# Fermat's Lemma on The Hyperbolic Plane

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**Abstract.** Generalizing classical mathematical problems to other spaces is an important research direction in mathematics. This paper studies Fermat's Lemma on the hyperbolic plane, where the set of hyperbolic numbers is a commutative ring with zero divisors generated by two real numbers, similar to the complex numbers but more complicated. Compared to the classical Fermat's Lemma, the Fermat's Lemma in this paper has a higher dimension and is built on a hyperbolic ring whose algebraic structure is more complex than the real number field. On the basis of a generalization of the classical Fermat's Lemma, the paper overcomes the difficulty that hyperbolic numbers contain zero divisors, and obtains the proof of the first and second Fermat's Lemma on the hyperbolic plane through the decomposition of hyperbolic numbers, which will give impetus to theoretical research in hyperbolic analysis.

**Keywords:** Fermat's Lemma, Hyperbolic Number, Partial Order, $C\ell_{0,1}$-differentiable.

## 1. Introduction

As a special subset of the set $\mathbb{BC}$ of bicomplex numbers, hyperbolic numbers which are also called duplex, double or bireal numbers, naturally inherit the algebraic definitions, operations and properties from $\mathbb{BC}$, but also have their own algebraic structure and properties within the framework of bicomplex numbers [1]. In relativistic physics, Minkowski spacetime is a fundamental conceptual framework, where hyperbolic complex numbers and their partially ordered structure provides a new mathematical language for understanding and describing the space-time relations in relativity [2]. In 2021, Gamaliel Tellez extended the Shannon entropy and the Chaos game algorithm to hyperbolic numbers plane, which provided a new possibility for the application of hyperbolic numbers in the field of image noise reduction [3]. And due to the property, that hyperbolic numbers have zero divisors, Masaki Kobayashi studied and determined all the reducibilities of hyperbolic neural networks in 2020 [4], from which we can see the wide application of hyperbolic numbers.

Meanwhile Fermat's Lemma occupies an important place in calculus and is the basis for many other important theorems such as Rolle's theorem and Lagrange's finite-increment theorem. As a classical result of single variable calculus, Fermat's Lemma has been extended to the two-variable situation. The proposal and proof of double Fermat's theorem provided a new perspective and tool for analyzing the extreme value of functions in two-dimensional space [5]. Fermat's Lemma also provides a key theoretical basis for the analysis of extreme values of functions of multiple variables through gradients, which having a wide range of applications in various fields such as optimization problems and machine learning [6, 7]. Fermat's Lemma can also be generalized to fractional derivatives, which is widely used in the theory of differential equations [8].

Hyperbolic numbers have attracted much attention because of their unique algebraic structure and mathematical properties, and it is very hot research to generalize the classical analytic results to the hyperbolic plane. This paper builds upon the above studies by investigating Fermat's Lemma on the hyperbolic plane, giving a sufficient condition for the derivative of a function to be zero on the hyperbolic plane, thus providing theoretical support in hyperbolic analysis, physics and other aspects of research.

## 2. Preliminaries

### 2.1. Hyperbolic numbers

The set of hyperbolic numbers is a commutative ring with zero divisors generated by two real numbers and has the following definition:

$$\mathbb{D} := \{\delta = x + \mathrm{k}y : x, y \in \mathbb{R}\}, \tag{1}$$

where $\mathbf{k}$ is the hyperbolic unit that satisfies $\mathbf{k}^2 = 1$. Similar to complex numbers, the real numbers $x$ and $y$ are the real and hyperbolic parts of $\delta$, and we write $x = \mathrm{Re}(\delta), \quad y = \mathrm{Im}(\delta)$.

Therefore, hyperbolic numbers can be defined as ordered pairs $\{1, \mathbf{k}\}$, which can be interpreted as points on the hyperbolic plane based on $\{1, \mathbf{k}\}$ with cartesian coordinates $x$ and $y$, resembling in $\mathbb{C}$. For $\delta_1 = x_1 + \mathbf{k}y_1$, $\delta_2 = x_2 + \mathbf{k}y_2$, the addition and multiplication are defined as

$$\begin{aligned} \delta_1 + \delta_2 &= (x_1 + \mathbf{k}y_1) + (x_2 + \mathbf{k}y_2) \\ &= (x_1 + x_2) + \mathbf{k}(y_1 + y_2), \end{aligned} \tag{2}$$

and

$$\begin{aligned} \delta_1 \delta_2 &= (x_1 + \mathbf{k}y_1)(x_2 + \mathbf{k}y_2) \\ &= (x_1 x_2 + y_1 y_2) + \mathbf{k}(x_1 y_2 + x_2 y_1), \end{aligned} \tag{3}$$

where $\mathbf{k}^2 = 1$.

Hyperbolic numbers also have conjugations:

$$\overline{\delta} = x - \mathbf{k}y. \tag{4}$$

In addition to the standard base $\{1, \mathbf{k}\}$, there is a special pair of idempotent bases $\{\mathbf{e}, \mathbf{e}^\dagger\}$ can be written as

$$\mathbf{e} := \frac{1 + \mathbf{k}}{2}, \quad \mathbf{e}^\dagger := \frac{1 - \mathbf{k}}{2}, \tag{5}$$

and

$$\mathbf{e}^2 = \mathbf{e}, \quad (\mathbf{e}^\dagger)^2 = \mathbf{e}^\dagger. \tag{6}$$

For any $\delta = x + \mathbf{k}y \in \mathbb{D}$, it can be represented by $\mathbf{e}$ and $\mathbf{e}^\dagger$ as

$$\delta = (x + y)\mathbf{e} + (x - y)\mathbf{e}^\dagger := a\mathbf{e} + b\mathbf{e}^\dagger, \tag{7}$$

where $a = x + y \in \mathbb{R}, \ b = x - y \in \mathbb{R}$.

Besides the idempotent property, it can also be noted that $\mathbf{e}$ and $\mathbf{e}^\dagger$ are two special zero divisors with

$$\mathbf{e}\mathbf{e}^\dagger = 0. \tag{8}$$

Thus, it is not difficult to find that, the zero divisors are non-zero real multiples of $\mathbf{e}$ and $\mathbf{e}^\dagger$ on the lines $y = x$ and $y = -x$. For any $\delta_1 = a_1\mathbf{e} + b_1\mathbf{e}^\dagger$ and $\delta_2 = a_2\mathbf{e} + b_2\mathbf{e}^\dagger$, there is also addition and multiplication but in a different form than before:

$$\begin{aligned} \delta_1 + \delta_2 &= (a_1\mathbf{e} + b_1\mathbf{e}^\dagger) + (a_2\mathbf{e} + b_2\mathbf{e}^\dagger) \\ &= (a_1 + a_2)\mathbf{e} + (b_1 + b_2)\mathbf{e}^\dagger, \end{aligned} \tag{9}$$

and

$$\begin{aligned}\delta_1\delta_2 &= \left(a_1\mathbf{e}+b_1\mathbf{e}^\dagger\right)\left(a_2\mathbf{e}+b_2\mathbf{e}^\dagger\right)\\ &= \left(a_1a_2\right)\mathbf{e}+\left(b_1b_2\right)\mathbf{e}^\dagger.\end{aligned} \tag{10}$$

Unlike complex numbers, hyperbolic numbers can compare sizes, for which the set of non-negative hyperbolic numbers

$$\mathbb{D}^+ := \{\delta = a\mathbf{e}+b\mathbf{e}^\dagger : a\geq 0,\ b\geq 0\}, \tag{11}$$

and the set of non-positive hyperbolic numbers

$$\mathbb{D}^- := \{\delta = a\mathbf{e}+b\mathbf{e}^\dagger : a\leq 0,\ b\leq 0\}. \tag{12}$$

are introduced. On the basis of formula (10), it can be seen that $\mathbb{D}^+$ is a multiplicatively closed set. As can be seen in Figure 1, the two lines $y=x$ and $y=-x$, where the zero divisors lie, divide the hyperbolic plane into four regions. What can be seen clearly is that a hyperbolic number can be decomposed not only onto the $x$-axis and the $y$-axis, but also onto the lines $y=x$ and $y=-x$, that is, any hyperbolic number $\delta = x+\mathbf{k}y$ can be written as $\delta = a\mathbf{e}+b\mathbf{e}^\dagger$. It can also be clearly seen that non-negative hyperbolic numbers are located in the right area and non-positive hyperbolic numbers are located in the left area. The points in the upper and lower regions correspond to those hyperbolic numbers which cannot be called either positive or negative [9].

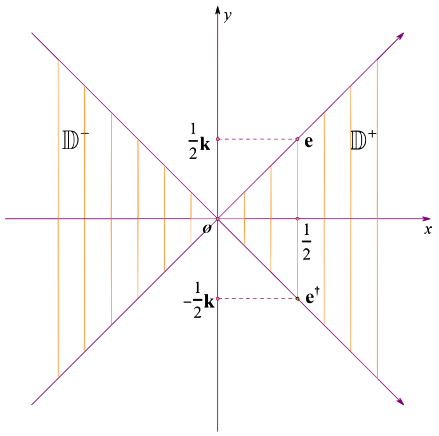



**Figure 1.** Non-negative hyperbolic numbers and non-positive hyperbolic numbers

Now define two partial orders $\succeq$ and $\preceq$ in $\mathbb{D}$. For $\delta_1 = a_1\mathbf{e}+b_1\mathbf{e}^\dagger$ and $\delta_2 = a_2\mathbf{e}+b_2\mathbf{e}^\dagger \in \mathbb{D}$, it is said that $\delta_1$ is $\mathbb{D}$-greater than $\delta_2$ if the following holds

$$\delta_1 \succeq \delta_2 \text{ iff } \delta_1-\delta_2 \in \mathbb{D}^+, \tag{13}$$

or $\delta_1$ is $\mathbb{D}$-less than $\delta_2$ if there holds

$$\delta_1 \preceq \delta_2 \text{ iff } \delta_1-\delta_2 \in \mathbb{D}^-, \tag{14}$$

which is shown in Figure 2. In Figure 2, through the point $\delta_1$ make two lines parallel to $y=x$ and $y=-x$, which also divide the hyperbolic plane into four regions. Combined with the definition and simple calculation, it can be seen that when $\delta_2 \succeq \delta_1$, there are $a_2 \geq a_1$ and $b_2 \geq b_1$, thus $\delta_2$ is in the region on the right. When $\delta_2 \preceq \delta_1$, there are $a_2 \leq a_1$ and $b_2 \leq b_1$, thus $\delta_2$ is in the region on the left.

In particular, for $\delta = a\mathbf{e}+b\mathbf{e}^\dagger$, there are

$$\delta \succeq 0 \text{ iff } \delta \in \mathbb{D}^+, \quad \delta \preceq 0 \text{ iff } \delta \in \mathbb{D}^-, \tag{15}$$

and

$$\delta \succ 0 \text{ iff } \delta \in \mathbb{D}^+ \setminus \{0\}, \quad \delta \prec 0 \text{ iff } \delta \in \mathbb{D}^- \setminus \{0\}. \tag{16}$$

Based on the partial order relations in $\mathbb{D}$, the closed hyperbolic interval $[\delta_1, \delta_2]_{\mathbb{D}}$ is defined as

$$[\delta_1, \delta_2]_{\mathbb{D}} = \{\omega \in \mathbb{D} : \delta_1 \preceq \omega \preceq \delta_2\}, \tag{17}$$

which is shown in Figure 3. In Figure 3, two lines parallel to $y = x$ and $y = -x$ are made respectively through point $\delta_1$ and $\delta_2$, and the four lines form a rectangular region. According to the above definition, it can be known that the region within the rectangle is the region where the closed hyperbolic interval $[\delta_1, \delta_2]_{\mathbb{D}}$ is located, that is, the geometric meaning of $[\delta_1, \delta_2]_{\mathbb{D}}$.

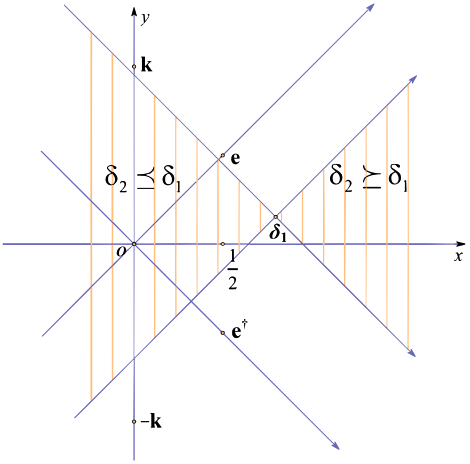



**Figure 2.** Geometric meaning of $\delta_2 \succeq \delta_1$, $\delta_2 \preceq \delta_1$

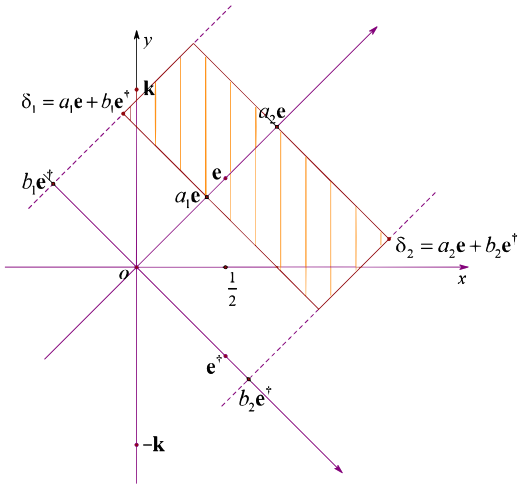



**Figure 3.** Geometric meaning of $[\delta_1, \delta_2]_{\mathbb{D}}$

### 2.2. Derivatives

Similar discussion as complex functions, mappings under hyperbolic functions are defined as: $f : \mathbb{D} \to \mathbb{D}$,

which can be represented by

$$f(\delta) = f_1(x, y) + \mathbf{k} f_2(x, y), \quad \delta = x + \mathbf{k}y \in \mathbb{D}. \tag{18}$$

Analogously, if the domain of a hyperbolic function $f$ is contained in a neighborhood of $\delta_0$, then the derivative of $f$ art $\delta_0$ has the following definition:

$$f'(\delta_0) = \lim_{\Delta\delta \to 0} \frac{f(\delta_0 + \Delta\delta) - f(\delta_0)}{\Delta\delta}, \tag{19}$$

where $\Delta\delta = \delta - \delta_0$. And when the limit exists, $f$ is said to be $C\ell_{0,1}$-differentiable at $\delta_0$. At the same time, it should be noted that if $L$ is the set of zero divisors, then $\Delta\delta \notin L$.

Just as hyperbolic numbers can be decomposed, the following introduces a theorem that hyperbolic functions can be differentiated.

**Theorem 2.1** [10] *For* $\delta = a\mathbf{e} + b\mathbf{e}^\dagger$, *the hyperbolic function* $f$ *is* $C\ell_{0,1}$*-differentiable if and only if*

$$f(\delta) = f_1(a)\mathbf{e} + f_2(b)\mathbf{e}^\dagger, \tag{20}$$

*where* $f_1$ *and* $f_2$ *are real-differentiable.*

From this, the derivative form is as follows:

$$f'(\delta)=f_1'(a)\mathbf{e}+f_2'(b)\mathbf{e}^{\dagger}. \tag{21}$$

## 3. Results

### 3.1. Fermat's Lemma on the hyperbolic plane

**Theorem 3.1 (The First Fermat's Lemma)** *If a hyperbolic function* $f$ *is* $C\ell_{0,1}$*-differentiable at* $\delta_0$, $\delta_0=a_0\mathbf{e}+b_0\mathbf{e}^{\dagger}\in\mathbb{D}$. *And* $\forall\varepsilon=\varepsilon_1\mathbf{e}+\varepsilon_2\mathbf{e}^{\dagger}\succ 0$, $f(\delta)\preceq f(\delta_0)$ *where* $\forall\delta=a\mathbf{e}+b\mathbf{e}^{\dagger}\in[\delta_0-\varepsilon,\delta_0+\varepsilon]_{\mathbb{D}}$, *then the derivative of* $f$ *at* $\delta_0$ *is* 0: $f'(\delta_0)=f_1'(a_0)\mathbf{e}+f_2'(b_0)\mathbf{e}^{\dagger}=0$, *where* $f(\delta)=f_1(a)\mathbf{e}+f_2(b)\mathbf{e}^{\dagger}$.

$\delta_0$ in theorem 3.1 is the maximum point redefined on the hyperbolic plane and the proof of this theorem refers to the proof of Fermat's Lemma in real analysis. On the basis of giving the definition of maximum point in hyperbolic plane, $f(\delta_0)$ and $f(\delta)$ are decomposed in accordance with the sufficient and necessary conditions of $C\ell_{0,1}$-differentiability. Then according to the definition of $C\ell_{0,1}$--differentiability and partial order, the partial order magnitude of $f_-'(\delta_0)$ and $f_+'(\delta_0)$ is obtained. Finally, the proof is completed by the sufficient and necessary conditions of the existence of the derivative.

*Proof Since* $f$ is $C\ell_{0,1}$ – differentiable at $\delta_0$, there are decompositions:

$$f(\delta_0)=f_1(a_0)\mathbf{e}+f_2(b_0)\mathbf{e}^{\dagger},\quad f(\delta)=f_1(a)\mathbf{e}+f_2(b)\mathbf{e}^{\dagger}. \tag{22}$$

Then

$$f(\delta)-f(\delta_0)=\left(f_1(a)-f_1(a_0)\right)\mathbf{e}+\left(f_2(b)-f_2(b_0)\right)\mathbf{e}^{\dagger}\preceq 0, \tag{23}$$

that is

$$\left(f_1(a)-f_1(a_0)\right)\le 0,\quad \left(f_2(b)-f_2(b_0)\right)\le 0. \tag{24}$$

By the definition of $C\ell_{0,1}$-differentiable,

$$\begin{aligned}\frac{f(\delta)-f(\delta_0)}{\delta-\delta_0}&=\frac{\left(f_1(a)-f_1(a_0)\right)\mathbf{e}+\left(f_2(b)-f_2(b_0)\right)\mathbf{e}^{\dagger}}{(a-a_0)\mathbf{e}+(b-b_0)\mathbf{e}^{\dagger}}\\&=\frac{\left(f_1(a)-f_1(a_0)\right)\mathbf{e}}{(a-a_0)\mathbf{e}+(b-b_0)\mathbf{e}^{\dagger}}+\frac{\left(f_2(b)-f_2(b_0)\right)\mathbf{e}^{\dagger}}{(a-a_0)\mathbf{e}+(b-b_0)\mathbf{e}^{\dagger}}\\&=\frac{\left(f_1(a)-f_1(a_0)\right)}{a-a_0}\mathbf{e}+\frac{\left(f_2(b)-f_2(b_0)\right)}{b-b_0}\mathbf{e}^{\dagger}.\end{aligned} \tag{25}$$

When $\delta\preceq\delta_0$, there is

$$(a-a_0)\mathbf{e}+(b-b_0)\mathbf{e}^{\dagger}\preceq 0, \tag{26}$$

thus

$$a-a_0\le 0,\quad b-b_0\le 0. \tag{27}$$

Consequently,

$$\frac{f(\delta)-f(\delta_0)}{\delta-\delta_0}\succeq 0. \tag{28}$$

When $\delta\succeq\delta_0$, there is

$$(a-a_0)\mathbf{e}+(b-b_0)\mathbf{e}^{\dagger}\succeq 0, \tag{29}$$

Thus

$$a-a_0\geq 0,\quad b-b_0\geq 0. \tag{30}$$

So, at this time,

$$\frac{f(\delta)-f(\delta_0)}{\delta-\delta_0}\preceq 0. \tag{31}$$

Due to $f$ is $C\ell_{0,1}$-differentiable at $\delta_0$,

$$f'(\delta_0)=f_-'(\delta_0)=f_+'(\delta_0). \tag{32}$$

And because

$$f_-'(\delta_0)=\lim_{\delta\to\delta_0^-}\frac{f(\delta)-f(\delta_0)}{\delta-\delta_0}\succeq 0,\quad f_+'(\delta_0)=\lim_{\delta\to\delta_0^+}\frac{f(\delta)-f(\delta_0)}{\delta-\delta_0}\preceq 0, \tag{33}$$

it follows that

$$f'(\delta_0)=f_1'(a_0)\mathbf{e}+f_2'(b_0)\mathbf{e}^{\dagger}=0 \tag{34}$$

From Figure 4, based on the geometric meaning of the closed interval $[\delta_0-\varepsilon,\delta_0+\varepsilon]_{\mathbb{D}}$, it can be seen that the neighborhood of $\delta_0$ is a rectangle centered on $\delta_0$. Through the function $f$, the neighborhood is mapped to the rectangular region where $f(\delta)$ is located. And combined with the meaning of partial order, it can be seen that the maximum point $\delta_0$ is mapped to the upper right corner of the rectangular region. Compared with the real analysis, the Fermat's Lemma on the hyperbolic plane has different geometric meanings, but all of them have $f'(\delta_0)=0$.

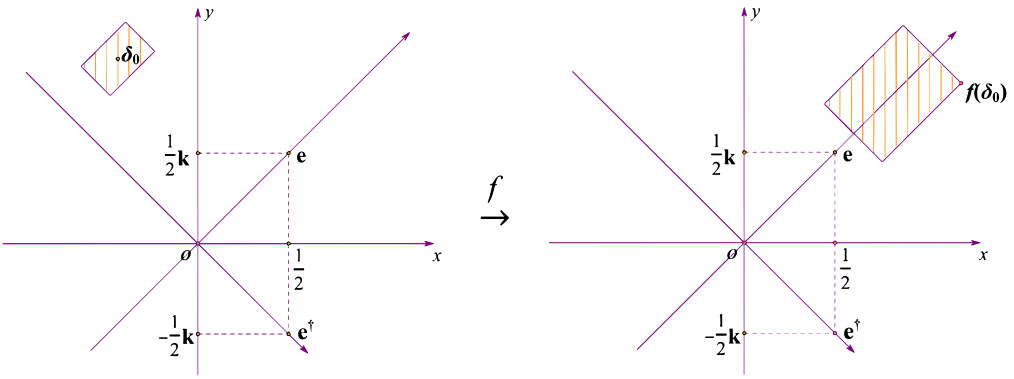



**Figure 4.** Geometric significance of the First Fermat's Lemma

Similarly, there is Theorem 3.2 below, and it is clear from Figure 5 that the minimum point $\delta_0$ is mapped to the lower left corner.

**Theorem 3.2 (The Second Fermat's Lemma)** *If a hyperbolic function* $f$ *is* $C\ell_{0,1}$ *-differentiable at* $\delta_0$ , $\delta_0 = a_0\mathbf{e} + b_0\mathbf{e}^\dagger \in \mathbb{D}$ . *And* $\forall \varepsilon = \varepsilon_1\mathbf{e} + \varepsilon_2\mathbf{e}^\dagger \succ 0$ , $f(\delta) \succeq f(\delta_0)$ *where* $\forall \delta = a\mathbf{e} + b\mathbf{e}^\dagger \in [\delta_0 - \varepsilon, \delta_0 + \varepsilon]_{\mathbb{D}}$ , *then the derivative of* $f$ *at* $\delta_0$ *is* 0 : $f'(\delta_0) = f_1'(a_0)\mathbf{e} + f_2'(b_0)\mathbf{e}^\dagger = 0$, *where* $f(\delta) = f_1(a)\mathbf{e} + f_2(b)\mathbf{e}^\dagger$.

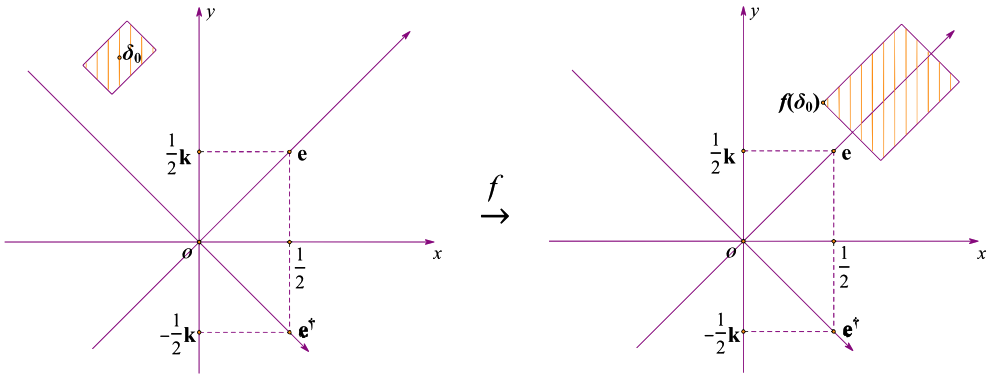



**Figure 5.** Geometric significance of the Second Fermat's Lemma

## 4. Conclusions

Hyperbolic number is a topic of great interest that has been studied by numerous scholars. This paper overcomes the characteristic that hyperbolic numbers have zero divisors, and generalizes the classical Fermat's Lemma to the hyperbolic plane according to the properties that hyperbolic numbers can be decomposed and have partial order relation. The proposal and proof of the first and second hyperbolic Fermat's Lemma will provide a theoretical foundation for the study of analytics, and may also contribute to applications in some fields, such as a deeper understanding of the geometric properties of space-time and the optimization of hyperbolic embeddings of network structures.

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