

Taylor's formula on the separating complex plane and its applications

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Abstract. This article begins by exploring split-complex numbers, also known as dual or double numbers, a mathematical concept that extends the real number system by creating a commutative ring with a zero divisor. Furthermore, the paper extends the Taylor series theory from the real analysis domain to the domain of split-complex numbers, thereby establishing the corresponding Taylor formula on the split-complex plane. The introduction of split-complex numbers opens up new perspectives and provides new tools for the modeling and analysis of quantum systems. In the field of electromagnetism, the application of split-complex numbers also demonstrates its potential, contributing to a more precise description of the behavior of electromagnetic fields under specific conditions. These theoretical advancements are expected to further promote research in the field of split-complex analysis, playing a more critical role in important branches of physics such as quantum mechanics and electromagnetism, thereby facilitating the development of related theories and the emergence of innovative practical applications.

Keywords: Separating Complex Numbers, Taylor's Formula, Hyperbolic Interval.

1. Introduction

The research by Deckelman and Robson [1] delved deeply into the split-complex number system, exploring its intrinsic connections with complex algebra. They employed linear operators to reveal how analytic functions map onto a set of smooth functions that satisfy hyperbolic Cauchy-Riemann equations, and further discussed the hyperbolic characteristics of split-complex numbers and their unique position in the definition of analytic functions. They pointed out that split-complex numbers do not form a division ring, thereby enriching the understanding of the applications and properties of this mathematical structure. Nadweh [2] introduced split-complex symbolic 2-plithogenic numbers into the study, combining them with split-complex numbers to explore their fundamental properties, operational rules, reversibility, and power equality. Through numerical settings and computational results, they showcased the features of this new generalized system, opening up new research directions in the field of algebraic structures. Shtawzen [3] particularly highlighted the potential applications of split-complex numbers in the field of algebra. Botos [4] studied the geometric structures induced by Hermitian forms in real algebraic linear spaces, elucidated the role of split-complex numbers in geometric descriptions, and conducted an in-depth analysis of the geodesic spaces in hyperbolic planes and the round 2-sphere. Abobala [5] researched linear Diophantine equations within the domain of split-complex numbers, derived the solvability conditions for these equations, and developed corresponding algorithms, providing new perspectives for solving such equations.

This study focuses on the application of Taylor series in the study of split-complex functions. As a powerful mathematical tool, Taylor series have found broad applications across various fields. Blaszczyk [6] utilized Taylor series in conjunction with weighted least squares to address computational issues in thermodynamic topological optimization, demonstrating the efficiency of this method. Choicely [7] proposed a numerical method based on Taylor series for solving initial

boundary value problems of linear partial differential equations with constant coefficients and analytic initial conditions. In the research on numerical solutions for ordinary differential equations, Alessandro [8] adopted Taylor series methods and Automatic Differentiation technology, focusing on solving vector field characteristics defined by finite combinations of basic and standard functions. They significantly improved computational efficiency through an exact quadratization preprocessing step, effectively reducing computational complexity, and providing an efficient approach for solving numerical solutions to ordinary differential equations.

These studies not only demonstrate the in-depth exploration of Taylor series and split-complex numbers in the fields of mathematics and computation but also pave new paths for the development of mathematical theory and practical applications, offering valuable references and insights for future research and practical applications.

2. Preliminaries

2.1. Split Complex Numbers $\mathbb{R}^{1,1}$

The separating complex numbers is a promotion of real numbers

$$\mathbb{R}^{1,1} = \{\zeta = x + yj | x, y \in \mathbb{R}, j^2 = 1\}, \quad (1)$$

is an exchangeable ring consisting of x and y and also the Clifford algebra $\mathcal{C}\ell_{1,0}$, we define the $Re(\zeta) = x$ as the real part of ζ and the $Im(\zeta) = y$ as the imaginary part of ζ and find that the unit j satisfies $1 - j^2 = 0$ and $j \neq \pm 1$.

Similarly, we can combine the principles of addition and multiplication in the domain of $\mathbb{R}$, while $\zeta_1 = x_1 + y_1j$ and $\zeta_2 = x_2 + y_2j \in \mathbb{R}^{1,1}$, it has

$$\begin{aligned} \zeta_1 + \zeta_2 &= (x_1 + y_1j) + (x_2 + y_2j) \\ &= (x_1 + x_2) + (y_1 + y_2)j \end{aligned} \quad (2)$$

and

$$\begin{aligned} \zeta_1 \zeta_2 &= (x_1 + y_1j)(x_2 + y_2j) \\ &= (x_1x_2 + y_1y_2) + j(x_1y_2 + x_2y_1) \end{aligned} \quad (3)$$

Inspiration from the $Re z = \frac{z+\bar{z}}{2}$ and $Im z = \frac{z-\bar{z}}{2}$, $z \in \mathbb{C}$, we can define

$$j_+ = \frac{(1+j)}{2} \text{ and } j_- = \frac{(1-j)}{2} \quad (4)$$

as another basis of $\mathbb{R}^{1,1}$. Additionally, the $j_+j_- = 0$ is proven and idempotent elements also exist in it,

$$j_+^2 = j_+ \text{ and } j_-^2 = j_- \quad (5)$$

and

$$j_+ + j_- = 1, j_+ - j_- = j \quad (6)$$

They form a whole ring. Because $\zeta = x + yj \in \mathbb{R}^{1,1}$, the idempotent representation can be also defined as:

$$\zeta = uj_+ + vj_- \quad (7)$$

where $u = x + y$ and $v = x - y$. In relation to the previous article on the laws of addition and multiplication, it has another form of expression:

$$\begin{aligned} \zeta_1 + \zeta_2 &= (u_1j_+ + v_1j_-) + (u_2j_+ + v_2j_-) \\ &= (u_1 + u_2)j_+ + (v_1 + v_2)j_- \end{aligned} \quad (8)$$

and

$$\begin{aligned}\zeta_1 \zeta_2 &= (u_1 j_+ + v_1 j_-)(u_2 j_+ + v_2 j_-) \\ &= u_1 u_2 (j_+)^2 + u_1 v_2 j_+ j_- + v_1 u_2 j_- j_+ + v_1 v_2 (j_-)^2 \\ &= u_1 u_2 j_+ + v_1 v_2 j_-\end{aligned}\quad (9)$$

which means $\mathbb{R}^{1,1}$ is isomorphous as $\mathbb{R} \oplus \mathbb{R}$.

2.2. Segmentation of the Region

Now, Let's turn our attention to the image, the different elements in $\mathbb{R}^{1,1}$ can be written as two parts:

$$\zeta = \alpha j_+ \text{ or } \zeta = \alpha j_- \quad (10)$$

for $\alpha \in \mathbb{R}$, is a special partition of a ring. Besides, these elements make up two areas and are demergered by L . Obviously, L consists of the union of two subspaces including $\forall \zeta \in \mathbb{R}^{1,1}$:

$$L_+ = \{\alpha j_+ : \alpha \in \mathbb{R}\} \text{ and } L_- = \{\alpha j_- : \alpha \in \mathbb{R}\} \quad (11)$$

When it defined, the inversion is given by

$$\zeta^{-1} = u^{-1} j_+ + v^{-1} j_- \quad (12)$$

It also makes sense to take the reverse order for only one of the components

$$\zeta_+^{-1} = u^{-1} j_+ + v j_- \quad (13)$$

and

$$\zeta_-^{-1} = u j_+ + v^{-1} j_- \quad (14)$$

Besides, inspiration by $\bar{z} = a - bi, z = a + bi \in \mathbb{C}$, we say that

$$\mathcal{C}(\zeta) = \bar{\zeta} = x - jy = v j_+ + u j_-, \zeta \in \mathbb{R}^{1,1} \quad (15)$$

And the following equation holds,

$$\begin{aligned}N(\zeta) &= \zeta \bar{\zeta} \\ &= x^2 - y^2 \\ &= uv\end{aligned}\quad (16)$$

The set

$$\mathbb{R}_+^{1,1} = \{\zeta = u j_+ + v j_- : u \geq 0, v \geq 0\} \quad (17)$$

divided by L is called the set of non-negative separating complex numbers, see Figure 1. Similarly,

$$\mathbb{R}_-^{1,1} = \{\zeta = u j_+ + v j_- : u \leq 0, v \leq 0\} \quad (18)$$

is called the set of non-positive separating complex numbers.

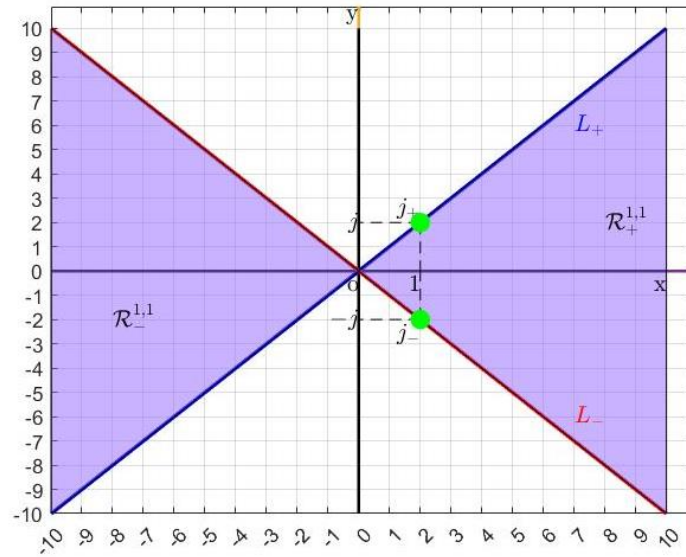


Figure 1. The positive and negative separate complex numbers.

2.3. Division of Intervals

If $\zeta_1, \zeta_2 \in \mathbb{R}_+^{1,1}$, we have

$$\zeta \preceq \omega \Leftrightarrow \omega - \zeta \in \mathbb{R}_+^{1,1} \quad (19)$$

and it's defined that ω is $\mathbb{R}_+^{1,1}$ less than ζ , or

$$\omega \preceq \zeta \Leftrightarrow \omega - \zeta \in \mathbb{R}_-^{1,1} \quad (20)$$

and it's defined that ω is $\mathbb{R}_-^{1,1}$ less than ζ , see Figure 2. The binary relationship between them shows that $\zeta \in \mathbb{R}_+^{1,1}$ is equivalent to $\zeta \succeq 0$ and specially, $\zeta \in \mathbb{R}_+^{1,1} - \{0\}$ is equivalent to $\zeta \succ 0$; In another set, it is valid that $\zeta \in \mathbb{R}_-^{1,1}$ is equivalent to $\zeta \preceq 0$ and $\zeta \in \mathbb{R}_-^{1,1} - \{0\}$ is equivalent to $\zeta \prec 0$. When we express ζ by its definition, for $\zeta_1 = u_1 j_+ + v_1 j_-$ and $\zeta_2 = u_2 j_+ + v_2 j_-$, we can see that $\zeta_1 \preceq \zeta_2 \Leftrightarrow u_1 \leq u_2$ and $v_1 \leq v_2$. Besides, the relation $\preceq$ has the properties of reflexive, transitive and antisymmetric, thus, $\preceq$ represents a partial order in $\mathbb{R}^{1,1}$.

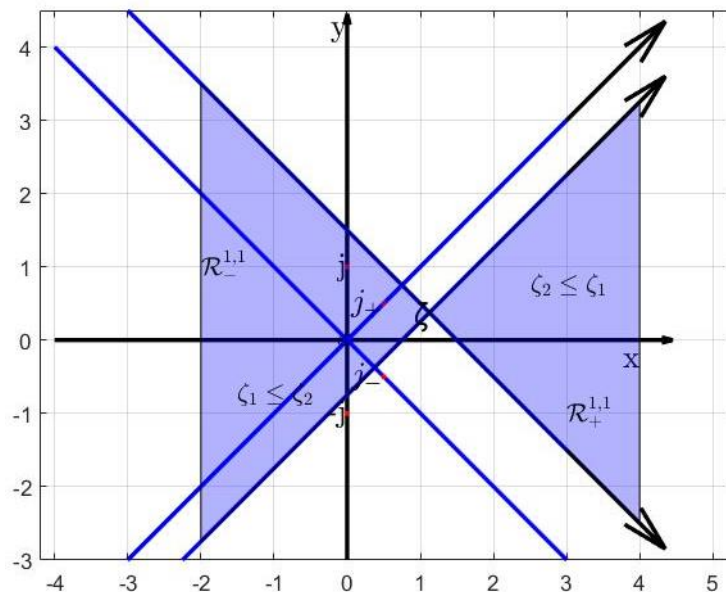


Figure 2. A partial order on $\mathbb{R}_+^{1,1}$

For $\zeta = u_1 j_+ + v_1 j_-$ and $\omega = u_2 j_+ + v_2 j_-$ in $\mathbb{R}^{1,1}$ fulfills $\zeta \preceq \omega$, the $[\zeta, \omega]_{\mathbb{R}^{1,1}}$ can be defined by (see Figure 3)

$$[\zeta, \omega]_{\mathbb{R}^{1,1}} = \{\theta \in \mathbb{R}^{1,1} : \zeta \preceq \theta \preceq \omega\}. \quad (21)$$

Equivalently, $\theta = \theta_1 j_+ + \theta_2 j_- \in [\zeta, \omega]_{\mathbb{R}^{1,1}}$ iff

$$u_1 \leq \theta_1 \leq u_2 \text{ and } v_1 \leq \theta_2 \leq v_2 \quad (22)$$

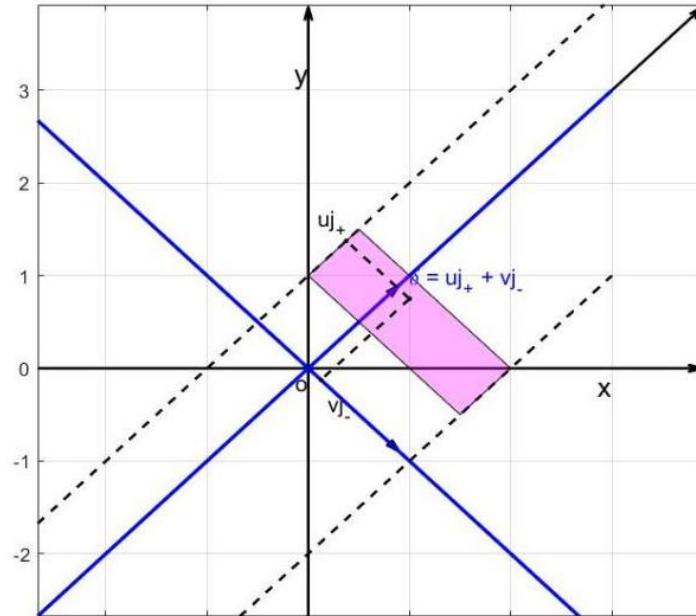


Figure 3. The hyperbolic interval $[j, 2]_{\mathbb{R}^{1,1}}$.

2.4. Modulus of $\mathbb{R}^{1,1}$

Now, we consider modulus of split complex numbers. For $\zeta = u j_+ + v j_-$, the positive split complex number

$$|\zeta|_{\mathbb{R}^{1,1}} = |u j_+ + v j_-|_{\mathbb{R}^{1,1}} = |u| j_+ + |v| j_- \quad (23)$$

Is called the modulus of ζ .

And, for any $\zeta, \omega \in \mathbb{R}^{1,1}$, It has

$$\begin{aligned} 1. |\zeta|_{\mathbb{R}^{1,1}} &= 0 \Leftrightarrow \zeta = 0 \\ 2. |\zeta \omega|_{\mathbb{R}^{1,1}} &= |\zeta|_{\mathbb{R}^{1,1}} |\omega|_{\mathbb{R}^{1,1}} \\ 3. |\zeta + \omega|_{\mathbb{R}^{1,1}} &\preceq |\zeta|_{\mathbb{R}^{1,1}} + |\omega|_{\mathbb{R}^{1,1}} \end{aligned} \quad (24)$$

2.5. Analysis of Functions of a Split-Complex Variable

Finally, we could consider a split complex variable's function:

$$f: U \subseteq \mathbb{R}^{1,1} \rightarrow \mathbb{R}^{1,1} \quad (25)$$

Here, the set U could be regarded as an open upset of $\mathbb{R}^{1,1}$. As with the definition of the R , it could be expressed as $f(\zeta) = f_1(u, v) + j f_2(u, v)$. Combined with the basis of $\mathbb{R}^{1,1}$,

$$f = f_1(u) j_+ + f_2(v) j_- \text{ and } f = f_1(v) j_+ + f_2(u) j_- \quad (26)$$

takes more intuitive representation of f .

Link the complex analysis, we could write the limit of the difference quotient

$$\lim_{\substack{h \rightarrow 0 \\ h \notin L}} \frac{f(z_0 + h) - f(z_0)}{h} \quad (27)$$

in this way to find its means.

Theorem 2.5.1[9, Theorem 2.7]The split complex valued function f is $Cl_{1,0}$ -differentiable if and only if

$$f = f_1(u)j_+ + f_2(v)j_- \quad (28)$$

where f_1 and f_2 are differentiable on the $\mathbb{R}$.

3. Results

3.1. Taylor's formula in the separating complex plane

Theorem 3.1(Taylor's theorem in $\mathbb{R}^{1,1}$) $f(\zeta) = f_1(u)j_+ + f_2(v)j_-$ is a separating complex function and when $\zeta = \zeta_0, \zeta_0 \in \mathbb{R}^{1,1}$, it is n -times $Cl_{0,1}$ -differentiable at the point ζ .

When $\zeta = uj_+ + vj_- \rightarrow \zeta_0 = u_0j_+ + v_0j_-$, the function f has

$$f(\zeta) = f(\zeta_0) + f'(\zeta_0)(\zeta - \zeta_0) + \dots + \frac{f^{(n)}(\zeta_0)}{n!}(\zeta - \zeta_0)^n + o((\zeta - \zeta_0)^n). \quad (29)$$

Here, $f^{(n)}(\zeta_0) = f_1^{(n)}(u_0)j_+ + f_2^{(n)}(v_0)j_-$.

$$o((\zeta - \zeta_0)^n) = \{g(\zeta) | \lim_{n \rightarrow \infty} \frac{g(\zeta)}{(\zeta - \zeta_0)^n} = \lim_{n \rightarrow \infty} \frac{g_1(u)j_+ + g_2(v)j_-}{(u - u_0)^n j_+ + (v - v_0)^n j_-} = 0\} \quad (30)$$

Proof

For this polynomial f , we have

$$\begin{aligned} & \lim_{\zeta \rightarrow \zeta_0} \frac{f(\zeta) - \sum_{k=0}^{\infty} \frac{f^{(k)}(\zeta_0)}{k!} (\zeta - \zeta_0)^k}{(\zeta - \zeta_0)^n} \\ &= \lim_{u \rightarrow u_0} \left[\frac{f(u) - \sum_{k=0}^{\infty} \frac{f^{(k)}(u_0)}{k!} (u - u_0)^k}{(u - u_0)^n} j_+ \right] + \lim_{v \rightarrow v_0} \left[\frac{f(v) - \sum_{k=0}^{\infty} \frac{f^{(k)}(v_0)}{k!} (v - v_0)^k}{(v - v_0)^n} j_- \right] \\ &= \frac{1}{n!} [f^{(n)}(u_0)j_+ + f^{(n)}(v_0)j_-] \\ &= \frac{1}{n!} f^{(n)}(\zeta_0), \end{aligned} \quad (31)$$

which shows that the coefficients before each term of the polynomial f are $\frac{f^{(n)}(\zeta_0)}{n!}$.

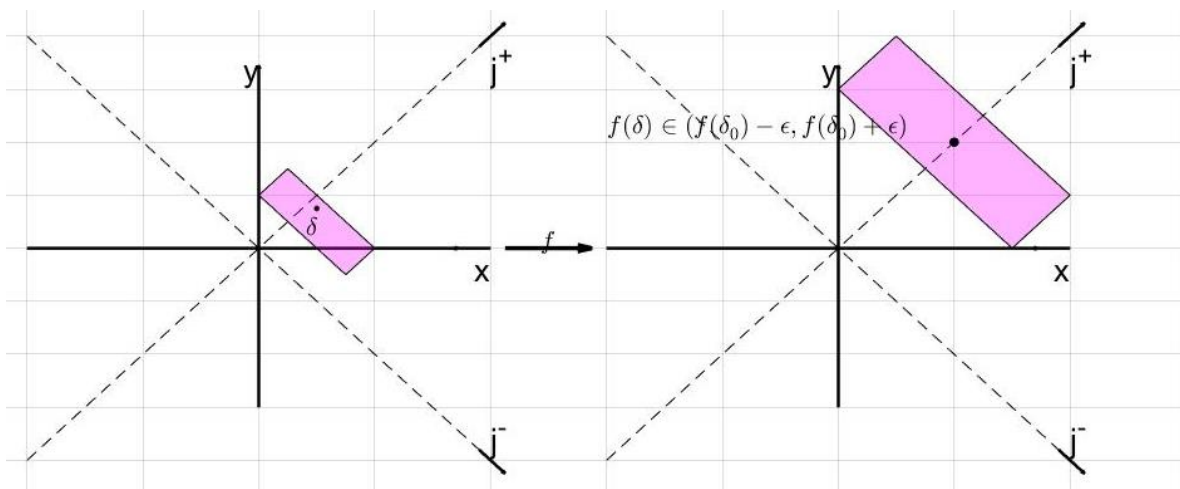


Figure 4. Geometric Interpretation of Taylor Series.

For the residual $(\zeta - \zeta_0)^n$ and Figure 4, if we define

$$T_n(\zeta) = f(\zeta_0) + f'(\zeta_0)(\zeta - \zeta_0) + \dots + \frac{f^{(n)}(\zeta_0)}{n!}(\zeta - \zeta_0)^n,$$

$R_n(\zeta) = f(\zeta) - T_n(\zeta)$ and $Q_n(\zeta) = (\zeta - \zeta_0)^n$, it has

$$\begin{aligned} f^{(n)} &= f_1^{(n)}(u_0)j_+ + f_2^{(n)}(v_0)j_- \neq 0 \lim_{\zeta \rightarrow \zeta_0} \frac{f(\zeta) - \sum_{k=0}^{\infty} \frac{f^{(k)}(\zeta_0)}{k!}(\zeta - \zeta_0)^k}{(\zeta - \zeta_0)^n} \\ &= \lim_{\zeta \rightarrow \zeta_0} \frac{R_n(\zeta)}{Q_n(\zeta)} = \lim_{\zeta \rightarrow \zeta_0} \frac{R'_n(\zeta)}{Q'_n(\zeta)} = \dots = \lim_{\zeta \rightarrow \zeta_0} \frac{R_n^{(n-1)}(\zeta)}{Q_n^{(n-1)}(\zeta)} \\ &= \lim_{u \rightarrow u_0} \frac{R_n(u)}{Q_n(u)}j_+ + \lim_{v \rightarrow v_0} \frac{R_n(v)}{Q_n(v)}j_- = \lim_{u \rightarrow u_0} \frac{R'_n(u)}{Q'_n(u)}j_+ + \lim_{v \rightarrow v_0} \frac{R'_n(v)}{Q'_n(v)}j_- = \dots \\ &= \lim_{u \rightarrow u_0} \frac{R_n^{(n-1)}(u)}{Q_n^{(n-1)}(u)}j_+ + \lim_{v \rightarrow v_0} \frac{R_n^{(n-1)}(v)}{Q_n^{(n-1)}(v)}j_- \\ &= \lim_{u \rightarrow u_0} \frac{f^{(n-1)}(u) - f^{(n-1)}(u_0) - f^{(n)}(u_0)(u - u_0)}{n(n-1) \dots 2(u - u_0)}j_+ \\ &\quad + \lim_{v \rightarrow v_0} \frac{f^{(n-1)}(v) - f^{(n-1)}(v_0) - f^{(n)}(v_0)(v - v_0)}{n(n-1) \dots 2(v - v_0)}j_- \\ &= \frac{1}{n!} \lim_{u \rightarrow u_0} \left[\frac{f^{(n-1)}(u) - f^{(n-1)}(u_0)}{(u - u_0)} - f^{(n)}(u_0) \right] j_+ \\ &\quad + \frac{1}{n!} \lim_{v \rightarrow v_0} \left[\frac{f^{(n-1)}(v) - f^{(n-1)}(v_0)}{(v - v_0)} - f^{(n)}(v_0) \right] j_- \\ &= 0 + 0 \\ &= 0. \end{aligned}$$

4. Conclusions

This paper has conducted an in-depth study of the differential theory on the separated complex plane, successfully deriving the Taylor's theorem on the set of $\mathbb{R}^{1,1}$. This theorem is an extension of the Taylor's theorem on the real plane and provides a solid theoretical foundation for the field of separated complex analysis. In the field of electromagnetism, particularly when analyzing the propagation of electromagnetic waves through inhomogeneous media and addressing complex electromagnetic field problems, the application of double numbers has provided us with a fresh perspective and a set of tools that significantly simplify and enhance the efficiency of the mathematical analysis of electromagnetic wave propagation. The introduction of these theories is expected to infuse new impetus into the applied research within the realm of physics.

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