

The Recent Progress and the State-of-art Applications of Navier Stokes Equation

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Abstract. Navier Stoke equation plays an important role in physics field to describe the movement of fluid. In description of movement of fluid, turbulent flow is difficult to describe because it cannot be predicted precisely for movement of every particle. In this paper, we present the basic information of Navier Stoke equation, history of developing Navier Stoke equation as well as solving method. In addition, the state-of-art applications in fluid mechanics are also demonstrated. Moreover, the limitation of Navier Stoke equation and its future prospect are proposed accordingly. These results shed light on guiding further exploration focusing on the fluid mechanics.

Keywords: Navier Stoke equation, physics, engineering.

1. Introduction

Navier Stoke equation is one of the basic equations in hydrodynamics, which describes how the velocity, pressure, temperature, and density of a moving fluid are related. In 1827, The equation of motion of viscous fluid is first formulated by Navier, which only consider the flow of incompressible fluid [1]. In 1831, Poisson put forward the motion equation of compressible fluid. In 1845, Saint-Venant and Stokes independently proposed that the viscosity coefficient should be a constant. All of these are called Navier-Stokes equation. The equations are extensions of Euler Equations and include the effects of viscosity on the flow [2].

The Navier Stokes equations consists of a time-dependent continuity equation for conservation of mass, three time-dependent conservation of momentum equations and a time-dependent conservation of energy equation. Moreover, these equations are a set of coupled differential equations and could. In theory, these equations can be solved for a given flow problem by using methods from calculus. However, in practice, these equations are too difficult to solve analytically [3-5].

In the past, it can be solved in 2D analytically, but solving these equations in 3D is still difficult even in nowadays. Previously, engineers made further approximations and simplifications to the equation set until they had a group of equations that they could solve. Contemporarily, high speed computers have been used to solve approximations to equations using a variety of techniques, e.g., finite difference, finite volume, finite element, and spectral methods [6]. It is considered as the most important unsolved problem as scientists cannot give general solutions to these equations.

The motivation of this paper is looking for the using of the N-S equation in any kinds of area and the future use of the N-S equation in other areas. This paper will illustrate Navier-Stokes equation. The rest part of the paper is organized as follows. The Sec. 2 will describe the basic information of Navier-Stokes equation. The Sec. 3 will show the simple solving method of Navier-Stokes equation by applying mathematic method of differentiation. Afterwards, the state-of-art applications will be demonstrated. Eventually, the Sec. 4 will make a brief conclusion.

2. Basic Description

A Navier-Stokes equations are a set of nonlinear partial differential equations that account for the character of nonlinearity in fluids due to convective acceleration [7-10]. For fluids in different dimensions, different assumptions are made to simplify the problem and generalize fluid flow for the application of Navier-Stokes. For example, one-dimensional flows simplify Navier-Stokes equations to a linear set of differential equations, whereas three-dimensional flows that model real fluids (e.g., Rayleigh-Benard flow) add layers of complexity to the calculation, leaving the problem often as an open research area rather than a solvable mathematical quandary [11]. Different solutions for the functions are schematically illustrated in Fig. 1.

Navier-Stokes equations are also seen as an accurate portrayal of the turbulence of fluids. For fluids of high Reynolds numbers (i.e., fluids whose behavior is dominated by the effects of inertia relative to resistance), the dynamical profile can be characterized using Navier-Stokes [12]. Although the numerical solutions cannot be known in certitude, due to the significantly different mixing-length scales involved in turbulence, and the fine changes in fluid motion over time that need to be accounted for in such an exact solution.

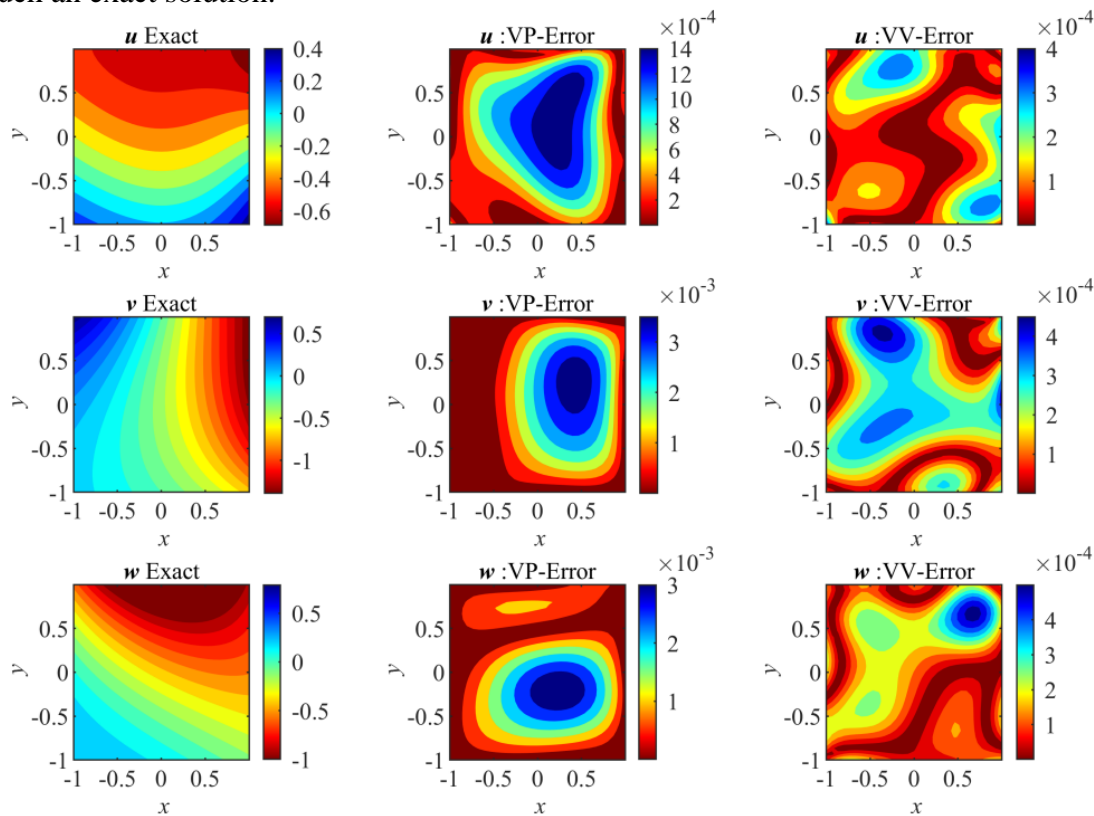


Figure 1. A sketch of the velocity solution [11].

3. Solving method

Mainly, we use the combination of physics and mathematical methods to solve Navier Stokes equations [7-9]. Following this is an example of solving Navier Stokes equations. As Navier Stokes equations are conservation of mass and momentum, one obtains:

$$\frac{\partial u_i}{\partial x_i} = 0 \quad (1)$$

Where t is the time, x_i is Cartesian coordinates, u_i is the corresponding velocity components, and p is the pressure. For Newtonian fluid, we have equation that can express τ_{ij} , which is viscous stress tensor is:

$$\tau_{ij} = v \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) \quad (2)$$

$$\frac{\partial u_i}{\partial t} = -\frac{\partial p}{\partial x_i} - \frac{\partial u_{iu_j}}{\partial x_j} + \frac{\partial \tau_{ij}}{\partial x_j} = -\frac{\partial p}{\partial x_i} + R_i \quad (3)$$

Here, v expresses the effective kinematic viscosity. For the interacted process, the time integration is dependent on operator splitting. It can be solved in many factors for the Navier Stokes equation including pressure, viscosity in momentum equations and so forth. In this case, it uses fractional method in differentiation to solve. It can be a complied in following two steps. Firstly, by applying momentum equations, it needs to solve auxiliary velocity field. To make auxiliary velocity onto divergence free velocity filed, it is needed to use pressure to do this work. By differentiating momentum equations, one derives

$$\frac{3u_i^* - 4u_i^n + u_i^{n-1}}{2\Delta t} = -\frac{\partial p^n}{\partial x_i} + R(u_i^*) \quad (4)$$

In this equation, represents auxiliary velocity field. The $R(u_i^*)$ is the equation of momentum in terms of convective and viscosity. It should take derivative. The velocity field is under the condition of incompressibility condition. It is gained by applying the following steps.

$$\frac{1}{\Delta t} (u^{n+1} - u^*) = -\nabla p' \quad (5)$$

In this equation $p' = p^{n+1} - p^n$, $n+1$ level of time, it is continuous equation.

$$\nabla \cdot u^{n+1} = 0 \quad (6)$$

By applying Poisson equation due pressure, we can get condition of incompressibility.

$$\nabla^2 p' = \frac{1}{\Delta t} \nabla \cdot u^* \quad (7)$$

By taking divergence of Eqs. (5) and (6), Poisson equation due pressure is gained. There is linearization error. In order to reduce Linearization error, we need focus on the purpose of sub-iteration procedure, which is different than in the artificial compressibility method. For the second method, it requires solution of steady-state problem at every moment of step.

4. Application

Navier-Stokes equations offer an accurate description of fluid motion under well-formulated boundary conditions. Unlike models which predict fluid behaviour with simplifications such as zero viscosity and incompressibility, Navier-Stokes equations model fluids in their real forms.

The specific problems that Navier-Stokes equations tackle concern fluids in a variety of flow states. The solution begins by setting some boundary values, then using scale analysis to simplify the problem. One such example is the movement of a fluid pass parallel plates. The scaled boundary value problem is,

$$\frac{d^2 u}{dy^2} = -1 \quad (8)$$

Where u is the initial velocity, y is the displacement relative to the vertical direction, and the boundary conditions given are $u(0) = u(1) = 0$. Solving the differential equation gives

$$u = \frac{1}{2} (y - y^2) \quad (9)$$

Which formulates the kinematic relationship between flow velocity and displacement for parallel flow. Another example is Rayleigh-Benard convection, a buoyancy-driven flow concerning the behaviour of fluids heated from below and cooled from above. Fluids in this state are turbulent, forming a regular pattern of spherical cells—Benard cells—on the surface. As fluids rise up from the bottom of the container, thermal diffusion, buoyancy, and internal friction interplay to influence the transfer of heat. Navier-Stokes equations set the boundary conditions to be such that the velocity at wall surfaces satisfy the condition of rigidity (i.e., $u(0) = 0$), and that the temperature is constant. The resulting dynamical system allows an accurate evaluation of the Rayleigh numbers of the fluid in different phases—during the onset of convection, and in plumes when the fluid is captured in a state of oscillatory instability.

4.1. The use of N-S equation in the aerodynamics

The continuous adjoint method is formulated in two-dimensional blade design by using N-S equation. Combining Thompson's theory of time-related boundary condition, we present the boundary conditions of the adjoint equation for internal flow and discuss the restrictions of cost function in the case of given surface temperature and adiabatic conditions on blade walls as illustrated in Fig. 2 [13].

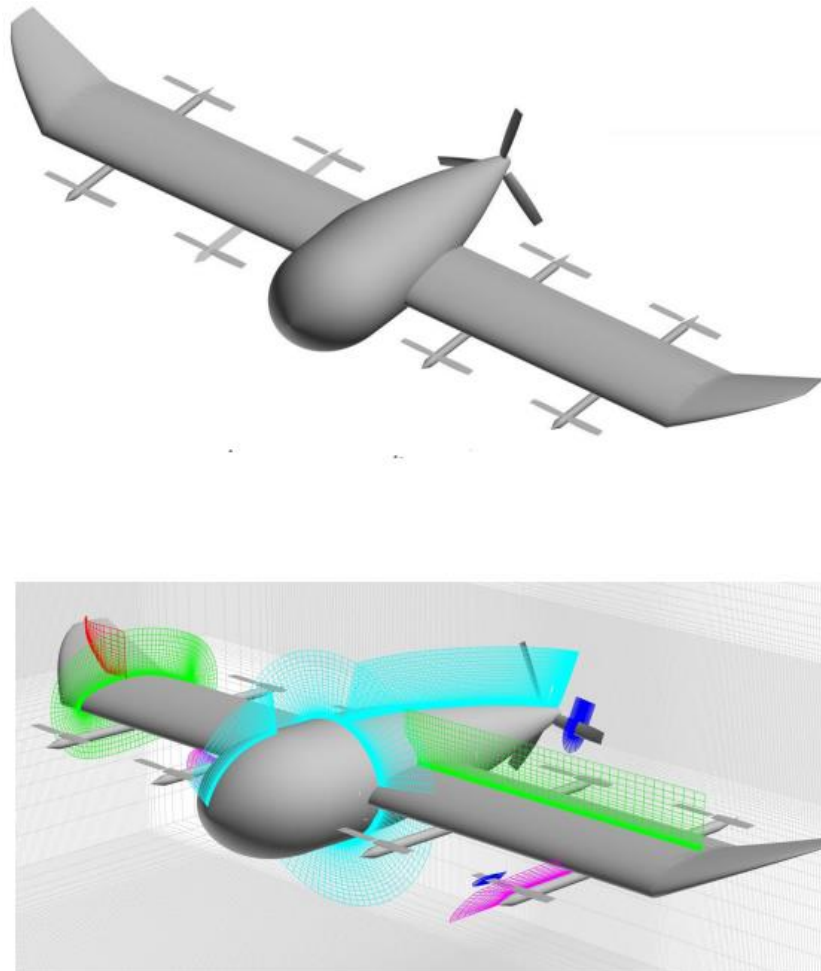


Figure 2. A sketch of the application in aerodynamics [13].

4.2. The use of N-S equation in Aerodynamic profile design

The use of the N-S equation has many kinds of use in many areas. Therefore, it also includes the use in the Aerodynamic profile design, such as design the size of the car for F1 race or the shape of a plane. Fig. 3 presents a sketch of the novel flying car with consideration of the N-S equations.

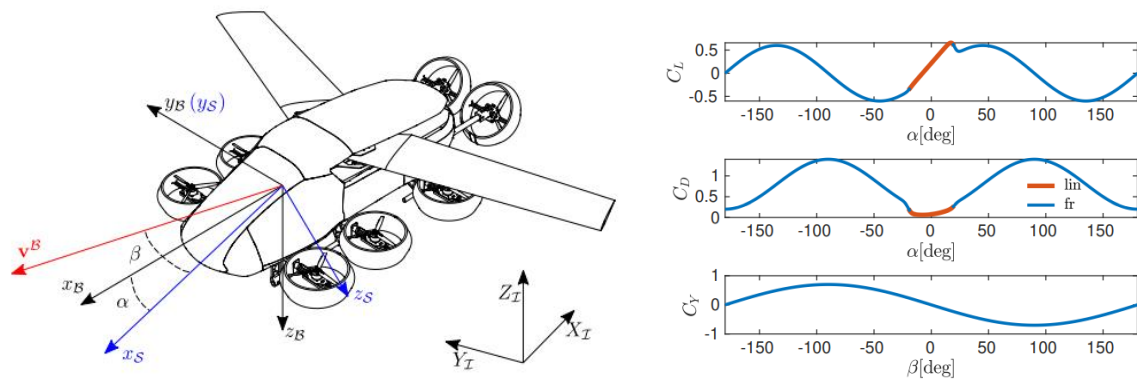


Figure 3. The implementation of N-S equations in flying cars [14].

4.3. The use of N-S equation in many kinds of marine disaster

The N-S equation can be used in many different kinds of conditions of the marine disaster because the N-S equation can make easy of many kinds of disaster such as tornado or tsunami, scholars can make the things into many lines and make it easier for people avoid the disasters. A numerical model is given in Ref. [15], where the simulations results are presented in Fig. 4.

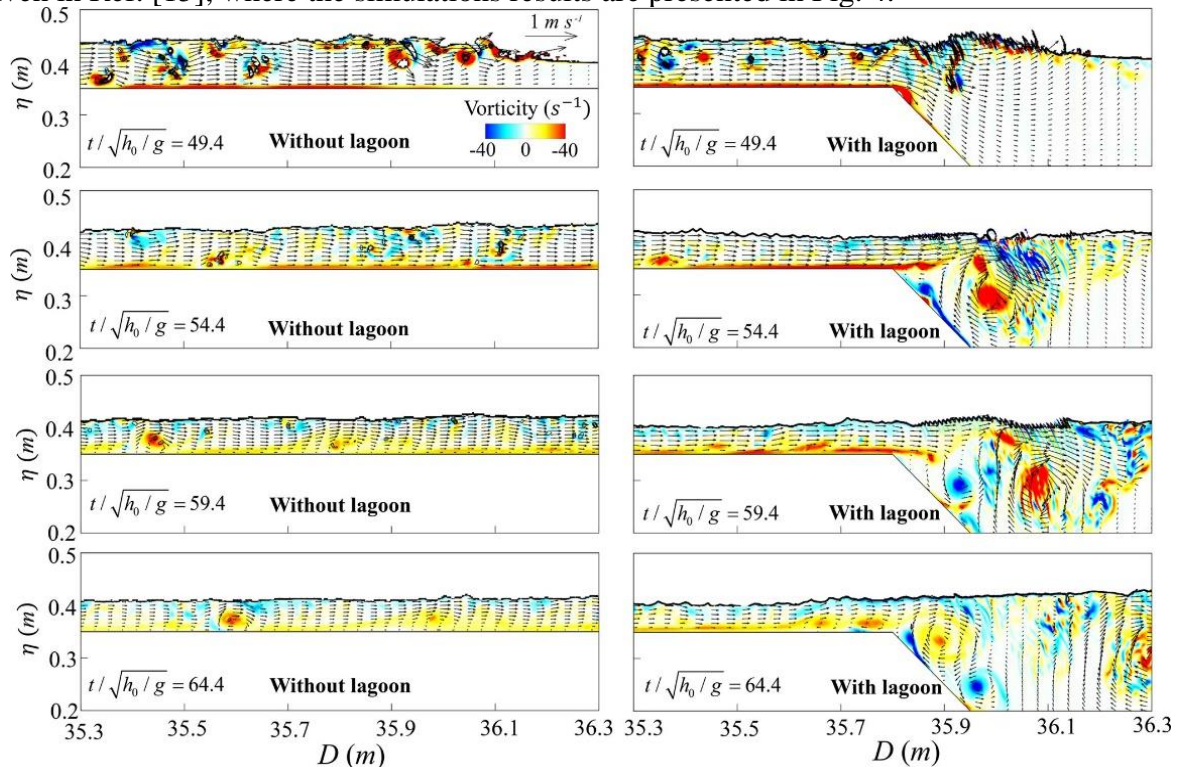


Figure 4. The simulations for tsunami based on N-S equations [15].

5. Limitation and future prospects

However, contemporary analysis for the equation still faces plenty of defects and drawbacks. To be specific, it is limited in the 2D applications, which means it cannot be applied in real life as we need 3D to solve problems in real phenomena because analytical method is impossible to solve. However, once Navier Stokes equations are solved in three dimensions, scholars can use it to build model in many fields such as weather forecast, the fluid movement in ocean current, and so forth. These models help people have better understanding of nature and can know movement about fluid earlier. It may help farmers who rely on ocean to have more preparation to face disasters. Therefore, this may develop global economy due to reduce supply-side shock by natural disasters.

Currently, scholars use computers to calculate Navier Stokes equation and want to make any progress on it. It has a few applications to push the process of solving Navier Stokes equation. In the future, Navier Stokes equation may be solved. Besides, when combined with other equations just like the law of conservation of mass with the good boundary conditions, the Navier-Stokes equation appears to be an accurate model of fluid motion; even in the turbulence conditions the equations just conform to its actual conditions. N-S equation is the equations that can focus on the research about the Fluid continuity (the object that made by the particle) and without the theory of relativity velocity. In some special conditions such as in very small space, that the object is made by discrete molecules it will be different with the describing of the N-S equations. It will cause some different results. Another limitation is the complexity of the equation. There is effective formula for characterizations of generally common fluid classes, but for rarer classes, when use the N-S equations will usually face some various complex conditions, even lead us some unsolved problems.

6. Conclusions

In summary, this paper discusses the N-S equation about the solution, future of the equation and the limitation of the equation from the perspective of us. In begin we talk about the solution of the equation, limitation and the future use of the equation. Besides, we describe the equation in various ways and the definition of the various equations. Third, we talk about the future use of the N-S equation in kinds of areas and the applications that already be used nowadays. N-S equation can solve many kinds of things in people's daily life such as whether forecast. However, there are still many kinds of problems that will be discovered and be solved by the N-S equation. In the future, Navier-Stokes equation may be applied in many fields to help scientists to know the better of the phenomenon which appear in the world. It can help to predict or model in the field of aerography, engineering, physics and so forth. Overall, these results offer a guideline for Navier-Stokes equation.

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