Mathematical Modeling and Analysis of Atmospheric CO_2 Concentration and Global Temperature Change: A Multi-Model Approach

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Abstract. This research works on the trend of atmospheric CO_2 levels and global temperature change by deployment multiple forecasting models. Historical data from 1959 to 2021 is used for analysis by fitting polynomials, exponential function, and autoregressive models in the forecasting of CO_2 concentration. ARIMA and hybrid ARMA-polynomial models are used in forecasting temperatures. The results showed that atmospheric CO_2 can achieve 512-513 ppm by 2050 and 688-835 ppm by 2100. Global land-ocean temperature will increase to 1.25°C, 1.50°C, and 2°C in the years 2030, 2038, and 2051, respectively. We have proven high correlation between CO_2 concentration and temperature change based on our model's having good reliability over approximately a seventy-year period. This research bears valuable information regarding future climate trends and underscores pressing needs for continued surveillance as well as mitigation efforts.

Keywords: CO_2 concentration; global warming; time series analysis; ARIMA; autoregressive model; climate change prediction.

1. Introduction

The levels of atmospheric carbon dioxide have been on a gradual increase since the onset of the Industrial Revolution, with more substantial acceleration noted in recent decades [1]. Such an increase in CO_2 has been associated with increases in global temperatures, eliciting fears regarding its potential effects on climate systems, ecosystems, and human societies [2]. It is therefore very important to have accurate forecasts of future CO_2 concentrations and temperature changes for informing policies related to the climate and strategies for mitigation.

There have been several modeling approaches to the forecasting of CO_2 levels and temperature changes. For example, Meinshausen *et al.* projected future CO_2 concentrations using a suite of climate models under different scenarios of future emissions [3]. In another related development, Hansen *et al.* developed models for the prediction of global temperature trends based on history and forcing factors [4].

This research tries to add a body of evidence to:

• Analysis of historical trends in atmospheric CO_2 concentrations from 1959 to 2021.

• Development and comparison of multiple models for the prediction of future CO_2 concentration levels.

• Forecasting global land-ocean temperature changes and the relationship that exists between them with CO_2 levels

• Examining the long-term reliability of these predictive models

A multi-model approach was used to ensure robust projections that can be used to inform climate science and policy decisions.

2. Literature Review

The atmospheric CO_2 concentrations and global temperature change have been at the center of the climate study for decades. Long-term CO_2 monitoring was pioneered by Keeling *et al.*, who started measurements at the Mauna Loa Observatory in 1958 [5]. The data are now key to following the rise of levels of atmospheric CO_2.

Different modeling approaches have been used to predict future CO_2 levels. This includes Lotka-Volterra models, which were originally for population dynamics but have been adapted to model CO_2 emissions and concentrations [6]. Time series analysis techniques have also been applied, including ARIMA models, to forecast levels of CO_2 [7].

Temperature predictions have used global climate models as the most common. These are complex models that use various components of the Earth system to carry out a simulation of future climate scenarios [8]. However, even simpler statistical models have been effective in forecasting temperatures [9].

The link between CO_2 levels and global temperature is not a new issue. There have been many mentions in several IPCC reports concerning the strong relationship between these variables [10]. But the exact sensitivity of temperature to CO_2 is still under research, for various feedback that complicates the basic relationship [11].

Recent studies have also explored potential climate system tipping points, which could take the form of CO_2 concentrations or levels of global mean temperature at which impacts become abrupt and/or irreversible. This further underline how essential reliable long-term forecasting is, and knowing just where current model features stop.

Our research is a contribution to the above-mentioned literature by combining more than one modeling approach and specifically addressing CO_2 concentration and temperature predictions and the long-term reliability of these estimates.

3. Methodology

3.1. Data Sources

This study utilized two primary datasets:

- Atmospheric CO_2 concentration data from 1959 to 2021, obtained from this source [12].
- Global land-ocean temperature change data from 1958 to 2021, relative to the 1951-1980 mean, obtained from this website [13].

3.2. CO_2 Concentration Models

Three distinct models were developed to analyze and predict CO_2 concentrations:

3.2.1 Polynomial Fitting Model

A polynomial was employed:

$$Y = a_0 + a_1 t + a_2 t^2 + \cdots + a_k t^k \tag{1}$$

where Y is the CO_2 concentration, t is the year, and a_i $(i = 0, 1, 2, \ldots, \mathrm{k})$ are unknown coefficients. These coefficients were determined using the least squares method:

$$Q(a_0, a_1, \ldots, a_k) = \sum_{i=1}^{N} \left(\sum_{j=0}^{k} a_j t_i^j - Y_i \right)^2 \tag{2}$$

where t_i = 1958 + i, i = 1, 2, ..., 63, and N = 63. The coefficients that minimize $Q(a_0, a_1, \ldots, a_k)$ were obtained by solving the following system of linear equations:

$$\frac{\partial Q}{\partial a_l} = 2 \sum_{i=1}^{N} \left(\sum_{j=0}^{t} a_j t_i^j - Y_i \right) t_i^l = 0, l = 0,1,2, \ldots, k \tag{3}$$

The specific form of linear equations is:

$$\begin{cases} a_0 N + a_1 \sum_{i=1}^{N} t_i + a_2 \sum_{i=1}^{N} t_i^2 + \cdots + a_k \sum_{i=1}^{N} t_i^k = \sum_{i=1}^{N} Y_i \\ a_0 \sum_{i=1}^{N} t_i + a_1 \sum_{i=1}^{N} t_i^2 + a_2 \sum_{i=1}^{N} t_i^3 + \cdots + a_k \sum_{i=1}^{N} t_i^{k+1} = \sum_{i=1}^{N} Y_i t_i \\ \vdots \qquad\qquad \vdots \\ a_0 \sum_{i=1}^{N} t_i^k + a_1 \sum_{i=1}^{N} t_i^{k+1} + \cdots + a_k \sum_{i=1}^{N} t_i^{2k} = \sum_{i=1}^{N} Y_i t_i^k \end{cases} \tag{4}$$

The unknown coefficients a_i $(i = 0, 1, 2, \ldots, \mathrm{k})$ can be determined by solving the linear equation (4).

3.2.2 Exponential Function Fitting Model

An exponential function was utilized:

$$Y = ae^{bt} + c \tag{5}$$

where a, b, and c are coefficients determined through iterative fitting. Due to the complexity of directly calculating the coefficients, an iterative approach was adopted:

1) The interval [100, 300] was divided into discrete points: $c_i = 100 + 0.1i, i = 1, 2, 3, \ldots, 2000$.

2) For each c_i, parameters a_i and b_i were estimated, and the corresponding fitting error was calculated.

3) The c_i yielding the smallest error was selected as the optimal c value, with the corresponding a_i and b_i as the best parameters for a and b.

3.2.3 Autoregressive Model

A *n*th-order autoregressive model at the first k times was implemented:

$$Y_n = b_0 + b_1 Y_{n-1} + b_2 Y_{n-2} + \cdots + b_k Y_{n-k} \tag{6}$$

where Y_n is the concentration of CO_2 in year n, $b_i, i = 0, 1, 2, \ldots, \mathrm{k}$, denote the unknown coefficients. Formula (6) can be written in matrix form as following:

$$\boldsymbol{Xb} = \boldsymbol{B} \tag{7}$$

were

$$\boldsymbol{X} = \begin{bmatrix} 1 & Y_k & Y_{k-1} & \cdots & Y_1 \\ 1 & Y_{k+1} & Y_k & \cdots & Y_2 \\ & \vdots & & \vdots & \\ 1 & Y_{N-1} & Y_{N-2} & \cdots & Y_{N-k} \end{bmatrix} \tag{8}$$

$$\boldsymbol{b} = \begin{bmatrix} b_0 \\ b_1 \\ \vdots \\ b_2 \end{bmatrix} \text{ and } \boldsymbol{B} = \begin{bmatrix} Y_{k+1} \\ Y_{k+2} \\ \vdots \\ Y_N \end{bmatrix} \tag{9}$$

Then, we have:

$$\boldsymbol{X^T X b} = \boldsymbol{X^T B} \tag{10}$$

and

$$\boldsymbol{b} = (\boldsymbol{X}^T \boldsymbol{X})^{-1} \boldsymbol{X}^T \boldsymbol{Y} \tag{11}$$

3.3. Temperature Change Models

Two models were developed for temperature predictions:

3.3.1 ARIMA Model

For stationary sequences R_t, the ARMA Model with the following form can be applied to fit them:

$$R_t = \beta_1 R_{t-1} + \beta_2 R_{t-2} + \cdots + \beta_p R_{t-p} - \varepsilon_t - \alpha_1 \varepsilon_{t-1} - \cdots - \alpha_q \varepsilon_{t-q} \tag{12}$$

where ε_t is Gaussian noise, β_i and α_j are unknown coefficients to be determined.

3.3.2 Hybrid ARMA-Polynomial Model

This model involved two steps:

1) Fitting a quadratic polynomial to remove the trend:

$$T_t = a + bt + ct^2 \tag{13}$$

2) Applying an ARMA model to the residuals:

$$R_t = Z_t - T_t \tag{14}$$

3.4. CO_2-Temperature Relationship

The relationship between CO_2 concentration and temperature change was analyzed using the Pearson correlation coefficient:

$$\rho(Y,Z) = \frac{\sum_{t=1}^{N}(Y_t-\bar{Y})(Z_t-\bar{Z})}{\sqrt{\sum_{t=1}^{N}(Y_t-\bar{Y})^2}\sqrt{\sum_{t=1}^{N}(Z_t-\bar{Z})^2}} \tag{15}$$

where Y_t is the concentration of CO_2 in the atmosphere, Z_t denote the land-ocean temperature changes and $\bar{Y}$ and $\bar{Z}$ are determined by the following equations:

$$\bar{Y} = \frac{1}{N}\sum_{t=1}^{N} Y_t \tag{16}$$

$$\bar{Z} = \frac{1}{N}\sum_{t=1}^{N} Z_t \tag{17}$$

4. Results and Discussion

4.1. Historical CO_2 Concentration Analysis

The analysis of historical data revealed that the claim of CO_2 increases in March 2004 being greater than any previous decade is inaccurate. The average increase of CO_2 concentration in the first 10 years of year t before 2004 was investigated using the formula:

$$I_t = \frac{Y_t - Y_{t-10}}{10} \tag{18}$$

where Y_t is the average CO_2 concentration in March of year t. Figure 1 illustrates these results. As shown in Figure 1, while the increase in CO_2 concentration in 2004 was significant, it was slightly lower than the increase observed in 2003. This finding highlights the importance of careful data analysis and the potential for misinterpretation of climate trends when focusing on short-term periods.

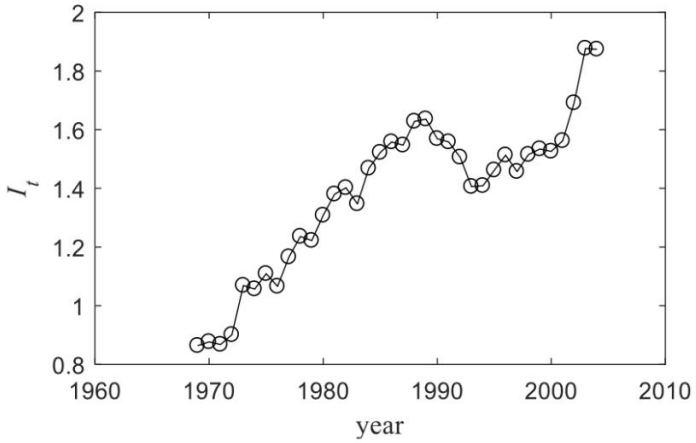


Figure 1. The average increase of CO_2 concentration in the first 10 years of year t=1963-2004.

4.2. CO_2 Concentration Predictions

Three models were developed to predict future CO_2 concentrations: polynomial fitting, exponential function fitting, and autoregressive models. Each model demonstrated different strengths in fitting the historical data and projecting future trends.

4.2.1 Polynomial Fitting Model

The polynomial fitting model was applied with different degrees to determine the best fit. Figure 2 shows the comparison between the fitting results of different polynomials and the measured data. It can be observed that the quadratic polynomial fitting model (Figure 2b) provides the most consistent results with the measured data, while the other polynomial fittings show less satisfactory outcomes. Based on this observation, the quadratic polynomial was chosen for further analysis. The exact quadratic polynomial fitting formula derived from all available data is:

$$Y = 48771.4 - 50.28t + 0.01304t^2 \tag{19}$$

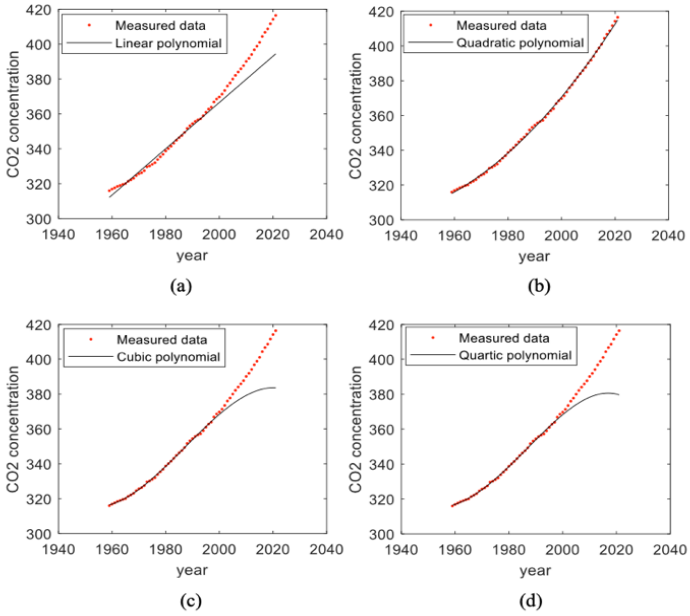


Figure 2. The CO_2 concentration generated by the measured data, (a) linear polynomial fitting, (b) quadratic polynomial fitting, (c) cubic polynomial fitting and (d) quartic polynomial fitting.

4.2.2 Exponential Function Fitting Model

The exponential function fitting model was also applied to the data. Figure 3 illustrates the performance of this model in fitting the historical data.

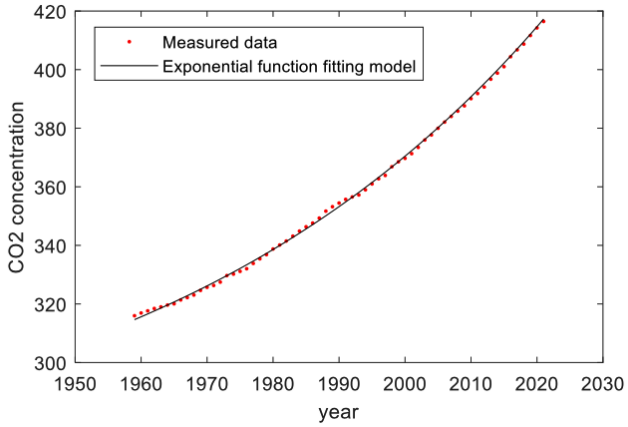


Figure 3. Trend of CO_2 concentration from the Exponential Point Fitting.

The good agreement between the measured and calculated results indicates that the exponential function fitting model is valid for the considered problem. When all data is used, the fitting formula is:

$$Y = 8.1762e^{0.0163t} + 256.5 \tag{20}$$

4.2.3 Autoregressive Model

Finally, the autoregressive model was implemented. Figure 4 shows the performance of this model.

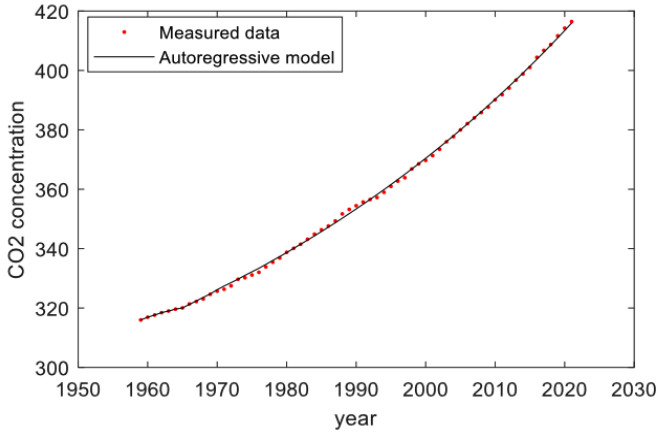


Figure 4. Trend of CO_2 concentration from the Autoregressive model.

The autoregressive model demonstrated excellent consistency with the measured data. The corresponding fitting formula is:

$$Y_n = -6.2235 + 0.9595Y_{n-1} - 0.1732Y_{n-2} + 0.3142Y_{n-3} - 0.0235Y_{n-4} \\ -0.2624Y_{n-5} + 0.0614Y_{n-6} + 0.1484Y_{n-7} \tag{21}$$

4.2.4 Comparison of Model Predictions

The results of the three models yielded the following predictions for CO2 concentrations are shown in Table 1.

Table 1. Predicted CO_2 Concentrations (ppm).

Model	2050	2100	Year reaching 685 ppm
Polynomial	496.81	688.33	2100
Exponential	512.97	835.42	2082
Autoregressive	512.82	832.31	2082

The autoregressive model demonstrated the lowest average error (0.4143) compared to the polynomial (0.6314) and exponential (0.6318) models. This suggests that the autoregressive model may be the most accurate for short-term predictions. The consistency between the exponential function fitting model and the autoregressive model adds confidence to these projections.

All models predict that CO_2 concentrations will not reach 685 ppm by 2050, contrary to some previous projections. However, this level is expected to be reached between 2082 and 2100, depending on the model used. This finding has significant implications for climate policy and mitigation efforts, suggesting that there may be more time than previously thought to implement effective CO_2 reduction strategies.

Figure 5 compares the CO_2 concentration predictions from all three models up to 2100. The divergence between the polynomial model and the other two models becomes more pronounced after 2050, highlighting the importance of model selection in long-term projections. The close agreement between the exponential and autoregressive models provides increased confidence in their predictions.

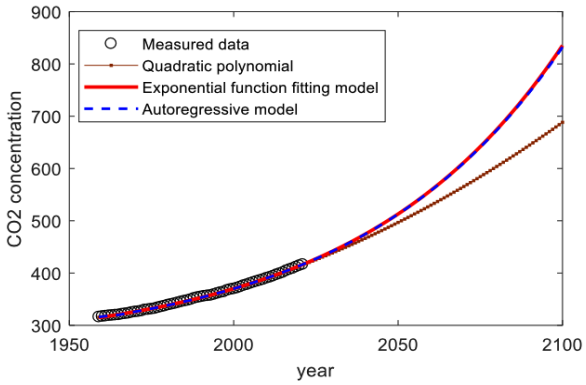


Figure 5. The change of CO_2 concentration from 2021 to 2100 predicted by using three models.

4.3. Land-Ocean Temperature Change Predictions

The ARIMA and hybrid ARMA-polynomial models were used to predict future land-ocean temperature changes. These models were chosen for their ability to capture both trend and cyclical components in time series data.

4.3.1 ARIMA Model

The ARIMA model was applied after differencing the original temperature data to achieve stationarity. Figure 6 shows the results of the ARIMA model predictions.

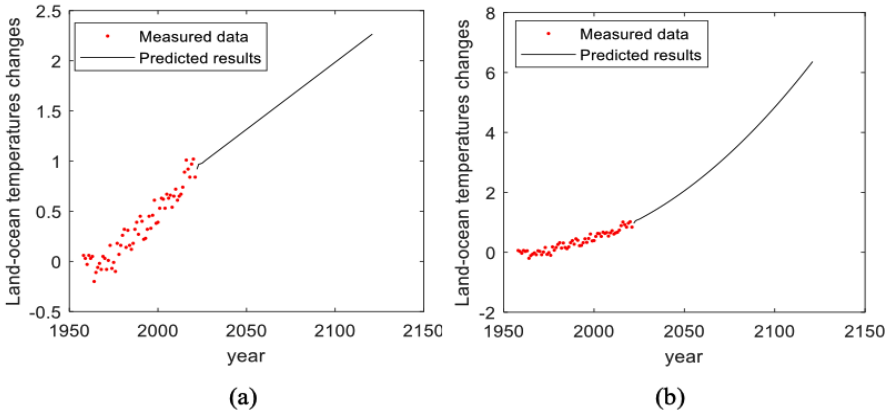


Figure 6. The prediction of land-ocean temperatures changes obtained by using the model combine the ARMA and (a) primary difference, and (b) second order difference.

4.3.2 Hybrid ARMA-Polynomial Model

The hybrid model combined a polynomial trend removal with an ARMA model for the residuals. Figure 7 illustrates the predictions from this hybrid approach.

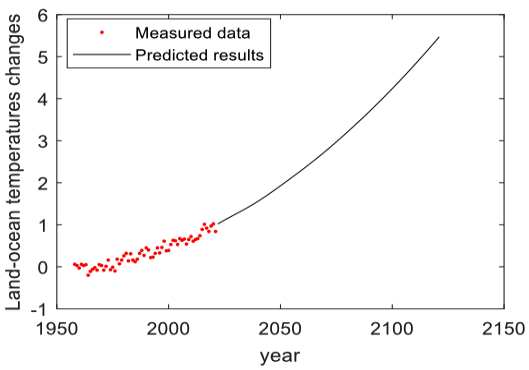


Figure 7. The prediction of land-ocean temperatures changes obtained by using the model combine the ARMA and polynomial fitting.

4.3.3 Temperature Change Milestones

The average of these two models yielded the following milestones for global land-ocean temperature change:

- 1.25°C increase: 2030
- 1.50°C increase: 2038
- 2.00°C increase: 2051

These years are consistent with projections made by more demure climate models, which signifies that warming might be happening faster than thought. The period from 1.25°C to 2.00°C (only 21 years) is especially worrying and underscores the need for efforts in climate mitigation to be immediate and aggressive.

4.3.4 Summary

As depicted in Figure 7, the hybrid model forecasts land-ocean temperatures to keep rising, with the increasing trend likely to accelerate further. Such acceleration is typical of many climate models that describe various self-reinforcing responses among component parts in Earth's climate system.

Note that, the ARIMA model and hybrid ARMA-polynomial model already portray a semblance of the same trends, and this serves to increase our prediction confidence. The fact that these two different modeling approaches agree on the trend itself implies that any underlying behavior in land-ocean temperature change is indeed robust; otherwise, it could not be captured by more than one statistical method.

The models also show that, if the current trends prevail, before the early fifties of this century the global land-ocean potential temperature might surpass more than 2°C relative to the 1951-1980 mean. Such a forecast is extremely alarming, taking into consideration that the Paris Agreement established the long-term global average temperature increase well below 2°C above pre-industrial levels and pursued to limit the increase to 1.5°C.

Yet, one should bear in mind that these are extrapolations based on historical trends and do not reflect any changes that might be introduced in the future in emissions scenarios or climate policy. Drastic reductions in greenhouse gas emissions could change the face of these projections, reiterating that immediate and sustained climate action are of the essence.

4.4. CO_2-Temperature Relationship

A strong correlation between CO_2 concentration and temperature change was found, with a Pearson correlation coefficient of 0.9613. This high correlation supports the well-established link between atmospheric CO_2 levels and global warming. Figure 8 illustrates this relationship.

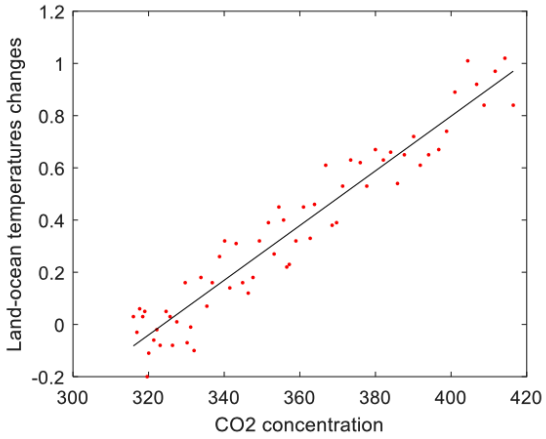


Figure 8. The relationship between land-ocean temperatures changes and CO_2 concentration.

The linear regression model yielded:

$$\Delta T = 0.0105 * [CO2] - 3.3926 \quad (22)$$

Where ΔT is the temperature change in °C and $[CO2]$ is the CO_2 concentration in ppm. This relationship suggests that for every 100-ppm increase in CO_2 concentration, approximately a 1.05°C increase in global temperature can be expected, assuming the relationship remains linear.

This sensitivity is slightly higher than some previous estimates, which typically range from 0.8°C to 1.2°C per 100 ppm CO_2. The higher sensitivity in this model could be due to the inclusion of recent years' data, which have shown accelerated warming trends. However, it's important to note that this linear relationship may not hold at higher CO_2 concentrations due to potential non-linear feedbacks in the climate system.

4.5. Model Reliability and Limitations

Based on the analysis, the models are estimated to maintain reliability for approximately 70 years into the future. This limitation is primarily due to the potential for significant changes in Earth systems as temperature increases approach 4°C above pre-industrial levels. Beyond this point, non-linear feedback and tipping points in the climate system may render simple extrapolation models unreliable.

Several factors contribute to this limitation:

• Potential changes in carbon cycle feedbacks, such as reduced oceanic CO_2 uptake or increased permafrost methane release.

• Possible alterations in global circulation patterns, affecting heat distribution and local climate regimes.

• Non-linear responses in ice sheet dynamics, potentially accelerating sea-level rise.

• Changes in albedo due to reduced snow and ice cover, amplifying warming trends.

These factors underscore the need for more complex Earth system models for long-term projections beyond the 70-year horizon identified in this study.

4.6. Implications for Climate Policy

This study has the following important implications for climate policy:

• The CO_2 concentrations projected to very high levels imply that current mitigation efforts are not strong enough for achieving their target according to international agreements, including the Paris Agreement.

• The rapid advancement within temperature enhancement emphasizes the need for very aggressive emission reductions more urgently.

• The high correlation between CO_2 and temperature has a clear message for policy actors: as CO_2 reduction will directly and predictably limit temperature rise.

• The models' 70-year horizon provides a reliability window within which policies can plan and implement, stressing the consideration of both immediate actions alongside long-term strategies.

So, while these models provide valuable insights for the coming decades, they also reinforce the need for further research and more advanced modeling techniques to be able to capture the multi-dimensionality of long-term climate change projections. The results underscore the urgency of climate action while also providing a nuanced understanding of the potential trajectories of CO_2 concentrations and global temperatures.

5. Conclusion

This paper analyzes atmospheric CO_2 concentration trends and how comprehensively they relate to global temperature changes. Our multi-model ensemble has good strength in predicting both CO_2 levels and temperature for the next few decades. The main results of our study are that CO_2 concentration peaks at 512-513 ppm in 2050 (and 688-835 ppm by 2100) while global land-ocean temperatures will rise by 1.25°C, 1.50°C, and 2°C in the years 2030, 2038, and 2051, respectively. We have found a very strong relationship between CO_2 concentrations and temperature changes: an increase of approximately 1.05°C for every additional 100 ppm of CO_2 added to the atmosphere. This

work also represents our sensitivity limit; we impose a priori that some climate responses do not go to infinity in terms of nonlinear increases unless given reasons.

These results drive home how the efforts to reduce greenhouse gas emissions and the need for climate change adaptation are urgent than ever. Although our findings are valuable for research and policy in climate science, some assumptions were based on historical data and linear relationships that may not always hold. Further research should try to incorporate more variables and feedback mechanisms so that predictive capacity over longer terms is enhanced, especially as we approach critical thresholds in the climate system. The work also needs to be extended for the monitoring of these models so that they can help make reasonable climatic decisions in several forthcoming years which could also show a fairly bleak picture under current trends.

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