\backslash\{a\}$and $t>0$, a is the isolated singularity of $g(z)$, then the definition of $g(z)$ in the residue of a is

$$\frac{1}{2\pi}\int_{|z-a|=p} g(z)dz = \mathrm{Res}(g,a),\ 0<p<t. \tag{1}$$

The definition given by Cauchy is still used today, and has been extended to differential equations [4], series theory, and other disciplines. In 2000, Beck derived an equation for the quantity of lattice points in a dilated n-dimensional tetrahedron using the residue theorem. The origin and lattice points on each coordinate axis are where the vertices are located [5]. In quantum field theory, scattering amplitudes often involve complex integrals which can be estimated using the residue theorem. By encircling the poles corresponding to intermediate particle states in the complex plane, one can compute the residues and extract the contributions from these intermediate states. Feynman diagrams involve the propagation of virtual particles, which can result in singularities in the complex plane [6]. In quantum mechanics, the path integral formulation provides a way to compute transition amplitudes and expectation values of observables. Path integrals involve integrals over all possible paths of a system in configuration space. The calculation of transition amplitudes and expectation values can be made easier by using the residue theorem to assess some contour integrals that arise in the route integral formulation. In statistical mechanics, partition functions and thermodynamic quantities such as free energy, entropy, and specific heat are often expressed as contour integrals. By applying the

residue theorem, these integrals can be evaluated, leading to expressions that relate the properties of the system to the singularities of the partition function [7].

In section 2, the definition of residue theorem is explained briefly, and the related proof is given. Secondly, the residue theorem is applied to two types of real variable function integrals, so that the applications of the residue theorem are not limited to complex variable function integrals. In section 3, the logarithmic residue theorem is extended and proved. The effective use for the lemma of the logarithmic residue theorem can quickly find the zeros and poles, which is very convenient for the study of complex functions and applications in different fields that need to use complex functions.

2. Cauchy Residue Theorem

An effective tool in complex analysis that enables the evaluation of certain contour integrals is the residue theorem. It relates the values of a function's residues, which are the complex residues at its singular points, to the value of the contour integral of the function around a closed curve.

Theorem 2.1 (Cauchy Residue Theorem) Assume $g(z)$ be a complex function which is analytic everywhere both inside and on a straightforward closed curve C, with the exception of a finite number of isolated singularities. If $g(z)$ has poles at those singularities, then the residue theorem tells the contour integral of $g(z)$ around C is equals to the total of the residues of $g(z)$ at its poles inside C:

$$\oint_c g(z)dz = 2\pi i \sum Res(g,a), \tag{2}$$

Where $\mathrm{Res}(g(z),a)$ represents the residue of $g(z)$ at the pole a. The sum is taken over all poles of $g(z)$ inside C. In other words, the theorem tells that the integral of a function around a closed curve is determined solely by the singularities of the function inside the curve, not by the details of the curve itself.

Proof: In order to illustrate the Residue theorem, The points should be the centers of the circles C_k that are positively orientated, interior to C, and that are so small that they don't share any points. The simple closed contour C and the circles C_k together define the boundary of a closed region when f is analytical. The points inside C and outside of each C_k make up the interior, a massively connected area. Consequently, based on the Cauchy-Goursat theorem,

$$\int_c g(z)dz - \sum_{k=1}^{n} \int_{C_k} g(z) = 0. \tag{3}$$

This is reduced to an equation because

$$\int_{Ck} g(z)dz = 2\pi i Res g(z), \tag{4}$$

And the proof is complete [8].

The Residue Theorem is widely used in physics, engineering, finance, biology and other fields. For example, the Residue Theorem is widely used in the calculation of transfer functions in circuits, the calculation of vorticity in liquid dynamics, and the calculation of volatility in financial markets. The generalized Residue Theorem can also be used to solve some complex integral problems, such as calculating the integral that is decomposed into multiple parts by fractions, or calculating the integral of a complex function on a curved path, etc.

The first sort of integral considered is a trigonometric integral in the range of 0 to 2π.

Example 2.1 ([9]) Compute the integral $\mathrm{I} = \int_0^{2\pi} \frac{cos\theta}{5-3cos\theta} d\theta$.

Proof: Firstly, let's consider the following integral

$$I = \int_{|z|=1} \frac{\frac{z+\frac{1}{z}}{2}}{5-3\left(\frac{z+\frac{1}{z}}{2}\right)} \frac{dz}{iz}. \tag{5}$$

By multiplying both the numerator and denominator by 2z, the formula can become

$$I = i\int_{|z|=1} \frac{z^2+1}{z(3z-1)(z-3)} dz. \tag{6}$$

Similarly, the value of the integral is equal to the residue at z=0 and 1/3. Hence, the final answer is

$$I=-2\pi\left(\frac{1}{3}-\frac{5}{12}\right)=\frac{\pi}{6}. \tag{7}$$

The second sort of integral is $\frac{P(x)}{Q(x)}$-type integral, a common integral is considered

$$\int_{-\infty}^{+\infty} \frac{x^2-x+2}{x^4+10x^2+9} dx. \tag{8}$$

It is easy to check that $f(z)=\frac{z^2-z+2}{z^4+10z^2+9}$ has two poles on the upper half, namely, z=i and z=3i. Since the residue is $\mathrm{Res}(f(z),i)=-\frac{1+i}{16}$ and $\mathrm{Res}(g(z),3i)=\frac{3-7i}{48}$, it follows that $I=\frac{5}{12}\pi$.

Example 2.2 Define $g(z)=\frac{az^3+bz^2+cz+d}{z^4-1}$ with a=2, b=4i+4, c=18, d=4-4i. Evaluate the integral $\int_\gamma g(z)dz$, when $\gamma(z)=i+\frac{e^{iz}}{2}, 0\le z\le 2\pi$, $\gamma(z)=\frac{i-1}{2}+\sqrt{2}e^{iz}, 0\le z\le 2\pi$, and $\gamma(z)=1+5e^{iz}, 0\le z\le 2\pi$.

Proof: The first step is to find the singularities and $g(z)=\frac{az^3+bz^2+cz+d}{(z-i)(z+1)(z-1)(z+i)}$, so the singularities are 1, -1, i and -i. Then g(z) can be writted as $g(z)=\frac{7}{z-1}+\frac{-2}{z-i}+\frac{3}{z+1}+\frac{-6}{z+i}$. It can be obviously seen that the residue at z=1, -1, i and -i are 7, 3, -2 and -6. After finding the residue at each value of z, it is easier to integral these three functions by sketching each graph in the complex plane. For $\gamma(z)=i+\frac{e^{iz}}{2}, 0\le z\le 2\pi$, it is a circle with the center (0, i) and radius 1. Only the singularity at z=i is in the circle. For $\gamma(z)=\frac{i-1}{2}+\sqrt{2}e^{iz}, 0\le z\le 2\pi$, it is a circle with -1 and i in it so the singularities at z=-1 and i are in it. As for the $\gamma(z)=\frac{i-1}{2}+\sqrt{2}e^{iz}, 0\le z\le 2\pi$, 1, -1, i and -i are all in the circle. Finally, the basic integral method is used to calculate these integrals. For $\gamma(z)=i+\frac{e^{iz}}{2}, 0\le z\le 2\pi$, $\oint_\gamma g(z)dz=2\pi i\times(-2)=-4\pi i$. For $\gamma(z)=\frac{i-1}{2}+\sqrt{2}e^{iz}$, $\oint_\gamma g(z)dz=2\pi i\times(3+(-2))=2\pi i$. As for the last one $\gamma(z)=\frac{i-1}{2}+\sqrt{2}e^{iz}, 0\le z\le 2\pi$, $\oint_\gamma g(z)dz=2\pi i\times(3+7+(-2)+(-6))=4\pi i$.

However, finding the residue does not have only one method. Another approach is to use Laurent series to expand the series and find the coefficient of negative first term. For example, the Laurent series of g(z) is equal to $g(z)=\frac{z_k}{4(z-z_k)}[1+b_1(z-z_k)+b_2(z-z_k)^2+b_3(z-z_k)^3\ldots][h(z_k)+h'(z_k)(z-z_k)+\ldots]$. The residue term of the function is $\frac{z_k\times g(z_k)}{4}\times\frac{1}{z-z_k}$. So $\mathrm{Res}_{z=-1}[g(z)]=\frac{-1}{4}(-2+4$ i+4-18+4-4i)=3; $\mathrm{Res}_{z=1}[g(z)]=\frac{1}{4}(2+4i+4+18+4-4i)=$

7; $Res_{z=i}[g(z)]=\frac{i}{4}(-2i-4i\text{-}4+18i+4\text{-}4i)=\text{-}2$; $Res_{z=-i}[g(z)]=\frac{-i}{4}(2i-4i\text{-}4\text{-}18i+4\text{-}4i)=\text{-}6$. The rest of the job that involves sketching and calculating the integrals are no different.

3. Logarithmic Residue Theorem

In residue theorem of complex functions, due to $\frac{g'(z)}{g(z)}=\frac{d}{dz}[Lng(z)]$, the residue in z=a in g(z) is called as logarithmic residue, which is written as $\mathrm{Res}\left[\frac{g'(z)}{g(z)},a\right]$. The logarithmic residue is an important and effective method to discuss the quantity of zeros and poles of analytic functions.

Corollary 3.1 Let Z=b be the n-th zero of g(z), and then z=b must be the first-order pole of the $\frac{g'(z)}{g(z)}$ and $\mathrm{Res}\left[\frac{g'(z)}{g(z)},b\right]=n$.

Proof: Let z=b be the nth zero of g(z), hence

$$g(z)=(z-b)^n g_1(z). \tag{9}$$

Where $g_1(z)$ is analysis in z=b and $f_1(b)\neq 0$.

$$\begin{aligned} g'(z)=[(z-b)^n g_1(z)]' &= (z-b)^n g_1(z)+n(z-b)^{n-1}g_1(z) \\ &= (z-b)^{n-1}[(z-b)g'(z)=ng_1(z) \\ &= (z-b)^{n-1}h(z). \end{aligned} \tag{10}$$

Where $h(z)=(z-b)g_1'(z)+ng_1(z)$ is analysis in z=b, and $h(b)=ng_1(z)\neq 0$, so z=b is the n-1th zero of g'(z). Hence,

$$\mathrm{Res}\left[\frac{g\prime(z)}{g(z)},b\right]=n\cdot\frac{[g\prime(z)]^{n-1}}{[fgz)]^n}=n. \tag{11}$$

Corollary 3.2 ([10]): Let z=b be the mth poles, then z=b must be the 1st pole of $\frac{h\prime(z)}{h(z)}$, and

$$\mathrm{Res}\left[\frac{h'(z)}{h(z)},a\right]=-m. \tag{12}$$

Proof: If z=b is h(z)'s mth pole, z=b is the mth zero of $\frac{1}{h(z)}$ and (m-1) th zero of $\frac{1}{h\prime(z)}$. Due to

$$\frac{\frac{1}{h\prime(z)}}{\frac{1}{h(z)}}=\frac{-\frac{1}{h^2(z)}\times h\prime(z)}{\frac{1}{h(z)}}=-\frac{h\prime(z)}{h(z)}. \tag{13}$$

From Corollary one, it is easy to get

$$\mathrm{Res}\left[\frac{h\prime(z)}{h(z)},b\right]=-\mathrm{Res}\left[\frac{\frac{1}{h\prime(z)}}{\frac{1}{h(z)}},b\right]=-m\frac{(\frac{1}{h\prime(z)})^{m-1}}{(\frac{1}{h(z)})^m}=-m. \tag{14}$$

4. Conclusion

This paper mainly studies the residue theorem. The residue theorem generalizes the Cauchy integral theorem and the Cauchy integral formula. The core principle of the residue theorem is the Cauchy integral theorem, which asserts that the integration of analytical functions is path independent. The generalization and application of the logarithmic residue theorem are discussed through several

application examples of the logarithmic residue theorem in real integrals. Calculating complex and difficult integrals is made easier by the residue theorem, especially for integrals that decay quickly and similarly to other integrals. Further research on this basic tenet of complex analysis is possible in the future. The number of poles and zeros of analytical functions can be discussed using the essential and useful concept of logarithmic residue. By using the aforementioned corollaries, it is possible to calculate the zeros and poles fast and effectively, which significantly reduces the amount of time needed to study complex functions.

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