Equivalent transformation of differential and integral equations in hyperbolic plane

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Abstract. Hyperbolic numbers, serving as an extension of real numbers, exhibit an algebraic structure akin to complex numbers, constituting commutative rings with zero divisors that stem from two real numbers. This paper delves into the exploration of the equivalence relation between the solutions of differential equations and integral equations within the realm of the hyperbolic plane. The pivotal finding of this study reveals that the solutions of hyperbolic differential equations and integral equations are inherently equivalent. Such a discovery not only solidifies the groundwork for delving into hyperbolic differential equations but also injects a renewed impetus into the advancement of physics. By bridging the gap between differential and integral equations in the hyperbolic domain, our research paves the way for a deeper understanding of hyperbolic numbers' mathematical structures and their applications in the physical sciences, thereby opening up new avenues for exploration and discovery.

Keywords: Hyperbolic numbers, Hyperbolic curves, Integral equations, Differential equations.

1. Introduction

This paper presents a systematic study of the theory of existence and uniqueness of solutions from a new point of view where the notion of differential equation defined on the set of hyperbolic numbers and hyperbolic intervals has been taken into consideration. The hyperbolic numbers were also called split-complex, paracomplex or double numbers [1]. They had been discussed universally, especially their applications in physics, such as the research on hyperbolic numbers with respect to the klein-gordon equation for fermions and bosons [2, 3]. If complex numbers are not fit for describing certain phenomena, or other physical [4, 5] processes then their generalizations or modifications may be interesting.

The differential equation shows up in many applications in different zones of science such as mechanics, physics, engineering, biology, economics, and so on. As mentioned earlier, previous works on differential equations have majorly coped with the problem of the existence of a solution to ordinary and partial differential equations. But recently, as this text gradually understand and improve hyperbolic numbers, our research on differential equations should also go further. Thus, in [6], it is explained in detail that hyperbolic differential equations is a significant mathematical model for the leaning of physics.

This paper applied hyperbolic numbers to differential equation which are becoming more and more important in some mathematical models of real processes and phenomena studied in physics, chemical technology, population dynamics, biotechnology and economics. Lots of authors studied the problem of existence and uniqueness of differential equations and obtained many results [7]. Our study is about the existence and uniqueness of the hyperbolic differential equations. This text will majorly study the equivalence relationship between initial value problem and integral equation. Theoretical and numerical applications on system are not the main subject of this article [8-10]. In the past few years, this text have found many applications in ordinary differential equation and partial

differential equation with applications in physics, such as fractional abstract, prediction of coronavirus, acoustic and so on[11-15]. Based on the above research, this paper further discusses the equivalence between differential equations and integral equations on hyperbolic numbers. This paper overcomes the difficulty of hyperbolic numbers containing zero factors and uses the decomposition properties of hyperbolic numbers to derive an important step in the unique existence theorem of solutions to differential equations on hyperbolic numbers. In this article, this text will majorly investigate the behavior of differential equation defined on the set of hyperbolic numbers.

2. Propaedeutics

Hyperbolic numbers are a kind of mathematical objects which extend the real number system. In this paper, the properties of hyperbolic numbers are studied by analogy with the properties of complex numbers.

Hyperbolic numbers are defined as follows:

$$H := \{\delta = x + ky : x, y \in \mathbb{R}\} \tag{1}$$

Where the hyperbolic unit $k^2 = 1$ and $k \neq \pm 1$, the set H is a commutative ring, we deine the real part of δ as $\mathrm{Re}(\delta) = x$ and the hyperbolic part of $\mathrm{Im}(\delta) = y$, here are its addition and multiplication operations

$$\begin{aligned} \delta_1 + \delta_2 &= (x_1 + ky_1) + (x_2 + ky_2) \\ &= (x_1 + x_2) + k(y_1 + y_2) \end{aligned} \tag{2}$$

And

$$\begin{aligned} \delta_1\delta_2 &= (x_1 + ky_1)(x_2 + ky_2) \\ &= (x_1x_2 + y_1y_2) + k(y_1x_2 + y_2x_1) \end{aligned} \tag{3}$$

Consider idempotent decomposition, where $\delta_1 = x_1 + ky_1$, $\delta_2 = x_2 + ky_2$ for any $\delta \in H$, the conjugate of δ is denoted by $\overline{\delta}$ and it is $\overline{\delta} = x - ky$, the ring of hyperbolic number has zero-divisors. Hyperbolic numbers can also be idempotent as

$$\begin{gathered} \eta = \frac{1+k}{2}, \eta^\dagger = \frac{1-k}{2} \\ \eta\eta^\dagger = 0 \end{gathered} \tag{4}$$

Where $\{\eta, \eta^\dagger\}$ is called the idempotent basis of H, they satisfy the following properties

$$\eta^2 = \eta,\ (\eta^\dagger)^2 = \eta^\dagger \tag{5}$$

And we have

$$\eta + \eta^\dagger = 1, \eta - \eta^\dagger = k \tag{6}$$

In addition, the zero factor also has the property of idempotent. These idempotent zero factors make up the whole ring. If $\delta \in H$, another expression of idempotent is

$$\begin{aligned} \delta &= (x + y)\eta + (x - y)\eta^\dagger \\ &=: (\alpha\eta + \beta\eta^\dagger) \end{aligned} \tag{7}$$

From the above argument, it is obvious that the zero factor in the set H is a real multiple of η and $\eta^\dagger$. Because the multiples of η and $\eta^\dagger$ are on the $y = x$ and $y = -x$ lines. Therefore δ is a zero divisor if and only if $\delta = x\eta$ or $\delta = y\eta^\dagger$; $x, y \in \mathbb{R} - \{0\}$

Also, for $\delta_1 = \alpha_1\eta + \alpha_2\eta^\dagger, \delta_2 = \beta_1\eta + \beta_2\eta^\dagger, \delta_1, \delta_2 \in H$ we have

$$\begin{aligned} \delta_1 + \delta_2 &= (\alpha_1 + \beta_1)\eta + (\alpha_2 + \beta_2)\eta^\dagger \\ \delta_1\delta_2 &= (\alpha_1\beta_1)\eta + (\alpha_2\beta_2)\eta^\dagger \end{aligned} \tag{8}$$

The set of non-negative hyperbolic numbers is defined as

$$H^+ := \{\delta = \alpha\eta + \beta\eta^\dagger : \alpha \geq 0, \beta \geq 0\} \tag{9}$$

Similarly, the set of non-positive hyperbolic numbers is defined as follows

$$H^- := \{\delta = \alpha\eta + \beta\eta^\dagger : \alpha \leq 0, \beta \leq 0\} \tag{10}$$

It is observer that if $\delta_1, \delta_2 \in H^+$, then $\delta_1\delta_2 \in H^+$, therefore, H^+ is closed under the multiplication. This relationship can be represented by Figure 1.

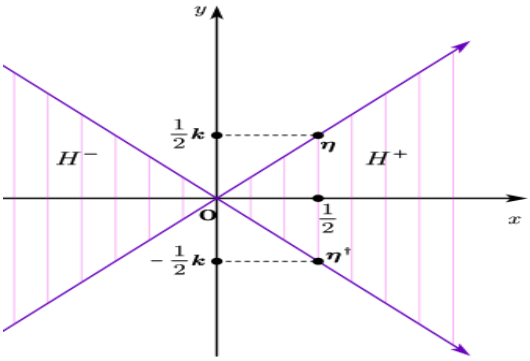


Figure 1. The concept of positive and negative hyperbolic numbers.

For any hyperbolic number $\delta_0 = x_0 + ky_0$, the axis η $\eta^\dagger$ divides the whole plane into four parts. Are the quarter planes of H – greater than or equal to $\delta_0 (\delta \succeq \delta_0)$, the quarter planes of H – less than or equal to $\delta_0 (\delta \preceq \delta_0)$, and the two quarter planes of neither $\delta \succeq \delta_0$ nor $\delta \preceq \delta_0$. For $\delta, \varpi \in H$, we have

$\delta \preceq \varpi$ iff $\varpi - \delta \in H^+$

And we say that ϖ is H – greater than δ or

$\varpi \preceq \delta$ iff $\varpi - \delta \in H^-$

We denote that ϖ is H – less than δ, moreover $\delta \in H^+$ is equivalent to $\delta \succeq 0$ and that $\delta \in H^+ - \{0\}$, is equivalent to $\delta \succ 0$; $\delta \in H^-$ is equivalent to $\delta \preceq 0$ and that $\delta \in H^- - \{0\}$ is equivalent to $\delta \prec 0$. For $\delta_1 = \alpha_1\eta + \alpha_2\eta^\dagger, \delta_2 = \beta_1\eta + \beta_2\eta^\dagger$ we denote that $\delta \preceq \varpi$ iff $\alpha_1 \leq \beta_1$ and $\alpha_2 \leq \beta_2$. And, the relation $\preceq$ is reflexive, the relation $\preceq$ denotes a partial order in H [16]. We can see this order of relationships in Figure 2.

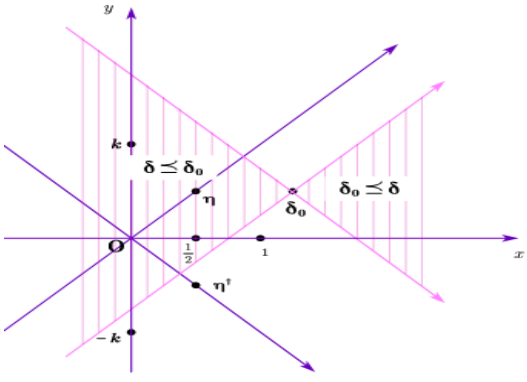


Figure 2. A partial order on H.

The decomposition transformation involved is carried out on the hyperbolic plane, and thus we use the real-valued norm to study the differentiability of differential equations in the hyperbolic plane.

The length of any hyperbolic interval is a non-negative hyperbolic number. Given a hyperbolic number $\delta = \alpha\eta + \beta\eta^{\dagger}$, the hyperbolic positive number [16, 17]

$$|\delta|_k = |\alpha\eta + \beta\eta^{\dagger}|_k = |\alpha|\eta + |\beta|\eta^{\dagger} \in H^{+} \tag{11}$$

Is called the hyperbolic modulus (or the hyperbolic value norm) of δ.

Definition 2.1 [18] A hyperbolic sequence $\{\delta_n\}_{n\in\square}$ is called convergence. If there is $\delta_0 \in H$ for any $\zeta > 0$, there exists $N \in \square$ such that for any $n \geq N$ there holds.

$$|\delta_n - \delta_0|_k \prec \zeta \tag{12}$$

We say that δ_0 is the limit of the sequence, and the sequence $\{\delta_n\}_{n\in\square}$ converges to δ_0. Let U be the open set of $f : U \subset D \to H$, decompose the hyperbolic function onto the $\eta, \eta^{\dagger}$ axis.

$$f(\delta) = f_1(x, y)\eta + f_2(x, y)\eta^{\dagger} \tag{13}$$

Where $D \subset H$ is not a zero factor and cannot be on the decomposition axis of the hyperbolic interval.

If the partial derivatives are continuous, then the hyperbolic function f is $\ell_{0,1} - differentiable$ at δ_0. If all partial derivatives at δ_0 exist and are continuous, then we say that f is $\ell_{0,1} - differentiable$ at that point, hyperbolic functions are convergent [19, 20], it's decomposable. It can be broken down into the following forms

$$f = f_1(\alpha)\eta + f_2(\beta)\eta^{\dagger} \tag{14}$$

Definition 2.2 [18] Let Γ be a surface in hyperbolic geometry, whose parameterization is

$$\phi : [a, b] \times [c, d] \subset \square \to H \tag{15}$$

ϕ Idempotent expression of

$$\phi(u, v) = \phi_1(u)\eta + \phi_2(v)\eta^{\dagger} \tag{16}$$

Where $\phi_1 : [a, b] \to \square$ and $\phi_2 : [c, d] \to \square$, the set of hyperbolic numbers is $\square$, the hyperbolic function studied is holomorphic on this two-dimensional surface region, that is, the hyperbolic function on our defined hyperbolic region is decomposable.

The hyperbolic integral of the hyperbolic function $\phi(\delta) = \phi_1(\alpha, \beta)\eta + \phi_2(\alpha, \beta)\eta^{\dagger}$ in the hyperbolic plane can be defined as a line integral, it is about the value of some curve ς in H. Defining the hyperbolic integral as

$$I = \int_{\varsigma} \phi(\delta) \otimes d\delta, d\delta = (d\alpha, d\beta) \tag{17}$$

Where ς is the piecewise continuously differentiable curve in H [20]. Decompose onto the $\eta, \eta^{\dagger}$ axis

$$\begin{aligned} I &= \int_{\varsigma} \phi(\delta) \otimes d\delta = \int_{\varsigma} \phi(\delta) \otimes d(\alpha\eta + \beta\eta^{\dagger}) \\ &= \eta \int_{\varsigma_1} \phi_1(\alpha) d\alpha + \eta^{\dagger} \int_{\varsigma_2} \phi_2(\beta) d\beta \end{aligned} \tag{18}$$

Where $\varsigma = \varsigma_1\eta + \varsigma_2\eta^{\dagger}$.

Definition 2.3 [16] for $\delta = \alpha_1\eta + \beta_1\eta^\dagger$ $\varpi = \alpha_2\eta + \beta_2\eta^\dagger$ in H such that $\delta \preceq \varpi$, the closed hyperbolic interval $[\delta, \varpi]_H$ is defined by

$$[\delta, \varpi]_H = \{\vartheta \in H : \delta \preceq \vartheta \preceq \varpi\} \tag{19}$$

Equivalently,

$$\vartheta = \mu_1\eta + \mu_2\eta^\dagger \in [\delta, \varpi]_H \quad \text{iff} \quad \alpha_1 \le \mu_1 \le \alpha_2 \text{ and } \beta_1 \le \mu_2 \le \beta_2 \tag{20}$$

If $\varpi - \delta$ is a non-negative zero-factor hyperbolic number, the object obtained in the hyperbolic region is (real) one-dimensional, the hyperbolic interval $[\delta, \varpi]_H$ is degenerate. If $\varpi - \delta$ is an invertible positive hyperbolic number, then the hyperbolic interval is non-degenerate. In order to ensure the rigor of the decomposition, require that the hyperbolic function studied cannot be on the decomposition axis, the hyperbolic function studied is a non-zero factor hyperbolic number [16].

3. Results

According to the decomposition and partial order relation of hyperbolic numbers, the following is given to prove the equivalent solution of differential equation and integral equation in hyperbolic plane, which will further lay the foundation for the study of hyperbolic plane.

Theorem 3.1 [Equivalent solution of hyperbolic differential equation] suppose the initial problem

$$\frac{d\varpi}{d\delta} = f(\delta, \varpi), \varpi(\delta_0) = \varpi(0) \tag{21}$$

$f(\delta, \varpi)$ Is a continuous and decomposable hyperbolic function over a hyperbolic rectangular field □ . The function on□ satisfies

$$f(\delta, \varpi) = f_1(\delta_1, \varpi_1)\eta + f_2(\delta_2, \varpi_2)\eta^\dagger \tag{22}$$

We say that the solution of the hyperbolic differential equation

$$\varpi(\delta) = \varpi(0) + \int_{[\delta_0, \delta]_H} f(\delta, \varpi(\delta)) \otimes d\delta \tag{23}$$

Is equivalent to the solution of the hyperbolic integral equation $\frac{d\varpi}{d\delta} = f(\delta, \varpi(\delta))$, where the hyperbolic rectangular interval□

$$□ = \{(\delta, \varpi), |\delta - \delta_0|_k \preceq a, |\varpi - \varpi_0|_k \preceq b\} \tag{24}$$

Proof If the hyperbolic differential equation

$\frac{d\varpi}{d\delta} = f(\delta, \varpi(\delta))$

Has a solution $\varpi = \varpi(\delta)$.

Integrate the hyperbolic rectangular field $[\delta_0, \delta]_H$ at both ends of the above equation

$$\int_{[\delta_0, \delta]_H} \frac{d\varpi}{d\delta} \otimes d\delta = \int_{[\delta_0, \delta]_H} f(\delta, \varpi(\delta)) \otimes d\delta \tag{25}$$

Decompose the resulting integral function onto the $\eta, \eta^\dagger$ axis

$$\begin{aligned}\int_{[\delta_0,\delta]_H} \frac{d\varpi}{d\delta} \otimes d\delta &= \int_{[\delta_0,\delta]_H} f(\delta,\varpi(\delta)) \otimes d(\alpha\eta+\beta\eta^{\dagger}) \\ &= \eta\int_{\alpha_0}^{\alpha} d\varpi(\alpha) + \eta^{\dagger}\int_{\beta_0}^{\beta} d\varpi(\beta) \\ &= (\varpi(\alpha)\eta+\varpi(\beta)\eta^{\dagger}) - (\varpi(\alpha_0)\eta+\varpi(\beta_0)\eta^{\dagger}) \\ &= \varpi(\delta) - \varpi(\delta_0)\end{aligned} \tag{26}$$

$$\int_{[\delta_0,\delta]_H} \frac{d\varpi(\delta)}{d\delta} \otimes d\delta = \varpi(\delta) - \varpi(\delta_0) \Rightarrow \varpi(\delta) = \varpi(\delta_0) + \int_{[\delta_0,\delta]_H} f(\delta,\varpi(\delta)) \otimes d\delta \tag{27}$$

Where $\varpi(\delta_0) = \varpi(0)$.

This shows that the hyperbolic integral equation has a solution $\varpi = \varpi(\delta)$.

Similarly, if the hyperbolic integral equation

$\varpi(\delta) = \varpi(0) + \int_{[\delta_0,\delta]_H} f(\delta,\varpi(\delta)) \otimes d\delta$

Has a solution $\varpi = \varpi(\delta)$.

$$\frac{d\varpi}{d\delta} = \frac{d\varpi}{d(\alpha\eta+\beta\eta^{\dagger})} = \frac{d\varpi_1}{d\alpha}\eta + \frac{d\varpi_2}{d\beta}\eta^{\dagger} \tag{28}$$

Where $\varpi = \varpi_1\eta + \varpi_2\eta^{\dagger}$.

On the hyperbolic rectangular interval $[\delta_0,\delta]_H$, take the derivative of both sides of the equation with respect to δ, according to the above equation

$$\begin{aligned}f(\delta,\varpi(\delta)) &= \frac{d\varpi(\delta)}{d\delta} \\ &= \frac{d(\varpi(\alpha)\eta+\varpi(\beta)\eta^{\dagger})}{d(\alpha\eta+\beta\eta^{\dagger})} \\ &= \frac{d\varpi(\alpha)}{d\alpha}\eta + \frac{d\varpi(\beta)}{d\beta}\eta^{\dagger} \\ &= \frac{d\varpi}{d\delta}\end{aligned} \tag{29}$$

This shows that the hyperbolic differential equation has a solution $\varpi = \varpi(\delta)$, and

$$\varpi(\delta_0) = \varpi(0) + \int_{[\delta_0,\delta_0]_H} f(\delta,\varpi(\delta)) \otimes d\delta = \varpi(0) \tag{30}$$

This shows that the differential equation has a solution $\varpi = \varpi(\delta)$. So the solution of a hyperbolic integral equation is equivalent to the solution of a hyperbolic differential equation.

4. Conclusions

This study delved into the algebraic structure of hyperbolic numbers, drawing parallels with complex numbers, and specifically focused on the equivalent form of solutions to differential equations on the hyperbolic plane. By successfully equating the solutions of differential equations to integral equations, a crucial step has been taken towards proving the existence and uniqueness of solution theorems. Not only has this research laid the theoretical groundwork for the development of hyperbolic theory, but it has also provided a roadmap for future exploration. Looking ahead, there is significant potential to further explore the applications of hyperbolic numbers across various fields. In physics, hyperbolic numbers may offer new mathematical tools for describing spacetime structures, while in engineering, their algebraic properties could inspire novel approaches to signal processing

and control system design. Additionally, exploring the application of hyperbolic numbers in machine learning and artificial intelligence may reveal their potential advantages in complex pattern recognition and data processing tasks. Overall, research on hyperbolic numbers is poised to continue playing a crucial role in shaping future developments in both mathematical theory and practical applications, offering fresh insights and opportunities for academia and industry.

References

[1] Huang, Teng. On Euler number of symplectic hyperbolic manifold [J]. Advances in Mathematics, 2024, 437: 0001-8708.

[2] Mehmet Merkepci, Mohammad Abobala. On Some Novel Results about Split-Complex Numbers, the Diagonalization Problem, and Applications to Public Key Asymmetric Cryptography [J]. Journal of Mathematics, 2023.

[3] BoGwang Jeon; on the number of hyperbolic Dehn fillings of a given volume [J].American Mathematical Society, 2021.

[4] Rooney, J. Generalized complex numbers in mechanics [J]. Mechanisms and Machine Science, 2014, 22(7): 55-62.

[5] Ulrich, S. Relativistic quantum physics with hyperbolic numbers [J]. Physics Letters B, 2005, 625(3-4): 313-323.

[6] Zijia Zhang, Yaoming Cai, Dongfang Zhang. Solving Ordinary Differential Equations with Adaptive Differential Evolution [J]. IEEE Access, 2020, 8: 128908-128922.

[7] Lucas Backes, Davor Dragičević, Masakazu Onitsuka, Mihály Pituk. Conditional Lipschitz Shadowing for Ordinary Differential Equations [J]. Journal of Dynamics and Differential Equations, 2023.

[8] Hou Yunhui, Limnios Nikolaos, Schön Walter. On the Existence and Uniqueness of Solution of MRE and Applications [J]. Methodol Comput Appl Probab, 2017, 19(4): 1241–1250.

[9] Juan E. Santos, José M. Carcione, Jing Ba. Existence and uniqueness of solutions of thermo-poroelasticity [J]. Journal of Mathematical Analysis and Applications, 2021, 499(1):124907.

[10] Feng Yibin, Hu Shengnan, Liu Weiru. Existence and uniqueness of solutions to the Orlicz Aleksandrov problem [J]. Calculus of Variations and Partial Differential Equations, 2022, 61(4):1-23.

[11] Boffi, Daniele. On the existence and the uniqueness of the solution to a fluid-structure interaction problem [J]. Journal of Differential Equations, 2021, 279:136-161.

[12] Feng, Y., Hu, S. & Liu, W. Existence and uniqueness of solutions to the Orlicz Aleksandrov problem [J]. Caculus of Variations and partial Differential Equations, 2022,148.

[13] Amira Rahai, Ali Slimani, Amar Guesmia. Existence and uniqueness of solution for a fractional thixotropic model [J]. Mathematical Methods in the Applied Sciences, 2023, 46.13684-13694.

[14] Shakhmurov V.B.: Coercive boundary value problems for regular degenerate differential- operator equations [J]. Journal of Mathematical Analysis and Applications, 2004, 292(2): 605–620.

[15] Mohsin, M.On existence and uniqueness of solutions to a pantograph type equation [J]. The anziam Journal. 2020; 62(4):489-512.

[16] Öztürk İskender, Özdemir Mustafa. Affine transformations of hyperbolic number plane [J]. Boletín de la Sociedad Matemática Mexicana, 2022, 28(3):1:20.

[17] Luna–Elizarrarás M. Elena, Golberg, Anatoly. More about Bicomplex Möbius Transformations: Geometric, Algebraic and Analitical Aspects [J]. Advances in Applied Clifford Algebras, 2024, 34(3): 1-35.

[18] M. Elena Luna-Elizarrarás, Michael Shapiro, Daniele C. Struppa, Adrian Vajiac. Bicomplex Holomorphic Functions [M]. Springer International Publishing Switzerland, 2015.

[19] Emanuello John A., Nolder Craig A. Projective Compactification of $\mathbb{R}^{1,1}$and Its Möbius Geometry [J]. Complex Analysis and Operator Theory, 2015, 9(2): 329-354.

[20] Ritu GOYAL. Bicomplex Polygamma Function [J]. Tokyo Journal of Mathematics, 2007, 30(2):523-530.