The summation method and application of power series

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Abstract. Summation, or the calculation of the sum of a series, is a fundamental operation in mathematics with widespread applications in various fields. It involves adding together a sequence of numbers or terms to obtain a single value. The ability to accurately determine the sum of a series is crucial for solving mathematical problems, analyzing data, evaluating functions, and making predictions. Power series are an integral part of mathematics and have found extensive applications in various fields. The study of power series summation, specifically the derivation and analysis of summation formulas, plays a crucial role in understanding the behavior and properties of these series. This paper aims to discuss the process of finding the sum function of power series by normalization method, definition method and the power series expansion, thus the power series establishes relations with several series which have common sum functions, and the advantages and limitations of each approach will be discussed, along with practical examples to illustrate the application of the sum function of power series in calculating π and integration.

Keywords: Power series, summation method, calculation, sum of series.

1. Introduction

Summation, or the calculation of the sum of a series, is a fundamental operation in mathematics that finds widespread applications in various fields. It involves adding together a sequence of numbers or terms to obtain a single value. Accurately determining the sum of a series is crucial for solving mathematical problems, analyzing data, evaluating functions, and making predictions [1]. Power series, which are an integral part of mathematics, have found extensive applications in various fields. The study of power series summation, specifically the derivation and analysis of summation formulas, plays a crucial role in understanding the behavior and properties of these series. By establishing relations with several series that have common sum functions, power series provide a powerful tool for mathematical analysis and problem-solving [2].

In recent years, there has been a growing interest in the application of power series in calculating π and integration. The value of π, which represents the ratio of a circle's circumference to its diameter, is a fundamental constant in mathematics with profound implications across various scientific disciplines [3]. Power series provide an efficient and versatile approach to approximating the value of π. By using the sum function of power series, one can iteratively compute increasingly accurate estimates of π. This iterative process converges towards the true value of π, allowing for precise calculations in practical applications [4].

Furthermore, the sum function of power series plays a crucial role in integration, which is the process of determining the area under a curve. By expanding functions into power series, complex functions can be transformed into simpler algebraic expressions, facilitating the evaluation of integrals. This technique, known as power series expansion, enables efficient and accurate integration in various mathematical and scientific problems [5]. The application of the sum function of power series in calculating π and integration extends beyond pure mathematics. It has significant implications in fields such as physics, engineering, computer science, and statistics. From modeling physical phenomena to analyzing data and designing algorithms, the ability to accurately compute π and perform integration using power series provides invaluable insights and practical solutions [6].

This paper aims to discuss the process of finding the sum function of power series using different methods, including the definition method, normalization method, and power series expansion. Each approach offers unique advantages and limitations, and their application in calculating π and integration will be explored. Practical examples will be provided to illustrate the effectiveness of the

sum function of power series in these mathematical operations [7]. By understanding the principles and techniques involved in the summation of power series, researchers and practitioners can enhance their ability to solve complex mathematical problems, analyze data, and make accurate predictions. The insights gained from this study will contribute to the development of advanced mathematical techniques and further deepen our understanding of the applications of power series in various fields.

2. Methods

2.1. The Method of Normalization

The method of Normalization is used to find the sum of a power series by transforming it into a known series form for easier calculation. This method requires us to express the given power series as a geometric series or another known series, so that the summation formula of the known series can be utilized to compute the sum function of the power series.

The key to the method of Normalization is to re-represent the power series in the form of a known series. Common techniques for achieving this include term-by-term integration and differentiation method, variable substitution method, term splitting and combination method, term splitting and cancellation method, function construction method and method of constructing differential equations [8]. Through these methods, the power series can be transformed into a known series and using the properties of the known series to solve it.

2.1.1. The Term-by-term Integration and Differentiation Method

Because the power series is in its convergence interval, it can be summed and derived item by item, therefore, the power series can be classified into the form of easy summation function, and then the resulting sum function is derived or integrated to get the sum function of the original series [9].

Example 1. Finds the sum function of the series $\sum_{n=1}^{\infty} nx^{n-1}$.

Solution 1. According to the sum function of the power series in its convergence interval, there is:

$$\sum_{n=1}^{\infty} nx^{n-1} = \sum_{n=1}^{\infty}(x_n)' = (\sum_{n=1}^{\infty} x_n)' \tag{1}$$

where $\sum_{n=1}^{\infty} x_n$ is a geometric series and

$$\sum_{n=1}^{\infty} x_n = \frac{x}{1-x}, where\ |x| < 1 \tag{2}$$

Then,

$$\sum_{n=1}^{\infty} nx^{n-1} = \left(\frac{x}{1-x}\right)' = \frac{1}{(1-x)^2}, where\ |x| < 1 \tag{3}$$

Solution 2. Term-by-term integration and then Term-by-term differentiation. Suppose:

$$s(x) = \sum_{n=1}^{\infty} nx^{n-1} \tag{4}$$

Since,

$$\int_0^x s(t)dt = \int_0^x \sum_{n=1}^{\infty} nt^{n-1}\, dt = \sum_{n=1}^{\infty} \int_0^x nt^{n-1}dt = \sum_{n=1}^{\infty} x^n = \frac{x}{1-x}, where\ |x| < 1 \tag{5}$$

Then,

$$s(x) = \left(\int_0^x s(t)dt\right)' = \left(\frac{x}{1-x}\right)' = \frac{1}{(1-x)^2}, where\ |x| < 1 \tag{6}$$

2.1.2. The term splitting and cancellation method

The term splitting and cancellation method is a technique used to find the sum of a series. The basic idea is to split and cancel terms within the series to simplify the process of summation [10]. Specifically, the Cancellation of Terms Method can be applied to series where adjacent terms have canceling relationships, either due to algebraic cancellations or properties of the function.

Example 2. Finds the sum function of the series $\sum_{n=1}^{\infty}\frac{1}{n^2+n}$.

Solution. Because $\frac{1}{n^2+n}=\frac{1}{n(n+1)}=\frac{1}{n}-\frac{1}{n+1}$, then,

$$\sum_{n=1}^{\infty}\frac{1}{n^2+n}=\sum_{n=1}^{\infty}\left(\frac{1}{n}-\frac{1}{n+1}\right)=\lim_{n\to\infty}\left(1-\frac{1}{2}+\frac{1}{2}-\frac{1}{3}+\cdots+\frac{1}{n}-\frac{1}{n+1}\right)=\lim_{n\to\infty}\left(1-\frac{1}{n+1}\right)=1 \quad (7)$$

2.2. The Method of Definition

The method of definition is used to find the sum of a power series through a direct and intuitive approach. It involves summing each term of the given power series to obtain the sum function of the entire series.

The key to the method of definition is to identify the general term expression of the power series and determine the range of the summation. Once the general term expression is known, calculate the sum of the first n terms, which is written by $\{S_n(x)\}$, then calculate the value when n tends to the infinite limit which is written by $\lim_{n\to\infty}S_n(x)$, that equals to the value of the sum function of the series.

Example 3. Finds the sum function of the series $\sum_{n=1}^{\infty}\frac{n}{2^n}x^n$, defined on$(-\infty,+\infty)$.

Solution. The general term expression is $\frac{n}{2^n}x^n$, and the range of the summation is $(-\infty,+\infty)$. Then the sum of the first n terms is:

$$S_n(x)=\frac{1}{2}x+\frac{2}{2^2}x^2+\cdots+\frac{n}{2^n}x^n \quad (8)$$

If both sides of the equation multiply $\frac{1}{2}x$, there is:

$$\frac{1}{2}xS_n(x)=\frac{1}{2^2}x^2+\frac{2}{2^3}x^3+\cdots+\frac{1}{2^{n+1}}x^{n+1} \quad (9)$$

If $(8)-(9)$, there is:

$$\left(1-\frac{1}{2}x\right)S_n(x)=\frac{1}{2}x+\frac{1}{2^2}x^2+\cdots+\frac{1}{2^n}x^n-\frac{n}{2^{n+1}}x^{n+1} \quad (10)$$

Which is:

$$S_n(x)=\frac{1}{\left(1-\frac{1}{2}x\right)}\left\{\frac{\frac{1}{2}x\left[1-\left(\frac{1}{2}x\right)^n\right]}{1-\frac{1}{2}x}-\frac{n}{2^{n+1}}x^{n+1}\right\} \quad (11)$$

When n tends to the infinite limit, there is:

$$\lim_{n\to\infty}S_n(x)=\lim_{n\to\infty}\frac{1}{\left(1-\frac{1}{2}x\right)}\left\{\frac{\frac{1}{2}x\left[1-\left(\frac{1}{2}x\right)^n\right]}{1-\frac{1}{2}x}-\frac{n}{2^{n+1}}x^{n+1}\right\}=\frac{\frac{1}{2}x}{\left(1-\frac{1}{2}x\right)^2} \quad (12)$$

The answer is:

$$\sum_{n=1}^{\infty}\frac{n}{2^n}x^n=\frac{\frac{1}{2}x}{\left(1-\frac{1}{2}x\right)^2},where\ x\in(-\infty,+\infty) \tag{13}$$

2.3. The Power Series Expansion

The power series expansion, or power series representation, is a mathematical technique used to represent a function as an infinite sum of terms. It is also commonly referred to as the Taylor series expansion. In this representation, a function is expressed as the sum of infinitely many terms, each multiplied by a corresponding power of a variable.

Mathematically, a power series can be written as $f(x)=a_0+a_1x+a_2x^2+a_3x^3+\cdots$. Where $f(x)$ is the function, x is the variable, and $a_0,a_1,a_2...$ are the coefficients multiplying each term.

The power series expansions of many elementary functions can be derived using the Taylor formula and subsequently simplified. There are the power series expansions of some common functions [5, 6].

$$\frac{1}{1+x}=\sum_{n=0}^{\infty}(-1)^nx^n,where|x|<1 \tag{14}$$

$$e^x=\sum_{n=0}^{\infty}\frac{x^n}{n!},where|x|<+\infty \tag{15}$$

$$sin\,x=\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}x^{2n+1},where|x|<+\infty \tag{16}$$

$$cos\,x=\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!}x^{2n},where|x|<+\infty \tag{17}$$

$$ln(1+x)=\sum_{n=0}^{\infty}\frac{x^n}{n},where\ x\in(-1,1] \tag{18}$$

Example 4. Finds the sum function of the series $\sum_{n=0}^{\infty}(-1)^n\frac{2n+3}{(2n+1)!}$.

Solution. Since $\sin x=\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}x^{2n+1}$, $|x|<+\infty$, if Both sides of the equation (16) multiply x^2, there is:

$$x^2sin\,x=\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}x^{2n+3} \tag{19}$$

Differentiating both sides of the equation (19) with respect to x, there is:

$$2x\,sin\,x+x^2\,cos\,x=\sum_{n=0}^{\infty}(-1)^n(2n+3)\frac{x^{2n+2}}{(2n+1)!} \tag{20}$$

Let $x=1$ in (20), there is:

$$\sum_{n=0}^{\infty}(-1)^n\frac{2n+3}{(2n+1)!}=2\,sin\,1+cos\,1 \tag{21}$$

3. Results and Discussion

3.1. Evaluation of the Summation Method

Methods for summing power functions can be evaluated based on the approach used, namely closed-form solutions or numerical approximation. In this paper, closed-form solutions include the method of definition, the method of Normalization, numerical approximation includes the power series expansion.

If a closed-form solution for the sum of the power series, represented as $\sum_{n=0}^{\infty} a_n x^n$ can be found, it provides an advantage of obtaining an exact analytical solution. For certain power series with known mathematical properties, such as geometric series or special series (e.g., exponential or trigonometric series), closed-form solutions can be derived using mathematical techniques and formulas. The benefits of this method include the accuracy of obtaining exact mathematical expressions and clear mathematical properties. However, finding closed-form solutions for more general power series can be challenging.

When a closed-form solution cannot be determined, numerical approximation methods can be used to approximate the sum of the power series. Common numerical approximation methods include Taylor series expansions, numerical integration, and numerical summation methods. These methods approximate the sum of the power series by truncating the series expansion, considering only a finite number of terms. The advantages of numerical approximation lie in the flexibility to control accuracy and computational complexity, making them suitable for a wide range of power functions. However, numerical approximation methods provide only approximate solutions, and the accuracy depends on the chosen number of terms and the precision of the algorithm.

Therefore, closed-form solutions provide high precision and clear mathematical properties when available, whereas numerical approximation methods offer flexibility for general power series summation but only provide approximate solutions. The choice of method depends on the specific problem, requirements, and the availability of closed-form solutions for the sum of the power series.

3.2. Application in Mathematical

The application of the sum function of power functions is diverse and can be found in various fields such as physics, engineering, economics, and computer science. Here are a few examples:

3.2.1. Calculate an approximate value for the mathematical constant π

The mathematical constant π is the key value used for accurately calculating the circumference, area, and volume of geometric shapes such as circles and spheres. The value of π has had a tremendous impact on the development of mathematics, providing important methods and tools for mathematical and scientific research, and possessing infinite fascination. Throughout history, mathematicians have devoted their lifetime and efforts to achieve greater precision in calculating π. It wasn't until the 17th century, with the aid of powerful mathematical tools like the expression of inverse tangent function in mathematical analysis, that the calculation of π entered a new stage. The method involves constructing power series using the inverse tangent function, finding the sum of the power series through differential and integral calculus, and approximating the value of π. Since

$$\sum_{n=0}^{\infty}(-x^2)^n = \frac{1}{1+x^2}, where\ x \in (-1,1) \tag{22}$$

Integrate both sides of equation (22) with respect to x simultaneously, there is:

$$arc\ tan\ x = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} x^{2n+1}, where\ x \in (-1,1] \tag{23}$$

Let $x = 1$ in (24), there is:

$$\frac{\pi}{4} = arc\ tan\ 1 = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots + \frac{(-1)^n}{2n+1} + \cdots \tag{24}$$

Then,

$$\pi = 4arc\ tan\ 1 = 4 \times \left(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots + \frac{(-1)^n}{2n+1} + \cdots\right) = 3.1415826535897198\cdots \tag{25}$$

Calculate the approximation values for each n value using code, and plot the change in approximation values using the plot function from the Matplotlib library. Additionally, to provide better comparison, draw a red dashed line to represent the true value of π.

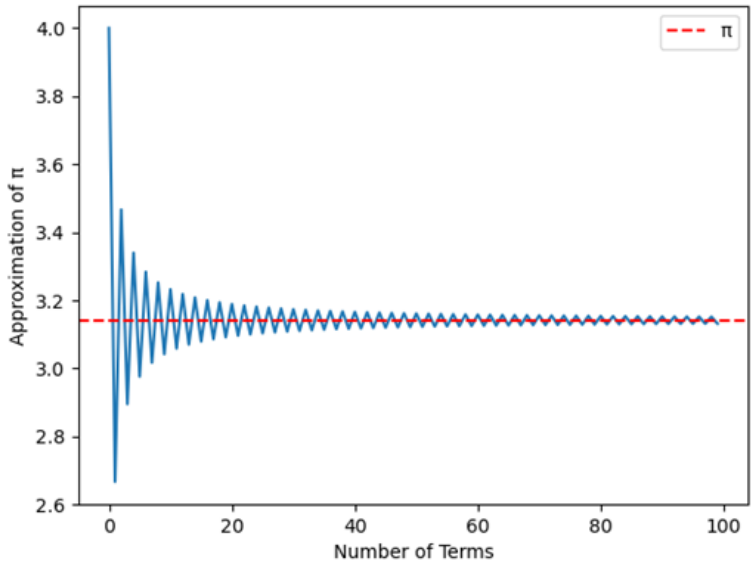


Figure 1. Approximation of π using Alternating Series [10]

Figure 1 provides a visualization of Adjusting the value of n to change the accuracy of the approximation and observe the trend of approximation values through the graph.

3.2.2. The application in integration

The computation of integrals plays a crucial role in higher mathematics, and integral calculus techniques are often highly skillful. The common methods for integral computation include change of variables and integration by parts. However, for certain integrals, utilizing the sum function of power series greatly simplifies and expedites the solving process [9].

Example 5. Calculate the integral $\int_0^1 \frac{\ln(1+x)}{x} dx$.

Solution. Since $\ln(1+x) = \sum_{n=0}^{\infty} \frac{x^n}{n}, where\ x \in (-1, 1]$, then:

$$\int_0^1 \frac{ln(1+x)}{x} dx = \int_0^1 \sum_{n=0}^{\infty} \frac{x^{n-1}}{n} dx = \sum_{n=0}^{\infty} \frac{1}{n} \cdot \int_0^1 x^{n-1} dx = \sum_{n=0}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6} \quad (26)$$

4. Conclusion

In conclusion, the summation of power series is a fundamental and essential operation in mathematics, with wide-ranging applications in various fields. The derivation and analysis of summation formulas for power series are crucial for understanding the behavior and properties of these series. Through the definition method, normalization method, and power series expansion, the relationships between power series and other series that share common sum functions are established. Each approach brings its advantages and limitations, and the choice of method depends on the specific problem or context.

The application of the sum function of power series extends to the calculating π and integration. It enables us to solve mathematical problems, analyze data, evaluate functions, and make predictions accurately. The practical examples presented in this paper have demonstrated the effectiveness of the sum function of power series in real-world scenarios.

In summary, the study of the sum function of power series provides valuable insights into the convergence, properties, and practical applications of these series. Further research in this area can lead to the development of new mathematical techniques and deepen the understanding of complex mathematical phenomena.

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