Analysis of Several Basic Formulae of Statistical Methods in Cosmology

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Abstract. As a matter of fact, statistic methods are widely applied in astrophysics in various aspects including particle motions, Big Bang theory, observations, and so on. With this in mind, this article focuses on statistical methods in cosmology. On this basis, the study aims to throw new light on understanding of these theorems. To be specific, it first discusses the history of statistical methods applied in physics especially in cosmology. In the main part, the paper analyzes some basic statistical theorems by modifying a current version of proof, or refining the proof through detailed analysis. For each analyzed formula, the paper introduces some recent applications in order to offer a better understanding of its applications in cosmology. In addition, the findings also suggest that even for proven theorems, the proof process still has room for optimization and improvements. Overall, this study comprehends the understanding of statistical methods in cosmology as well as sheds light on guiding further simplification and derivation of theoretical astrophysics.

Keywords: Statistical method, cosmology, virial theorem, Poisson-Vlasov equations, Liouville's theorem.

1. Introduction

The applications of statistical methods to physics can date back to a long time ago, and go into nearly every branch of physics. There are two main statistical methods applied in physics: statistical data analysis and analysis of statistical quantities. This article mainly talks about the latter one. In many circumstances, physicists tend to use statistical languages such as probability density function, average and variance, in order to give more detailed descriptions of quantities. Statistical methods explain macroscopic phenomena by using statistics and possibility approaches on microscopic units like single molecules or differential volume units.

In the study of laws of multi-object motions, physicists discovered some invariable systematic laws regardless of variable motions of specific particles in a system. The Newton's second law in the systematic form $\sum \boldsymbol{F}_{\boldsymbol{external}} = \sum_i (m_i \boldsymbol{a}_i)$, in the center-of-mass form $\sum \boldsymbol{F}_{\boldsymbol{external}} = (\sum_i m_i)\boldsymbol{a}_c$, together with momentum and energy formulae in these two forms (these perhaps also inspired the idea of virial theorem) announced the beginning of this kind of research, both successively and logically. Then, Hamiltonian and Lagrangian mechanics proposed the concept of ensemble, which completed the framework of system, leading to an enhancement of systematic analysis method, which made it easier to probe into statistical laws of a system. A systematic motion can be worked out by solving the equations corresponding to these laws decoupled with other necessary equations. In the middle of the 19th century, physicists like J. C. Maxwell, L. Boltzmann and J. W. Gibbs applied statistical methods to thermodynamics, causing an unprecedented revolution to the branch of physics. After the establishment of quantum physics, physics finally showed a much more dependent relationship with statistics because some basic quantum formulae are written in statistical languages, for example, the uncertainty equation $\Delta x \Delta p \leq \hbar$ in which Δ means standard deviations. Statistics and probability approaches even lay the foundation for knowing stochasticity in the physics world.

Cosmology, a subject probing into the cosmos and derived from astronomy, entered the physics field very early. Based on astronomical observation data from institutes and astronomers such as Tycho Brahe, physicists, e.g., Copernicus, Kepler and Galileo, concluded many laws of celestial bodies and managed to use the language of physics to describe and predict cosmological phenomena; and cosmology even boosted the development of physics like leading to the formulation of Newton's

law of universal gravitation, in turn. As theoretical mechanics and modern physics were established and developed, it is natural to see more and more statistical methods applied in cosmology because of increasing demands for and approaches to a more detailed theoretical framework. As a result, statistical methods are playing an increasingly significant role in cosmology nowadays. For cosmological simulations such as Illustris and its updated versions IllustrisTNG and THESAN, solving Poisson-Vlasov Equations has become a paradigm [1-3]. There are also some recent achievements via statistical methods in cosmology. An article by L. Verde in 2010 gives a comprehensive overview of statistical methods and applies them to several cosmological problems as examples [4]. In November 2022, Luciano published a paper about Kaniadakis statistics, which extended some physical quantities to non-Euclidean worlds, in cosmology, beyond traditional Maxwell-Boltzmann statistics [5]. Luciano's paper broadens the view of the cosmos.

The other kind of statistical method, data analysis, is also of great use. As technology updates, data obtaining gets more and more easy and regular. Scientific theorems and hypotheses are based on experiments; with high-frequency occurrence in cosmological experimental data, data analysis plays an increasingly important role in cosmology. Two typical examples are discoveries of Hubble expansion and the *J* particle. The data analysis method continued, like global statistics used in Illustris [6].

There are many theorems and formulae related to statistical quantities in mechanics and many of them show a wide range of applications in cosmology. The main part of this article will especially focus on some of these, the virial theorem and Poisson-Vlasov equations. Current proofs of these theorems are mostly not very accessible to new learners and readers can see modifications and in-depth analysis of proving steps of these theorems. For virial theorem, current proofs that can be seen are either complex or not rigorous; a proof of virial theorem, nearly the only understandable version for new learners [7], has its faults, and this article will do a correction and give an explicit and rigorous proof based on it. For Poisson-Vlasov equations, mainly based on Poisson equation and Liouville's theorem, many versions of proof are too sketchy to leave out some interpretable points, which may confuse; this article will refine the proof steps and try to interpret the mathematical and physical meaning in detail. Moreover, this article will review the applications of these formulae. This study analyzes these basic formulae in order to make the equations more accessible.

2. Virial Theorem and Applications

2.1. Brief Description

In a gravitational virial system (of finite positions and velocities), the average potential energy is always the negative of twice the amount of kinetic energy. Then a commonsense-contradicting property can be obtained: if the virial system absorbs energy, the particles' kinetic energy will decrease, in turn, to satisfy the increase of potential energy. However, there are still some prerequisites for a system to become a virial system: finite positions and vectors, central forces, and the formula type like $E_p = cr^{\alpha}$. Furthermore, the most basic prerequisite is that the system must be equivalent to an isolated system, e.g., the ideal gas in a container is not a virial system, i.e., the kinetic energy is not zero while the potential energy is zero, which is because the gas interacts with the wall of the container.

2.2. Proof

As to an isolated system, one defines

$$G = \sum_{i=1}^{n} \boldsymbol{p_i} \cdot \boldsymbol{r_i} \tag{1}$$

Here, *n* is the number of particles, *p*: momentum, ***r***: position vector. Due to the linear property of derivation, it can be derived that

$$\frac{dG}{dt} = \sum_{i=1}^{n} \dot{\boldsymbol{p}}_{\boldsymbol{i}} \cdot \boldsymbol{r}_{\boldsymbol{i}} + \sum_{i=1}^{n} \boldsymbol{p}_{\boldsymbol{i}} \cdot \dot{\boldsymbol{r}}_{\boldsymbol{i}} \tag{2}$$

Since

$$\sum_{i=1}^{n} \dot{\boldsymbol{p}}_{\boldsymbol{i}} \cdot \boldsymbol{r}_{\boldsymbol{i}} = \sum_{i=1}^{n} \boldsymbol{F}_{\boldsymbol{i}} \cdot \boldsymbol{r}_{\boldsymbol{i}} \tag{3}$$

And on the other hand,

$$\sum_{i=1}^{n} \boldsymbol{p}_{\boldsymbol{i}} \cdot \dot{\boldsymbol{r}}_{\boldsymbol{i}} = \sum_{i=1}^{n} m_i \boldsymbol{v}_{\boldsymbol{i}} \cdot \dot{\boldsymbol{r}}_{\boldsymbol{i}} = \sum_{i=1}^{n} m_i \boldsymbol{v}_{\boldsymbol{i}}^{2} = 2E_k \tag{4}$$

It can be derived that

$$\frac{dG}{dt} = 2E_k + \sum_{i=1}^{n} \boldsymbol{F}_{\boldsymbol{i}} \cdot \boldsymbol{r}_{\boldsymbol{i}} \tag{5}$$

One integrates and average both sides of Eq. (5), then:

$$\frac{1}{\tau}[G(\tau) - G(0)] = 2\overline{E_k} + \overline{\sum_{i=1}^{n} \boldsymbol{F}_{\boldsymbol{i}} \cdot \boldsymbol{r}_{\boldsymbol{i}}} \tag{6}$$

For periodic motion, one selects the interval length of integral τ as the period T; the left side = 0; For finite positions and velocities of each particle: selecting $\tau \to \infty$; the left side$\to$0. Then

$$\overline{E_k} = -\frac{1}{2}\overline{\sum_{i=1}^{n} \boldsymbol{F}_{\boldsymbol{i}} \cdot \boldsymbol{r}_{\boldsymbol{i}}} \tag{7}$$

For a conservative system,

$$\boldsymbol{F}_{\boldsymbol{i}} = -\boldsymbol{\nabla} E_{pi} \tag{8}$$

Then,

$$\overline{E_k} = \frac{1}{2}\overline{\sum_{i=1}^{n} \boldsymbol{\nabla} E_{pi} \cdot \boldsymbol{r}_{\boldsymbol{i}}} \tag{9}$$

In regards of a central force,

$$\boldsymbol{\nabla} E_p \cdot \boldsymbol{r} = -\boldsymbol{F} \cdot \boldsymbol{r} = -F\boldsymbol{e}_{\boldsymbol{r}} \cdot \boldsymbol{r} = \left(\frac{dE_p}{dr}\right) r = r\frac{dE_p}{dr} \tag{10}$$

This is because the force is parallel with the line connecting the force particle and the forced particle. If the potential function:

$$E_p = cr^{\alpha} \tag{11}$$

According to the derivations of power functions,

$$r\frac{dE_p}{dr} = \alpha E_p \tag{12}$$

Based on Eq. (10)-Eq. (12), one sums the right side of Eq. (9):

$$\overline{E_k} = \frac{\alpha}{2}\overline{E_p} \tag{13}$$

These are threads of proof in Zhou Yanbo's book Theoretical Mechanics Tutorial [7], together with the article's interpretation. However, the process to derive Eq. (13) is wrong, because the centers of forces are moving, not static! Therefore, a revision is needed. The author proposes a method to divide the total potential energy with the potential energy of each pair of particles as the basic unit, which is a correction to the proof process mentioned above. A sketch is shown in Fig. 1.

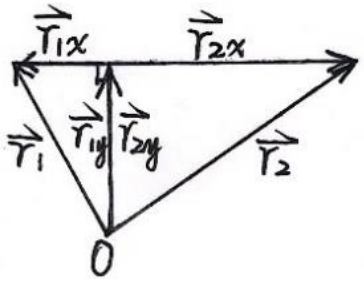


Figure 1. A sketch for variable.

According to Eq. (8), the value $\boldsymbol{\nabla} E_{pi}$ appeared in $\sum_{i=1}^{n} \boldsymbol{\nabla} E_{pi} \cdot \boldsymbol{r_i}$ on the right side of Eq. (9) means the gradient of E_{pi}, under the premise that all force particles do not move. Since $\boldsymbol{r_i}$ remains a constant as long as the origin point *O* is definite, the quantity $\boldsymbol{\nabla} E_{pi} \cdot \boldsymbol{r_i}$ can be split into the sum of $\boldsymbol{\nabla} E_{pi} \cdot \boldsymbol{r_i}$ in which E_{pi} means the potential energy each force particle stimulates on the particle *i*. Considering the forces and potentials of a pair of particles, $\boldsymbol{e_r}$: unit vector 2→1, 1&2 moving away as the positive direction. Though $E_{p1} = E_{p2}$,

$$\boldsymbol{\nabla} E_{p1} = -\boldsymbol{\nabla} E_{p2} \tag{14}$$

As mentioned above, $\boldsymbol{\nabla} E_{p1}$ is actually $\boldsymbol{\nabla_{r_2}} E_{p1}$ (the object 2 do not move); $\boldsymbol{\nabla} E_{p2}$ is actually $\boldsymbol{\nabla_{r_1}} E_{p2}$, the both sides' derivatives are with respect to different position vectors. In addition, because $\boldsymbol{\nabla} E_{pi} \parallel \boldsymbol{e_r}$,

$$\boldsymbol{\nabla} E_{p1} \cdot \boldsymbol{r_1} + \boldsymbol{\nabla} E_{p2} \cdot \boldsymbol{r_2} = r \frac{dE_p}{dr} \tag{15}$$

Whose derivation process is

$$\begin{aligned} \boldsymbol{\nabla} E_{p1} \cdot \boldsymbol{r_1} + \boldsymbol{\nabla} E_{p2} \cdot \boldsymbol{r_2} &= -\boldsymbol{F_{2\to1}} \cdot \boldsymbol{r_{1x}} - \boldsymbol{F_{2\to1}} \cdot \boldsymbol{r_{1y}} - \boldsymbol{F_{1\to2}} \cdot \boldsymbol{r_{2x}} - \boldsymbol{F_{1\to2}} \cdot \boldsymbol{r_{2y}} \\ &= -\boldsymbol{F_{2\to1}} \cdot \boldsymbol{r_{1x}} - \boldsymbol{F_{1\to2}} \cdot \boldsymbol{r_{2x}} = -F\boldsymbol{e_r} \cdot \boldsymbol{r_{1x}} + F\boldsymbol{e_r} \cdot \boldsymbol{r_{2x}} \\ &= -F\boldsymbol{e_r} \cdot (\boldsymbol{r_{1x}} - \boldsymbol{r_{2x}}) = -F\boldsymbol{e_r} \cdot (r\boldsymbol{e_r}) = -Fr = \left(\frac{dE_p}{dr}\right) r = r\frac{dE_p}{dr} \end{aligned} \tag{16}$$

That is to say, the value $\boldsymbol{\nabla} E_{p1} \cdot \boldsymbol{r_1} + \boldsymbol{\nabla} E_{p2} \cdot \boldsymbol{r_2}$ is independent of the selection of the origin point *O*. Process from Eq. (7)-Eq. (9) and that in Eq. (15) is in essence a set of transformation and inverse transformation because of the parallel relationship between the pair of central forces and the line connecting the two particles. Then

$$\sum_{i=1}^{n} \boldsymbol{\nabla E_{pi}} \cdot \boldsymbol{r_i} = \sum_{j}^{\boldsymbol{Each\ Pair}} \left(\boldsymbol{\nabla E_p} \cdot \boldsymbol{r}\right)_j = \sum_{j}^{\boldsymbol{Each\ Pair}} \boldsymbol{\alpha E_{pj}} = \boldsymbol{\alpha} \sum_{j}^{\boldsymbol{Each\ Pair}} \boldsymbol{E_{pj}} = \boldsymbol{\alpha E_p} \tag{17}$$

Therefore, Eq. (13) is rigorously proved. As a result, for a system satisfying $F \propto r^{-2}$, like a gravitational force system, $\alpha = -1$, then

$$\overline{\boldsymbol{E_k}} = -\frac{1}{2}\overline{\boldsymbol{E_p}},\ \boldsymbol{E} = \boldsymbol{E_k} + \boldsymbol{E_p} = \frac{1}{2}\overline{\boldsymbol{E_p}} \tag{18}$$

For a system satisfying $F \propto r$, like a simple harmonic force system, $\alpha = 2$, then

$$\overline{E_k} = \overline{E_p},\ E = 2\overline{E_k} = 2\overline{E_p} \tag{19}$$

As to an electromagnetic force system, if radiation is ignored, the force of particles will approach $F \propto r^{-2}$ and the virial theorem can be applied.

2.3. Applications in Cosmology

The virial theorem has a lot of important applications in cosmology. It can be used to understand dynamics and properties of cosmic structures and large-scale distribution of matter in the universe. The virial theorem is a magic weapon for understanding the behaviors of galaxies: the births, the motions, and the fates. For the formation of stars, the particles of initial nebulas are in constant collisions, and their mechanical energy is diffused by heat diffusion and direct electromagnetic radiation, then the distances of particles shrink and the average speed of particles increases; with the majority of kinetic energy transformed into internal energy, high enough to cause a nuclear fusion, the particles finally cluster and form a star; this is the application of that commonsense-contradicting property, as the 2.1 part said above; an important prerequisite is that the energy change process needs to be slow so that $\tau \to \infty$ can be satisfied, which is to say, the process of energy change is quasi-steady, and since the energy change of nebula motion is slow compared with the system energy, the virial theorem can be applied without doubt. As to galaxy dynamics, for example, regarding the motion of particles or planets going around a star, which can be simplified to the two-bodies problem, the virial theorem clarifies an invariant relation between different kinds of energy, which brings out insights into the two-bodies problem; however, the prerequisite of virial theorem is the total energy of the particle or planet is less than zero, otherwise the motion will be both neither periodic nor of finite position vector.

Furthermore, the virial theorem can even explain more complex problems like galaxy rotation curves, modified in consideration of Hubble expansion. In a cosmoscopic scale, the radius of particles that obey the virial theorem (virial radius) is finite. When the distance to the star is too far, the position vector and velocity may be infinite and break the prerequisites due to Hubble expansion. The virial radius satisfies the formula $\rho(< r_{vir}) = \Delta_c \rho_c(t)$ in which $\rho(< r_{vir})$ is the average density of space inside virial radius, Δ_c is a constant related to cosmological parameters and ρ_c is the critical density of the cosmos. Then, the virial theorem can be applied to analyze the likelihood of mergers, collapse or expansion of galaxies by comparing the kinetic and potential energies over time.

The virial theorem is also of great use to research into mass distribution. By kinetic and potential energies in a cosmological structure derived from observable properties like positions and velocities, some of which can be described by galaxy rotation curves, scientists can estimate total mass of the structure, find dark matter and propose reasonable hypotheses about dark matter, including estimation of dark matter content (seen from Fig. 2). The virial theorem also attaches much to specific structures of dark matter. In a dark matter halo, there are clumps which are virial parts called subhalos in a dark matter halo and have many properties different from outer halo parts. The virial theorem is also used to test cosmological models [8] and explain thermodynamical laws of stars [9].

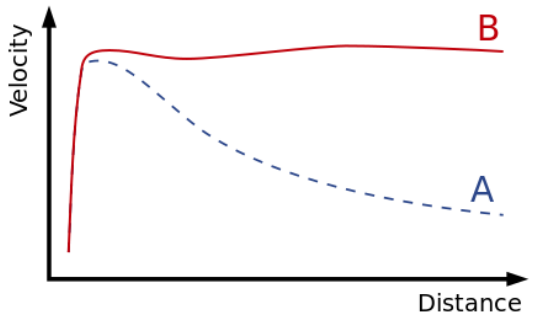


Figure 2. Reasons for existence of dark matter.

3. Poisson-Vlasov Equations

3.1. Brief Description

The Poisson-Vlasov equations consist of Poisson equation and Vlasov Equation. Poisson equation is equivalent to Newton's law of universal gravitation and relates the potential to the mass or charge density of the particles, specifically, in a cosmological background, the gravitational Poisson equation attaches the gravitational potential to the mass density. The Vlasov equation, which is derived from Liouville's theorem, is a function that describes the phase space density variation of particles with time. The Poisson-Vlasov equations are the key to solving problems of particle motion in the universe.

3.2. Poisson Equation

The Poisson Equation's proof is not too difficult, and what is needed is just to imitate the electric Gauss theorem:

$$\oiint_S \boldsymbol{E} \cdot \boldsymbol{dS} = \frac{Q_e}{\varepsilon_0} = 4\pi k Q_e \tag{20}$$

According to definition of divergence,

$$\boldsymbol{\nabla} \cdot \boldsymbol{E} = \frac{\rho_e}{\varepsilon_0} = 4\pi k \rho_e \tag{21}$$

Just do a similar calculation, then Poisson equation can be derived:

$$\boldsymbol{\nabla^2}_{\Phi G} = \boldsymbol{\nabla} \cdot \frac{\boldsymbol{F}}{\boldsymbol{m}} = 4\pi G \rho \tag{22}$$

Although its derivation is not troublesome, its solution is a problem.

3.3. Liouville's Theorem and Vlasov Equation

First, introduce some basic definitions. The mass density ρ in the 3D space:

$$\boldsymbol{\rho} = \frac{\boldsymbol{dm}}{\boldsymbol{dV}} = \frac{\boldsymbol{d^3 m}}{\boldsymbol{dxdydz}} \tag{23}$$

The mass density f in the 6D phase space:

$$\boldsymbol{f} = \frac{\boldsymbol{d^6 m}}{\boldsymbol{dxdydzdv_x dv_y dv_z}} \tag{24}$$

Then, it can be derived that

$$\boldsymbol{f} = \frac{\boldsymbol{d^3 \rho}}{\boldsymbol{dv_x dv_y dv_z}} \tag{25}$$

That is equivalent to

$$\boldsymbol{\rho} = \iiint_{R^3} \boldsymbol{f}(\boldsymbol{x}, \boldsymbol{v}, \boldsymbol{t}) \boldsymbol{dv} \tag{26}$$

The difference between the two quantities $\frac{\partial f}{\partial t}$ and $\frac{\mathrm{d}f}{\mathrm{d}t}$ also needs to be clarified: $\partial f/\partial t$ is phase space density's derivative to time at the position $\boldsymbol{x}$, $\boldsymbol{v}$; df/dt is phase space density's derivative to time of the matter whose initial position is $\boldsymbol{x}$, $\boldsymbol{v}$. Obviously, $\partial f/\partial t$ is not equal to zero generally;

when the initial phase position is given, its final position is definite; however, what one selects is a differential unit, whose matter may concentrate or diffuse, so df/dt is not essentially equal to zero as well. However, Liouvelle's Theorem says that as in a single-phase system, the phase density df/dt maintains a constant. To prove the Liouville's theorem, first review some basic quantities in analytical mechanics. The independent variables are generalized coordinates $q_\alpha(1 \le \alpha \le n)$ and in a conservative mechanical system, Lagrange quantity is $L = T - V$, where *T* is kinetic energy and *V* is potential energy, *T*, *V* is the function of $\cup_{i=1}^{n} q_\alpha$, $\cup_{i=1}^{n} \dot{q_\alpha}$. The generalized momentums

$$\boldsymbol{p_\alpha = \frac{\partial L}{\partial \dot{q_\alpha}} = \frac{\partial T}{\partial \dot{q_\alpha}} (1 \le \alpha \le n)} \tag{27}$$

Because the potential energy *V* is independent of generalized velocity $\dot{q_\alpha}$ For instance, if q_α is selected as *x*, *y*, *z*, p_α will be p_x, p_y, p_z, since

$$\boldsymbol{p_x = m v_x} \tag{28}$$

And

$$\boldsymbol{T = \frac{1}{2} m v^2 = \frac{1}{2} m (v_x{}^2 + v_y{}^2 + v_z{}^2)} \tag{29}$$

It can be derived that

$$\boldsymbol{p_x = \frac{\partial T}{\partial v_x}} \tag{30}$$

The Hamilton quantity:

$$\boldsymbol{H = -L + \sum_{i=1}^{n} p_\alpha \dot{q_\alpha}} \tag{31}$$

Which satisfy

$$\boldsymbol{\dot{q_\alpha} = \frac{\partial H}{\partial p_\alpha}, \ \dot{p_\alpha} = -\frac{\partial H}{\partial q_\alpha}} \tag{32}$$

Select any differential unit in the phase space of q_α, $p_\alpha(1 \le \alpha \le n)$, so the differential unit $\mathrm{d}\tau = \prod_{\alpha=1}^{n} (\mathrm{d}q_\alpha \mathrm{d}p_\alpha)$. Then,

$$\boldsymbol{\frac{dN}{d\tau} = f_N(t; \cup_{i=1}^{n} q_\alpha ; \cup_{i=1}^{n} \dot{q_\alpha})} \tag{33}$$

And because

$$\boldsymbol{\frac{df_N}{dt} = \frac{\partial f_N}{\partial t} + \sum_{\alpha=1}^{n} \left(\frac{\partial f_N}{\partial q_\alpha} \dot{q_\alpha} + \frac{\partial f_N}{\partial p_\alpha} \dot{p_\alpha} \right)} \tag{34}$$

It can be derived that

$$\boldsymbol{\frac{\partial f_N}{\partial t} = \frac{\partial^2 N}{\partial \tau \partial t}} \tag{35}$$

In regards of a pair of "surfaces" q_α, $q_\alpha + \mathrm{d}q_\alpha$, its corresponding base area (differential unit) $\mathrm{d}A_\alpha = \prod_{\substack{i=1 \\ i \ne \alpha}}^{n} \mathrm{d}q_i \prod_{i=1}^{n} \mathrm{d}p_i$

The incoming flux from q_α equals to $(f_N \dot{q_\alpha} \mathrm{d}t\mathrm{d}A_\alpha)|_{q_\alpha}$ while the outcoming flux from $q_\alpha + \mathrm{d}q_\alpha$ equals to $[(f_N \dot{q_\alpha})|_{q_\alpha} + \frac{\partial(f_N \dot{q_\alpha})}{\partial q_\alpha}\mathrm{d}q_\alpha]\mathrm{d}t\mathrm{d}A_\alpha$, then the net flux from the pair equals to $\frac{\partial(f_N \dot{q_\alpha})}{\partial q_\alpha}\mathrm{d}q_\alpha \mathrm{d}t\mathrm{d}A_\alpha = \frac{\partial(f_N \dot{q_\alpha})}{\partial q_\alpha}\mathrm{d}t\mathrm{d}\tau$. Then:

$$\frac{\partial f_N}{\partial t} = \frac{\partial^2 N}{\partial \tau \partial t} = -\sum_{\alpha=1}^{n} \frac{\partial(f_N \dot{q_\alpha})}{\partial q_\alpha} + \frac{\partial(f_N \dot{p_\alpha})}{\partial p_\alpha} = -\sum_{\alpha=1}^{n} \left(\frac{\partial f_N}{\partial q_\alpha}\dot{q_\alpha} + \frac{\partial f_N}{\partial p_\alpha}\dot{p_\alpha}\right) - f_N \sum_{\alpha=1}^{n} \left(\frac{\partial \dot{q_\alpha}}{\partial q_\alpha} + \frac{\partial \dot{p_\alpha}}{\partial p_\alpha}\right) \quad (36)$$

So,

$$\frac{df_N}{dt} = \frac{\partial f_N}{\partial t} + \sum_{\alpha=1}^{n} \left(\frac{\partial f_N}{\partial q_\alpha}\dot{q_\alpha} + \frac{\partial f_N}{\partial p_\alpha}\dot{p_\alpha}\right) = -f_N \sum_{\alpha=1}^{n} \left(\frac{\partial}{\partial q_\alpha}\frac{\partial H}{\partial p_\alpha} - \frac{\partial}{\partial p_\alpha}\frac{\partial H}{\partial q_\alpha}\right) \quad (37)$$

If the system is conventional, q_α, $\dot{q_\alpha}$, p_α, $\dot{p_\alpha}$ is independent to each other,

$$\frac{df_N}{dt} = 0 \quad (38)$$

The process above is this article's interpretation and modification based on Zhou's proof [7], and below is the proof of Vlasov equation. As to a single-phase system, there is

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + \nabla f \cdot v + \nabla_v f \cdot (-\nabla_{\Phi G}) = 0 \quad (39)$$

Because let n=3, q_1=x, q_2=y, q_3=z, then $p_1 = p_x$, $p_2 = p_y$, $p_3 = p_z$, $\mu :=$ the molar mass, then

$$m = \mu N \quad (40)$$

So,

$$f = \frac{d^6 m}{dxdydzdv_x dv_y dv_z} = \mu m^3 f_N \frac{d^6 N}{dxdydzdp_x dp_y dp_z} \quad (41)$$

Due to Eq. (38)

$$\frac{df}{dt} = 0 \quad (42)$$

Performing a simple identity transformation, the Vlasov equation can be finally derived.

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + \nabla f \cdot v + \nabla_v f \cdot (-\nabla_{\Phi G}) = 0 \quad (43)$$

3.4. Poisson-Vlasov Equations' Solution and Applications in Cosmology Simulation

Cosmological simulations like IllustrisTNG and THESAN (seen from Fig. 3) use Poisson-Vlasov equations to imitate dark matter: the Poisson equation Eq. (22) and Vlasov equation Eq. (43). In the ΛCDM model, the lack of thermal effect of dark matter means there is no collision and no heat diffusion but only gravitational force. And because dark matter halos usually take up the outer space of a galaxy and occupy most of the space mass, they can be regarded as an isolated system without external force. So, the dark matter halos are conservative mechanical systems and obey the Vlasov equation Eq. (38). What's more, the boundary condition comes to $\Phi_G = 0$, which belongs to that of

the 1st kind, the simplest one. The problem can even be simplified by that the dark matter particles in the same halo are the same, like WIMPs, then, a simpler form of Vlasov equation Eq. (39) can be applied. By solving Poisson-Vlasov Equations, analytical or numerical solution of the complete motion process can be obtained. The reason can be easily known by a classical numerical solution [2, 10]. For a certain moment, the 1st step to solute the equations is to solute $\boldsymbol{\nabla\Phi_G}$ by Poisson equation, while the 2nd step is to solute Vlasov equation by $\boldsymbol{\nabla\Phi_G}$ and initial conditions of distribution in $\boldsymbol{x}$, $\boldsymbol{v}$, which can be given by dark matter distribution like [11]:

$$\boldsymbol{\rho(r) = \frac{\rho_0}{1+\left(\frac{r}{r_c}\right)^2}} \tag{44}$$

Since the only unknown quantity is f, $\frac{\partial f}{\partial t}$ in Vlasov equation can be solved and the quantities in the next time step can be derived, step by step, the complete process can be imitated. And for the analytical solution, theoretically, the Poisson-Vlasov equations has 3 unknown quantities Φ_G, f, ρ. By introducing a constraint Eq. (26) about the relation between f and ρ, the solution's existence and uniqueness are confirmed. Reviewing the two equations, what can also be noticed is what appears in the two equations is $\Phi_{Gravity}$ not $F_{Gravity}$; that is because what is needed is to do scalar superposition if talking about $\Phi_{Gravity}$, which can make the calculation easier. If $F_{Gravity}$ is chosen instead, a vector superposition is necessary and the calculation is complicated.

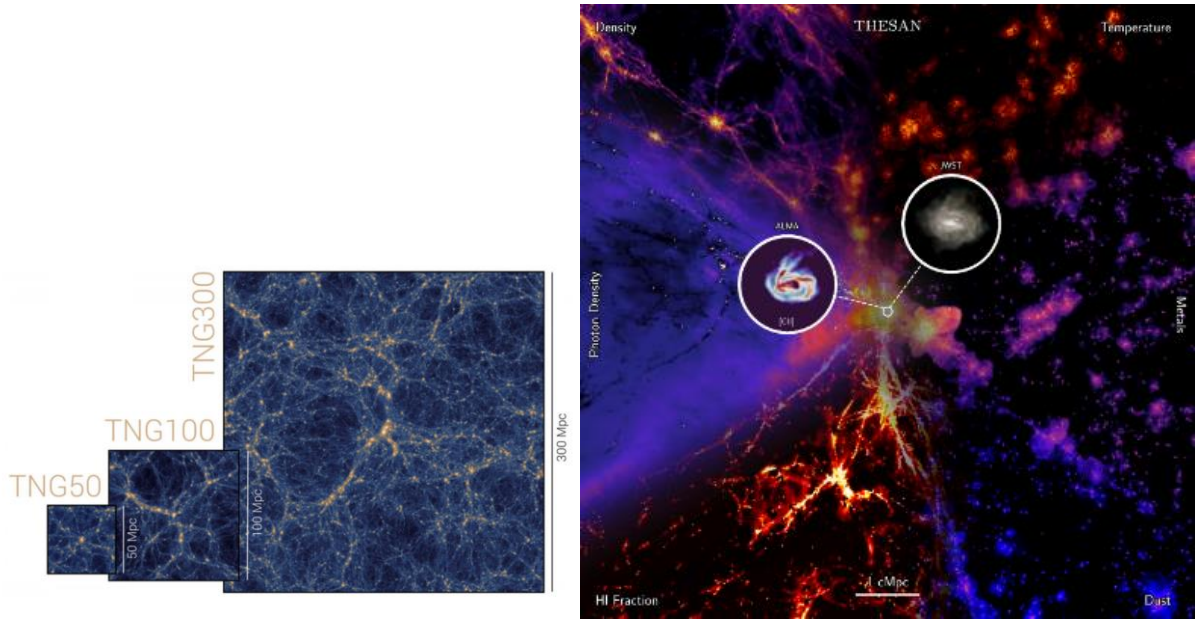


Figure 3. A sketch of IllustrisTNG and THESAN.

Here's the solution of Poisson Equation, partly learn from Liang Kunmiao's version [12]. One thinks about the divergence of any gravitational field ($\boldsymbol{\nabla^2\Phi_G}$). As to a constant distribution of mass, Gauss theorem is available, however, as to an asymmetrical gravitational field it cannot go to every space point, so the total mass needs to be divided into differential units; but for a gravitational field stimulated by differential units, the corollary of Gauss theorem can be used directly to describe the motion. As to a field stimulated by a differential mass that can be regarded as a single particle, the divergence goes to infinity while that of other points goes to zero, so it is needed to import a δ-function like that in the proof. One defines an abstract function

$$\boldsymbol{\delta(r-r_0) = \begin{cases} \infty & (r_0 = r) \\ 0 & (r_0 \neq r) \end{cases}} \tag{45}$$

Which satisfies:

$$\iiint_V \boldsymbol{\delta}(\boldsymbol{r}-\boldsymbol{r_0})\boldsymbol{dr_0} = \begin{cases} \mathbf{1} & (\boldsymbol{r}\in\boldsymbol{V}) \\ \mathbf{0} & (\boldsymbol{r}\notin\boldsymbol{V}) \end{cases} \tag{46}$$

A single particle's gravitational field can be described as

$$\boldsymbol{\nabla^2}_{\boldsymbol{\Phi G}} = \boldsymbol{4\pi G m \delta}(\boldsymbol{r}-\boldsymbol{r_0}) \tag{47}$$

That's because these three factors: at the particle point, divergence of the gravitational field is infinity since the mass remains a constant (not zero) as the volume comes to zero; at any other point in the space, the divergence is zero; and another formula derived from a similar calculation imitating Eq. (20):

$$\iiint_V \boldsymbol{\nabla^2}_{\boldsymbol{\Phi G}}\, \boldsymbol{dV} = \oiint_S \boldsymbol{\nabla}_{\boldsymbol{\Phi G}} \cdot \boldsymbol{dS} = \oiint_S \frac{\boldsymbol{F}}{\boldsymbol{m}} \cdot \boldsymbol{dS} = \boldsymbol{4\pi G m} \tag{48}$$

The δ-function meets the formula. Write Eq. (47) in a differential form:

$$\boldsymbol{d\nabla^2}_{\boldsymbol{\Phi G}} = \boldsymbol{4\pi G \delta}(\boldsymbol{r}-\boldsymbol{r_0})\boldsymbol{dm} = \boldsymbol{4\pi G \rho \delta}(\boldsymbol{r}-\boldsymbol{r_0})\boldsymbol{dr_0} \tag{49}$$

Then, an equivalence can be done. As to a gravitational field,

$$\begin{cases} \boldsymbol{\nabla^2}_{\boldsymbol{\Phi G}} = \boldsymbol{4\pi G \rho}, & \boldsymbol{r}\in\boldsymbol{V} \\ \boldsymbol{\Phi_G} = \mathbf{0}, & \boldsymbol{r}\in\boldsymbol{\partial V} \end{cases} \tag{50}$$

One defines $g(\boldsymbol{r},\boldsymbol{r_0})$ as

$$\begin{cases} \boldsymbol{\nabla_{r_0}}^2 \boldsymbol{g}(\boldsymbol{r},\boldsymbol{r_0}) = \boldsymbol{\delta}(\boldsymbol{r}-\boldsymbol{r_0}), & \boldsymbol{r},\boldsymbol{r_0}\in\boldsymbol{V} \\ \boldsymbol{g}(\boldsymbol{r},\boldsymbol{r_0}) = \mathbf{0}, & \boldsymbol{r_0}\in\boldsymbol{\partial V} \end{cases} \tag{51}$$

In which $\boldsymbol{r}$ means the point of discussion of physical quantities, $\boldsymbol{r_0}$ means each point in the space, ∂V is the boundary of the domain. Differentiate $\boldsymbol{\nabla}^2\Phi_G$ into the sum of $\mathrm{d}\boldsymbol{\nabla}^2\Phi_G$ excited by the mass on each microelement. From Eq. (49) and Eq. (50):

$$\boldsymbol{d}_{\boldsymbol{\Phi G}}(\boldsymbol{r}) = \boldsymbol{4\pi G \rho}(\boldsymbol{r_0})\boldsymbol{dr_0} \tag{52}$$

So,

$$\boldsymbol{\Phi_G}(\boldsymbol{r}) = \boldsymbol{4\pi G} \iiint_V \boldsymbol{\rho}(\boldsymbol{r_0})\boldsymbol{g}(\boldsymbol{r},\boldsymbol{r_0})\boldsymbol{dr_0} \tag{53}$$

This is because according to superposition principle of potential, the potential each differential portion of mass stimulates is summarized, and another rigorous proof of the formula Eq. (53) is

$$\begin{aligned} &\iint_{\partial V}\left[\Phi_G(\boldsymbol{r_0})\boldsymbol{\nabla}_{\boldsymbol{r_0}}g(\boldsymbol{r},\boldsymbol{r_0}) - g(\boldsymbol{r},\boldsymbol{r_0})\boldsymbol{\nabla}_{\boldsymbol{r_0}}\Phi_G(\boldsymbol{r_0})\right]\cdot \mathrm{d}\boldsymbol{S_0} = \iiint_V \boldsymbol{\nabla}_{\boldsymbol{r_0}}\cdot\left[\Phi_G(\boldsymbol{r_0})\boldsymbol{\nabla}_{\boldsymbol{r_0}}g(\boldsymbol{r},\boldsymbol{r_0}) - g(\boldsymbol{r},\boldsymbol{r_0})\boldsymbol{\nabla}_{\boldsymbol{r_0}}\Phi_G(\boldsymbol{r_0})\right]\mathrm{d}\boldsymbol{r_0} \\ &= \iiint_V\left[\boldsymbol{\nabla}_{\boldsymbol{r_0}}\Phi_G(\boldsymbol{r_0})\cdot\boldsymbol{\nabla}_{\boldsymbol{r_0}}g(\boldsymbol{r},\boldsymbol{r_0}) + \Phi_G(\boldsymbol{r_0})\boldsymbol{\nabla}_{\boldsymbol{r_0}}{}^2 g(\boldsymbol{r},\boldsymbol{r_0}) - \boldsymbol{\nabla}_{\boldsymbol{r_0}}\Phi_G(\boldsymbol{r_0})\cdot\boldsymbol{\nabla}_{\boldsymbol{r_0}}g(\boldsymbol{r},\boldsymbol{r_0}) - g(\boldsymbol{r},\boldsymbol{r_0})\boldsymbol{\nabla}_{\boldsymbol{r_0}}{}^2\Phi_G(\boldsymbol{r_0})\right]\mathrm{d}\boldsymbol{r_0} \\ &= \iiint_V\left[\Phi_G(\boldsymbol{r_0})\boldsymbol{\nabla}_{\boldsymbol{r_0}}{}^2 g(\boldsymbol{r},\boldsymbol{r_0}) - g(\boldsymbol{r},\boldsymbol{r_0})\boldsymbol{\nabla}_{\boldsymbol{r_0}}{}^2\Phi_G(\boldsymbol{r_0})\right]\mathrm{d}\boldsymbol{r_0} = \iiint_V \Phi_G(\boldsymbol{r_0})\delta(\boldsymbol{r}-\boldsymbol{r_0})\mathrm{d}\boldsymbol{r_0} - \iiint_V g(\boldsymbol{r},\boldsymbol{r_0})\boldsymbol{\nabla}_{\boldsymbol{r_0}}{}^2\Phi_G(\boldsymbol{r_0})\mathrm{d}\boldsymbol{r_0} \\ &= \Phi_G(\boldsymbol{r}) - \iiint_V g(\boldsymbol{r},\boldsymbol{r_0})\boldsymbol{\nabla}_{\boldsymbol{r_0}}{}^2\Phi_G(\boldsymbol{r_0})\mathrm{d}\boldsymbol{r_0} = \Phi_G(\boldsymbol{r}) - 4\pi G\iiint_V \rho(\boldsymbol{r_0})g(\boldsymbol{r},\boldsymbol{r_0})\mathrm{d}\boldsymbol{r_0} \end{aligned} \tag{54}$$

Which needs Eq. (55) below.

Only $\Phi_G(\boldsymbol{r_0})$ at $\boldsymbol{r}$ counts, so in the left side of Eq. (55), $\Phi_G(\boldsymbol{r_0})$ can be seen as a constant $\Phi_G(\boldsymbol{r})$ and can be transferred outside. Then,

$$\iiint_V \boldsymbol{\Phi_G}(\boldsymbol{r_0})\boldsymbol{\delta}(\boldsymbol{r}-\boldsymbol{r_0})\boldsymbol{dr_0} = \boldsymbol{\Phi_G}(\boldsymbol{r}) \tag{55}$$

The boundary ∂V is selected as too far to be affected by the gravitational field, $\boldsymbol{\nabla_{r_0}} g(\boldsymbol{r}, \boldsymbol{r_0}) = 0$, $g(\boldsymbol{r}, \boldsymbol{r_0}) = 0$. Then Eq. (53), which means a convolution, is proven.

As Liang's book proves [12], there is a theorem of convolution about Fourier transformation:

$$\boldsymbol{F}[\boldsymbol{f_1} * \boldsymbol{f_2}] = \boldsymbol{2\pi F}[\boldsymbol{f_1}]\boldsymbol{F}[\boldsymbol{f_2}] \tag{56}$$

In which *F* means Fourier transformation (the *k* can be selected as any non-zero number):

$$\boldsymbol{F}[\boldsymbol{f}(\boldsymbol{x})] = \frac{\boldsymbol{1}}{\boldsymbol{2\pi}} \int_{-\infty}^{+\infty} \boldsymbol{f}(\boldsymbol{\xi}) \boldsymbol{e^{-ik\xi}} \boldsymbol{d\xi} \tag{57}$$

Whose inverse transformation is

$$\boldsymbol{f}(\boldsymbol{x}) = \int_{-\infty}^{+\infty} \boldsymbol{F}\,[\boldsymbol{f}(\boldsymbol{x})] \boldsymbol{e^{ikx}} \boldsymbol{dk} \tag{58}$$

And "*" means convolution. Likewise,

$$\boldsymbol{F}[\boldsymbol{f_1 f_2}] = \boldsymbol{F}[\boldsymbol{f_1}] * \boldsymbol{F}[\boldsymbol{f_2}] \tag{59}$$

Furthermore, one defines 3D Fourier transformation as

$$\boldsymbol{F}[\boldsymbol{f}(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z})] = \frac{\boldsymbol{1}}{\boldsymbol{8\pi^3}} \iiint_{\infty} \boldsymbol{f}(\boldsymbol{\xi}, \boldsymbol{\mu}, \boldsymbol{\eta})\, \boldsymbol{e^{-i(k_1\xi + k_2\mu + k_3\eta)}} \boldsymbol{d\xi d\mu d\eta} \tag{60}$$

Whose inverse transformation is

$$(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}) = \iiint_{\infty} \boldsymbol{F}[\boldsymbol{f}(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z})] \boldsymbol{e^{i(k_1 x + k_2 y + k_3 z)}} \boldsymbol{dk_1 dk_2 dk_3} \tag{61}$$

Then in 3D Fourier transformation,

$$\boldsymbol{F}[\boldsymbol{f_1} * \boldsymbol{f_2}] = \boldsymbol{4\pi^2 F}[\boldsymbol{f_1}]\boldsymbol{F}[\boldsymbol{f_2}] \tag{62}$$

The article's proof thread is to consider 3D Fourier transformation into three 1D Fourier transformations: Obviously, there is

$$\boldsymbol{F}[\boldsymbol{f}] = \boldsymbol{F_z F_y F_x}[\boldsymbol{f}] \tag{63}$$

Then the formula can be deduced:

$$\begin{gathered} \boldsymbol{F_x}[\boldsymbol{f_1} * \boldsymbol{f_2}] = \boldsymbol{2\pi F_x}[\boldsymbol{f_1}]\boldsymbol{F_x}[\boldsymbol{f_2}] \Rightarrow \boldsymbol{F_y F_x}[\boldsymbol{f_1} * \boldsymbol{f_2}] = \boldsymbol{2\pi F_y}(\boldsymbol{F_x}[\boldsymbol{f_1}]\boldsymbol{F_x}[\boldsymbol{f_2}]) = \boldsymbol{2\pi F_y F_x}[\boldsymbol{f_1}] * \boldsymbol{F_y F_x}[\boldsymbol{f_2}] \\ \Rightarrow \boldsymbol{F_z F_y F_x}[\boldsymbol{f_1} * \boldsymbol{f_2}] = \boldsymbol{2\pi F_z}(\boldsymbol{F_y F_x}[\boldsymbol{f_1}] * \boldsymbol{F_y F_x}[\boldsymbol{f_2}]) = \boldsymbol{4\pi^2 F_z F_y F_x}[\boldsymbol{f_1}]\boldsymbol{F_z F_y F_x}[\boldsymbol{f_2}] \end{gathered} \tag{64}$$

Since Fourier transformation is a bijection, some quantities can be calculated by transforming forth and back in this way. Moreover, due to Fourier transformation turning a convolution into a simple multiplication, it can simplify the calculation. Furthermore, although the δ-function is strange, it can do Fourier transformation because Fourier transformation is defined by integration which the function meets. On the other hand, in the differential process, *m* turns to *ρ*, so the final solution doesn't contain infinity, which meets reality. Applied in Poisson Equation:

$$\boldsymbol{F[\Phi_G(r)] = 4\pi G F\left[\iiint_V \rho(r_0) g(r, r_0) dr_0\right] = 16\pi^3 F[\rho(r_0)] F[g(r, r_0)]} \quad (65)$$

Rewriting the formula as

$$\boldsymbol{\hat{\Phi}(k_1, k_2, k_3) = 16\pi^3 G \hat{g}(k_1, k_2, k_3) \hat{\rho}(k_1, k_2, k_3)} \quad (66)$$

By inversed Fourier transformation, one solutes

$$\boldsymbol{\Phi_G = \iiint_\infty \hat{\Phi}(k_1, k_2, k_3) e^{i(k_1 x + k_2 y + k_3 z)} dk_1 dk_2 dk_3} \quad (67)$$

For Vlasov Equation, there is no general expression of solution even if Φ_G is known; Poisson-Vlasov equations' solution is even more complex so that at most times it cannot be expressed in an analytical form. Anyway, numerical solutions are the paradigm for solutions of Poisson-Vlasov equations in cosmological simulations.

4. Conclusion

There are many applications of statistical theorems in cosmology, and the virial theorem and Poisson-Vlasov equations are important representatives of statistical theorems. Just like that new cosmological models are based on these traditional theorems, however physics advances and changes, it cannot break away from the original foundations. For virial theorem, after the Hubble expansion discovered, it was not be eliminated but revised in advanced forms to adapt to cosmoscopic backgrounds like the FLRW metric. This paper aims to go back to the original proofs and make them more accessible, by revising, interpreting and integrating. The paper has three limitations. First of all, there are only a few relevant basic theorems mentioned in the paper, a very small part. This means that the knowledge involved is not systematic enough. Second, there are not enough specific examples on problem solutions of these theorems. These omissions of details makes confusion and the issue mentioned in 1.3 not fully resolved. Third, for application parts, the article doesn't go into details very well. For example, due to space limitations, there is no mention of whether the solution to the Poisson-Vlasov equations has chaotic effects and how to deal with them. In the new era of physics and cosmology, with the increasing demands of accuracy and data processing, statistical methods applications in physics and cosmology will be more and more. New discoveries of statistical laws and establishments of statistical systems like Kaniadakis statistics, are around the corner.

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