Total Least Squares Method to Solve the Parameter Estimation of Autoregression Model

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Abstract. Parameter estimation of autoregressive models is a fundamental problem in time series analysis. The total least squares estimation method can simultaneously take into account the errors of the coefficient matrix X and the observation vector Y, while correcting for both. In this paper, the total least squares is applied to the parameter estimation of the autoregressive AR model. The application example verifies the feasibility and effectiveness of the total least squares method used for the parameter estimation of the of the autoregressive model.

Keywords: Autoregressive model, parameter estimation, total least squares, least squares.

1. Introduction

The time series analysis method is a dynamic data processing method, which can use the autocorrelation characteristics among the observed data to establish the corresponding parametric model, and use the parametric model to scientifically analyze and process the observed data, and forecast the change trend in the future.Autogressive model is a model commonly used in time-series statistical analysis. Autogressive model is a function model to predict the future trend of variables based on their past changes, and is widely used in the processing of deformation monitoring data. For the parameter estimation methods of autoregressive models, there are mainly moment estimation method (Yule-Walker estimation), least squares estimation method, and maximum likelihood estimation method. The least squares estimation method is the most commonly used method, but this method only considers the error of vector Y and ignores the error of coefficient matrix X when solving the parameters of the autoregressive model. Therefore, when performing parameter estimation, the errors of both vector Y and coefficient matrix X should be considered and both should be corrected at the same time. Based on this idea, the total least squares method is applied to the parameter estimation of autoregressive model in this paper. The application example verifies the feasibility and effectiveness of the total least squares method used for the parameter estimation of the of the autoregressive model.

2. Parameter estimation of autoregressive model

The p-order autoregressive model can be described as follows:

$$x_t = \varphi_1 x_{t-1} + \varphi_2 x_{t-2} + \cdots + \varphi_p x_{t-p} + a_t \tag{1}$$

Where a_t is white noise, $a_t \sim NID(0, \sigma_a^2)$, φ_i (i=1, 2, …p)is autoregressive parameter; σ_a^2 is the variance of a_t ,p is the order of the model, Equation (1) is called p order autoregressive model and is recorded as AR (P).

Substituting t = P + 1, P + 2,…N into equation (1), the following linear equations can be obtained:

$$\begin{cases} x_{p+1} = \varphi_1 x_p + \varphi_2 x_{p-1} + \cdots + \varphi_p x_1 + a_{p+1} \\ x_{p+2} = \varphi_1 x_{p+1} + \varphi_2 x_p + \cdots + \varphi_p x_2 + a_{p+2} \\ \vdots \\ x_N = \varphi_1 x_{N-1} + \varphi_2 x_{N-2} + \cdots + \varphi_p x_{N-p} + a_N \end{cases} \tag{2}$$

Let $Y=\begin{pmatrix} x_{p+1} \\ x_{p+2} \\ \vdots \\ x_N \end{pmatrix}$, $X=\begin{pmatrix} x_p & x_{p-1} & \cdots & x_1 \\ x_{p+1} & x_p & \cdots & x_2 \\ \cdots & \cdots & \cdots & \cdots \\ x_{N-1} & x_{N-2} & \cdots & x_{N-p} \end{pmatrix}$, $\varphi=\begin{pmatrix} \varphi_1 \\ \varphi_2 \\ \vdots \\ \varphi_p \end{pmatrix}$, $a=\begin{pmatrix} a_{p+1} \\ a_{p+2} \\ \vdots \\ a_N \end{pmatrix}$

Then equation (2) can be expressed in matrix form as follows:

$$Y = X\varphi + a \tag{3}$$

Considering that both coefficient matrix X and observation vector y contain accidental errors, the model can be expressed as:

$$\underset{n-p,1}{Y} + \underset{n-p,1}{\varepsilon} = (\underset{n-p,p}{X} + \underset{n-p,p}{E_X})\,\underset{p,1}{\hat{\varphi}} \tag{4}$$

Let the mean vector and covariance matrix of the error vector be

$$\begin{bmatrix} \varepsilon \\ vec(E_X) \end{bmatrix} \sim N\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \sigma_0^2 \begin{bmatrix} I_{n-p} & 0 \\ 0 & I_p \otimes I_{n+p} \end{bmatrix}\right)$$

Where $vec(\bullet)$ represents the straightening transformation of the matrix, $\otimes$ represents the direct product of the matrix.

The above formula can also be rewritten as:

$$([X,Y]+[E_X,\varepsilon])\begin{bmatrix} \varphi \\ -1 \end{bmatrix} = 0 \tag{5}$$

The total least squares estimation method for solving the above equations can be transformed into the following constrained optimization problems:

$\min\limits_{E_X,\varepsilon} \|(E_X,\varepsilon)\|_F$

Constraint condition: $Y+\varepsilon \in Range(X+E_X)$

Where $\|\bullet\|_F$ is the Fronenius norm of the matrix.

In this paper, the singular value decomposition (SVD) method of matrix is used to solve the parameters. The solving steps are as follows:

(1)Establishing the function model.

$$\underset{n-p,1}{Y} + \underset{n-p,1}{\varepsilon} = (\underset{n-p,p}{X} + \underset{n-p,p}{E_X})\,\underset{p,1}{\hat{\varphi}}$$

(2)Singular value decomposition of augmented matrices

$$B=\begin{bmatrix} X & Y \end{bmatrix}=\begin{bmatrix} \underset{p+1}{U_1} & \underset{n-2p-1}{U_2} \end{bmatrix}\cdot\begin{bmatrix} \Sigma \\ 0 \end{bmatrix}\cdot V^T = U_1\cdot\Sigma\cdot V^T$$

Where $U_1=\begin{bmatrix} \underset{p}{U_{11}} & \underset{1}{U_{12}} \end{bmatrix}$

$$\Sigma\cdot V^T=\begin{bmatrix} \Sigma_1 & 0 \\ \underset{p}{0} & \underset{1}{\Sigma_2} \end{bmatrix}\begin{bmatrix} V_{11} & V_{12} \\ V_{21} & V_{22} \end{bmatrix}\begin{matrix} p \\ 1 \end{matrix}$$

Where $U=\begin{bmatrix} U_1 & U_2 \end{bmatrix}\in R^{(n-p)\times(n-p)}$ is an orthogonal matrix composed of (n-p) eigenvectors of matrix $\begin{bmatrix} X & Y \end{bmatrix}\cdot\begin{bmatrix} X & Y \end{bmatrix}^T = XX^T+YY^T$, $V=\begin{bmatrix} V_{11} & V_{12} \\ V_{21} & V_{22} \end{bmatrix}\begin{matrix} p \\ 1 \end{matrix}=\begin{bmatrix} v_1,v_2,\cdots,v_{p+1} \end{bmatrix}\in R^{(p+1)\times(p+1)}$ is an orthogonal matrix composed of (p+1)eigenvectors of matrix

$[X \quad Y]^T \cdot [X \quad Y] = \begin{bmatrix} X^T X & X^T Y \\ Y^T X & Y^T Y \end{bmatrix}$. $\Sigma = \begin{bmatrix} \Sigma_1 & 0 \\ 0 & \Sigma_2 \\ p & 1 \end{bmatrix} p = diag(\sigma_1, \sigma_2, \cdots, \sigma_{p+1})$ is the singular value of matrix B.

(3)If V_{22} is nonsingular, then the parameter estimation is as follows:

$$\hat{\varphi} = -V_{12} \cdot V_{22}^{-1}$$

(4) Accuracy evaluation

The unit weight variance is as follows: $\sigma_0^2 = \dfrac{\hat{v}}{n-2p}$

The covariance matrix of the parameters is as follows: $D(\hat{\varphi}) \approx \sigma_0^2 (S - \hat{v} I_p)^{-1} S (S - \hat{v} I_p)^{-1}$.

3. Experiment

In order to verify the feasibility and validity of the total least squares method proposed in this paper for the parameter solution of the autoregressive model, a one-dimensional AR(p) model was established by using the observed data from a bridge pier in the assessment section of a high speed railway settlement observation project, and the parameters were estimated by using the least squares estimation method and the total least squares estimation method respectively.

In this paper, experiments were conducted using observation data from a high speed railway settlement observation project with 45 periods of observation data. The model was established based on the data from the first 40 periods, and the data from the last 5 periods were predicted and analyzed.

The least squares estimation method and the total least squares estimation method based on SVD decomposition were used for calculation respectively. The prediction results were shown in Table 1.

Table 1. Comparison of the prediction results of the two methods.

Least Squares Estimation					Total Least Squares Estimation				
Period	Measured Value (dmm)	Predicted Value (dmm)	Absolute Error (dmm)	Relative Error (%)	Period	Measured Value (dmm)	Predicted Value (dmm)	Absolute Error (dmm)	Relative Error (%)
41	-20.6	-20.3	0.3	1.46	41	-20.6	-20.5	0.1	0.48
42	-20.5	-20.3	0.2	0.97	42	-20.5	-20.5	0.0	0
43	-20.5	-20.0	0.5	2.44	43	-20.5	-20.4	0.1	0.49
44	-20.2	-19.7	0.5	2.47	44	-20.2	-20.0	0.2	0.99
45	-20.1	-19.4	0.7	3.48	45	-20.1	-19.6	0.5	2.49

Mean square error (MSE) and mean absolute percentage error (MAPE) were used to judge the accuracy of the model. The accuracy evaluation of the two algorithms based on the resolution results of the least squares and the total least squares was shown in Table 2.

Table 2. Accuracy evaluation of the two methods.

Accuracy Index	Least Squares Estimation	Total Least Squares Estimation
MSE	0.224	0.062
MAPE	2.164	0.890

When solving the parameters of the AR model, since both the coefficient matrix X and the observation vector Y are matrices composed of observations, the least squares estimation method only considers the errors of the observation vector Y and ignores the errors of the coefficient matrix X. But the total least squares method considers the error of coefficient matrix X and observation vector Y at the same time and corrects both. Therefore, the AR model established by the total least squares method can better reflect the real situation of data. According to the prediction results of the two methods in Table 1 and the accuracy evaluation of the two methods in Table 2, compared with

the AR model established by the least squares method, the model established by the total least squares method has higher prediction accuracy, is closer to the actual trend of the data, and can better reflect the real situation of the data.

4. Conclusions

When solving the parameters of the autoregressive model, the traditional least squares estimation method has obvious shortcomings because it does not consider the error of the coefficient matrix. While the total least squares estimation takes into account the error of the coefficient matrix X and the observation vector y, and corrects both of them. The established autoregressive model can better reflect the real situation of the data. This paper studies the method of parameter estimation of autoregressive model based on total least squares. Application examples show the feasibility and effectiveness of this method.

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