Newton's Methods: Constraints and Higher Order Convergence

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Abstract. Newton's method is designed to approximate the roots of functions with iterative process. The basic idea of Newton's method is repeatedly calculating the next approximation with a favorable initial guess which has a tangent line. Under some cases, such as indifferentiable functions, Newton's method does not work. Since Newton's method is not always desirable, it is varied to fit in different situations, and hence, different variants of Newton's method prosper. One kind of variants aims to enhance the speed and the efficiency of Newton's method, which is mainly discussed in this paper. Also, this kind of variants extends the applicability in different practical problems like chemicals and increase the order of convergence. This paper also discusses the efficiency of variants with efficiency index to normalize different methods and finds out real world problems need solutions with higher-order convergence and less computational cost. Nowadays, Newton's method and its variants are still a powerful tool in diverse scientific applications.

Keywords: Newton's method; order of convergence; efficiency index.

1. Introduction

Linear functions like $f(x) = kx + b$ have predictable and easily understandable solutions [1]. On the contrary, many nonlinear functions don't have a root-finding formula that offers explicit and exact solutions [2]. In real life, numerous phenomena like population growth and fluid dynamics are modeled by nonlinear functions, and hence, it is crucial to get a sense what is happening to the implicit solutions. To solve the problem of nonlinear functions, Sir Isaac Newton developed an ingenious method, known as Newton's method or Newton-Raphson method. The method uses iterative technique to better approximate the solution to the equation $f(x) = 0$. The idea behind the Newton's method picks a favorable point at the beginning with the corresponding tangent line as a direction to the next point intersecting the x-axis, repeating this process multiple times. As a result, the number should be close to the exact solution. An example using Newton's method is to directly approximate $\sqrt{2}$ with initial condition 1 from the function $f(x) = x^2 - 2$ equated to 0 [3]. After 3 iterations, the number of correct digits reaches 5, which is efficient and useful. However, if the initial condition is not chosen properly, some unexpected situations can happen. One case is the orbit of the first point does not converge, so the root is not found. Another case is the desired zero is lost, and another different solution is discovered. The trick of selecting the initial conditions and the differentiability of the function make the refinements of Newton's method indispensable. Another factor that people refine the method is the allure of more efficient way to converge [4]. Especially with the emergence of computers, the accuracy and speed of computing are guaranteed, and experts focus on the efficiency more. Besides, as nonlinear functions in other industries like engineering intensify, Newton's method has many variations to adapt to those industries.

This paper introduces how Newton's method works and the limitations of Newton's method. To converge to the root efficiently, this review paper also discusses some effort in refinement of Newton's method.

2. Preliminaries

2.1. Newton's Method

Newton's method aims to find the root of a function $f(x)$ given by an initial guess. The function should be continuously differentiable, and the initial point starts with $(x_0, f(x_0))$. What to do next

is using the equation $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$ to find the next point where $n \in N_0$ and $f'(x_n) \neq 0$. Here is the figure of the process (Figure 1) [3].

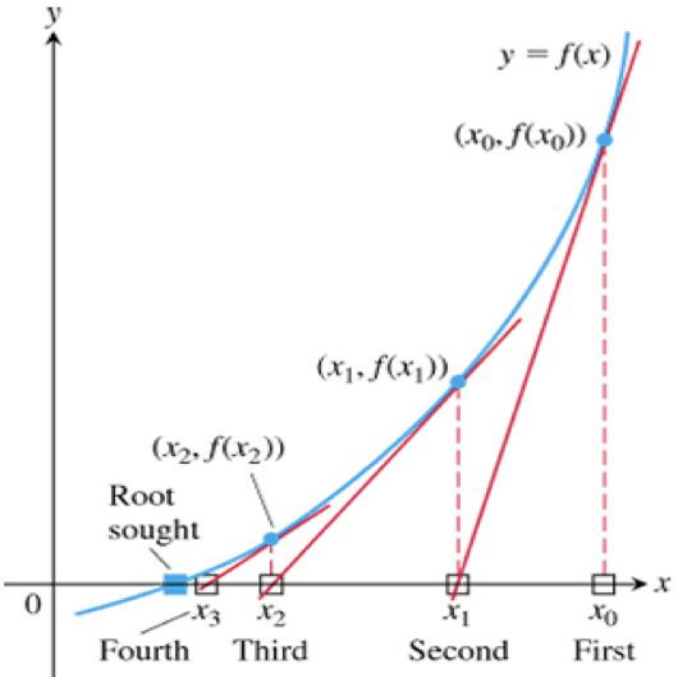


Fig. 1 The initial guess approaching the desired root with the equation $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$ [3]

It's also a good question that when the process of such iteration stops. It depends on the desired requirements, so it's flexible. One example is from the paper setting the threshold $|x^{(k+1)} - x^{(k)}| + |f(x)^{(k)})| < \varepsilon$ [5].

2.2. The Fundamental Idea behind Newton's Method

The method can be explained by the Taylor series of the function $f(x)$ expanded at x_0.

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n \quad (1)$$

And the function can be approximately expressed if only two terms are chosen, $f(x) \approx f(x_0) + f'(x_0)(x - x_0)$. To find the root, $f(x)$ is equal to 0, i.e., $f(x_0) + f'(x_0)(x - x_0) = 0$. Then the equation can be simplified as $x = x_0 - \frac{f(x_0)}{f'(x_0)}$. Here, x becomes the input for the next iteration, so it can be also labeled as x_1. As the formula iterates, it can be written as $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$ [6].

2.3. Order of Convergence

Let the function $f(x)$ has the desired root r, and $\{x_n\}$ be the sequence convergent to r, then the order of convergence p is expressed as $\lim_{n\to\infty} \frac{x_{n+1}-r}{(x_n-r)\hat{}p} = c \neq 0$ where p belongs to real numbers and c is the asymptotic error constant [6]. The error can be understood that each iteration is almost proportional to p-th power of the previous error. If c is smaller, then the method of convergence is faster.

From the definition, Newton's method has the order of 2 [6]. It doesn't come from the first two terms of the Taylor Series. Also with higher convergence order, the sequence converges to the root faster.

2.4. Efficiency Index

Efficiency index e is defined as $e = \sqrt[a]{p}$ where a computational cost measuring the number of operations during the iterative method, and p is defined above [6]. Although higher order of convergence makes the sequence converge to the root faster, it may cost too much. And efficiency index normalizes the methods. Newton's method has efficiency index $e = \sqrt[2]{2} \approx 1.414$.

3. Constraints and Variations of Newton's Method

With the emergence of technology, people pursue the speed of convergence and consider some special cases. Based on Newton's method, scholars come up with more methods like Halley's method to approximate the root more efficiently.

3.1. Constraints of Newton's Method

Newton's method has some constraints like it cannot work if the function is not differentiable. Also, the orbit can be divergent if the point is chosen inappropriately [3]. This is an interesting case that the sequence generated by Newton's algorithm diverges (Figure 2).

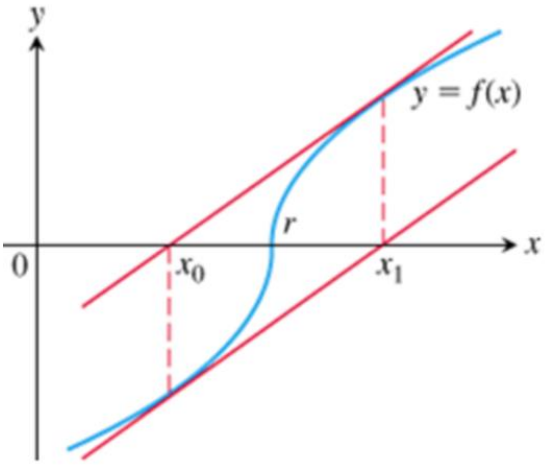


Fig. 2 The orbit of the initial condition jumps between x_0 and x_1 [3]

3.2. Third-Order Convergence

In this situation, let p be 3. Some common methods with third-order convergence are introduced.

3.2.1 Halley's method

This method is firstly introduced because this method has a similar idea to Newton's method. Newton's method adopted first two terms to approximate the root. In this case, first three terms are chosen to be 0, i.e., $f(x) \approx f(x_0) + f'(x_0)(x - x_0) + 0.5 \cdot f''(x_0)(x - x_0)^2 = 0$ with the equation from Newton's method $x = x_0 - \frac{f(x_0)}{f'(x_0)}$. Two equations together, the next point will be $x = x_0 - \frac{2f(x_0)f'(x_0)}{2[f'(x_0)]^2 - f(x_0)f''(x_0)}$ and then put in iteration $x_{n+1} = x_n - \frac{2f(x_n)f'(x_n)}{2[f'(x_n)]^2 - f(x_n)f''(x_n)}$ [8].

Halley's method has third-order convergence [8]. Since this method requires $f(x_n)$, $f'(x_n)$, and $f''(x_n)$, which are three required operations, so the computational cost is 3. The efficiency index is calculated as $e = \sqrt[a]{p} = \sqrt[3]{3} \approx 1.442$.

3.2.2 Chebyshev's method

This method is similar to Halley's method. Chebyshev's method can also be deducted from the same condition $\begin{cases} f(x_0) + f'(x_0)(x - x_0) + 0.5 \cdot f''(x_0)(x - x_0)^2 = 0 \\ x = x_0 - \frac{f(x_0)}{f'(x_0)} \end{cases}$.

Finally, the iterative algorithm becomes $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} - \frac{f^2(x_n)f''(x_n)}{2[f'(x_n)]^3}$. The efficiency index is the same as Halley's method because the computational cost is also 3, and the order of convergence is also 3. $e = \sqrt[a]{p} = \sqrt[3]{3} \approx 1.442$.

3.2.3 Other variants of Newton's Method with Third-Order Convergence

There are many methods with convergence order of 3. Here are two examples [9].

$$\begin{cases} y_n = x_n - \frac{f(x_n)}{f'(x_n)} \\ x_{n+1} = x_n - \frac{f(x_n)}{2f'(x_n)} - \frac{f(x_n)}{2f'(y_n)} \end{cases}, \begin{cases} y_n = x_n - \frac{f(x_k)}{f'(x_k)} \\ x_{k+1} = x_k - \frac{2f(x_k)}{f'(x_k)} - \frac{f(x_k)}{2f'(y_k)} \end{cases} \quad (2)$$

This article also points out that their efficiency indexes are the same as Halley's method's, which is 1.442. One noticeable point is that an iterative method $x_{k+1} = 0.5 \cdot x_k - \frac{f(x_k)}{f'(x_k)} + \frac{f^2(x_k)}{[f'(x_k)]^2}$ stands out because of the lowest computational cost[9]. From observation, only $f(x_k)$ and $f'(x_k)$ requires computation. In comparison, it costs the same operations as Newton's method but converges to the target faster. Therefore, it's efficiency index is $\sqrt[2]{3} \approx 1.732$. It should be the best performance in the third-order convergence so far because the order of convergence is fixed and 2 is the smallest number to operate so far.

3.3. Fourth-Order Convergence

To converge to the target faster, it is natural to try fourth-order convergence based on Newton's method.

3.3.1 Doubled Newton's method

This method is interesting because it continuously calculated the root with Newton's method twice as one step. It is like the same distance originally needs two steps but now only takes one big step.

This method is $\begin{cases} y_n = x_n - \frac{f(x_n)}{f'(x_n)} \\ x_{n+1} = y_n - \frac{f(y_n)}{f'(y_n)} \end{cases}$ [9]. Although it takes one big step equal to two steps, there are four things to compute in the process: $f(x_n)$, $f'(x_n)$, $f(y_n)$, $f'(y_n)$. Therefore, it's efficiency index is $e = \sqrt[4]{4} \approx 1.414$.

This method takes the same steps, but it erases half of the points estimated in the figure 1. The sequence of doubled Newton's method can be seen as a subsequence of Newton's method, i.e., $\{a_{2n}\}$.

3.3.2 Jarratt's method

This method is $\begin{cases} y_k = x_k - \frac{2f(x_k)}{3f'(x_k)} \\ x_{k+1} = x_k - \frac{3f'(y_k)f(x_k)+f(x_k)f'(x_k)}{6f'(y_k)f'(x_k)-2[f'(x_k)]^2} \end{cases}$ [10]. During the process, only $f(x_k)$, $f'(y_k)$ and $f'(x_k)$ needs computation, so $a = 3$, and efficiency index $e = \sqrt[3]{4} \approx 1.587$.

The article points out that there exist three other methods with the same efficiency, but they so far perform the best [9] with efficiency index 1.587.

3.4. Fifth-Order Convergence

In this case, p is 5. One method with fifth-order convergence is introduced.

$$\begin{cases} y_n = x_n - \frac{f(x_n)}{2f'(x_n)} \\ z_n = x_n - \frac{f(x_n)}{f'(y_n)} \\ x_{n+1} = z_n - \frac{2f(z_n)}{f'(y_n)} + \frac{f(z_n)}{f'(x_n)} \end{cases} \quad (3)$$

Obviously, only $f(x_n),\ f'(x_n),\ f'(y_n),\ f(z_n)$ need calculating, and hence, the efficiency index $e = \sqrt[4]{5} \approx 1.495$. The article points out that there also exist methods like this case with fifth-order convergence applied in practical ways like two-point boundary value problem and portfolio optimization [9]. This means the solutions will be given quickly with less effort. With swift change in markets, a resolution with quicker convergence enables people to operate to repeatedly correct the decisions based on the latest data. As a result, the market can perform better and reduce risks.

The efficiency is comparatively the highest in the fifth-order convergence, but it is utilized in real life because it fits in the problem.

3.5. Higher-Order Convergence of Newton's Method

For higher-order convergence, there also exists sixth-order convergence and even tenth-order convergence. They have higher speed of convergence to the desired solution, and they can be applied in real life case like chemicals [11].

Table 1. Efficiency Index

a p	2	3	4
2	1.414	1.732	2
3	1.26	1.442	1.587
4	1.189	1.316	1.414

Table 1 above illustrates the efficiency index gets larger as p is larger, and the efficiency index becomes smaller as a goes larger. Therefore, ideally, to solve a function more efficiently with Newton's method and its variations, the order of convergence should be enlarged, and the computational cost should be less. Normally higher-order convergence means the approximation is speedier under each iteration, but computational cost will increase as the convergence order multiply. However, it is still worth using methods with higher-order convergence because the rate of computational cost is far less than that of order of convergence [9]. Looking for a method with higher-order convergence and with lower computational cost will be a guide for solving nonlinear functions although initial guesses far from the desired root can undermine the advantages of the methods with higher-order convergence.

4. Conclusion

Newton's method is used to seek the root of a nonlinear function. This iterative process can approximate the values with little errors in high speed, especially with the help of technology. It is ensured by a threshold set by humans. From the figure 3, it is easy to see the correct decimals can double with each iteration. However, it does have constraints. From the definition, it is obvious that the function should be differentiable, and Newton's method does not work in such cases. Moreover, the initial guess is another trouble because this selected point can diverge or head to an undesired root even if the function is differentiable.

To converge faster, methods with higher-order convergence based on Newton's method has prospered. These methods use higher derivatives or more iteration formulas in order to approach the real root faster. They are also applied in real life successfully. Another bad thing of selecting an initial point far from the desired root can undermine the advantage of the methods with higher-order convergence. The efficiency index shows higher-order convergence enlarges the efficiency with lower computational cost. However, smaller-order convergence is destined to converge slower and higher-order convergence should have more computational steps. Therefore, it is a challenge pursuing swifter convergence and low computational cost at the same time, and the challenge will be a great direction applied to practical problems.

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