# Stochastic Recursive Zero-sum Differential Games under Model Uncertainty

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**Abstract.** In this paper, we study stochastic recursive zero-sum differential game problem where the payoff function is described by the solutions of a class of backward stochastic differential equations with uncertainty parameter $\boldsymbol{\theta}$, which are used to represent different market conditions. In such case, the existence of saddle points for stochastic differential game problems above-mentioned is proved.

**Keywords:** Backward stochastic differential equations; Stochastic recursive zero-sum differential game; Saddle point; Model uncertainty.

## 1. Introduction

Peng and Pardoux [1] introoduced the nonlinear backward stochastic differential equations (BSDEs), and at the same time, they proved the uniqueness and existence of corresponding solutions. From economic background, Dufffie and Epstein [2] propose a stochastic differential recursive utility in [2]. Especially, In the article [3] from El Karoui and Pengr, within a finite time range, the cost function is defined by y(0):

$$\begin{cases} x(t) = x_0 + \\ \qquad \int_o^t b\big(s, x(s), u(s)\big)ds + \int_0^t c(s, x(s), u(s))dB(s), \\ y(t) = \varphi\big(x(T)\big) + \\ \qquad \int_t^T f\big(s, x(s), y(s), z(s), u(s)\big)ds - \int_t^T z(s)dB(s), \end{cases}$$

Where $B = (B(t))_{0\le t\le T}$ is given as a standard d-dimensional Brownian motion on the space $(\Omega, \mathcal{F}, \mathrm{P})$.

Especially, due to the uncertainty of the model, it is difficult to know the actual diffusion coefficient and drift coefficient. However, these coefficients may be different in different markets. Given the probability $\lambda$ of a bull market, the probability $\lambda$ is unknown. So we can measure the cost in the following ways,

$$J(u) = \sup_{\lambda\epsilon[0,1]} (\lambda y_1\big(0\big) + (1-\lambda)y_2(0))$$

$$= \max\big(y_1(0), y_2(0)\big).$$

In what follows, $\Theta$ is defined as a locally compact Polish space, we use $\theta\epsilon\Theta$ to indicate different market conditions. The corresponding cost $y_\theta(0)$ is given by

$$\begin{cases} x_\theta(t) = x_0 + \\ \quad \int_o^t b_\theta\big(s, x_\theta(s), u(s)\big)ds + \int_0^t c_\theta(s, x(s), u(s))dB(s), \\ y_\theta(\,t) = \varphi_\theta\big(x_\theta(T)\big) + \\ \quad \int_t^T f_\theta\big(s, x_\theta(s), y_\theta(s), z_\theta(s), u(s)\big)ds - \int_t^T z_\theta(s)dB(s), \end{cases}$$

Where the cofficients depend on the market uncertainty parameter $\theta$. Let us say $\mathcal{Q}$ is the set of $\theta$.

Game theory can help us understand finance, securities, biology, economics, political science, computer science and international relations by considering the predicted and actual behavior of individuals in games and studying their optimization strategies. Stochastic differential game problem has been widely used in various social and scientific problems.

In markov framework, we briefly describe stochastic differential games. The dynamic of a controlled system is a process $x = (x_t)_{t\leq T}$, which is the solution of the following standard stochastic differential equation (SDG)

$$x(t) = x_0 + \int_o^t b\,(s, x(s), u(s), v(s))ds + \int_0^t \sigma\,(s, x(s), u(s), v(s))dB(s)\,, t \leq T$$

There are two agents $c_1$ and $c_2$ intervence in a zero-sum stochastic differential game (ZSDG). Each agent carries out an admissible control towards the system, we define $u = (u_t)_{t\leq T}(resp. v = (v_t)_{t\leq T})$ for $c_1(resp. c_2)$. As we can see in the following example

$$J(u, v) = E[g(x_T) + \int_0^T h(s, x(s), u(s), v(s))ds]$$

A saddle point for a ZSDG is an admissible control $(u^*, v^*)$ that satisfies $J\,(u^*, v) \leq J\,(u^*, v^*) \leq J\,(u, v^*)$ for any $(u, v)$. The next work we want to do is solve for the saddle points of the zero-sum differential game.

Many scholars at home and abroad have done a lot of research on zero-sum stochastic differential game, especially using the tool of backward stochastic differential equation. Hamadne and Lepeltier [4] obtained the saddle point existence results of zero-sum stochastic differential games with the following payoff.

$$J(u, v) = E^{(u,v)}\left[g(x_T) + \int_0^T h\big(s, x(s), u(s), v(s)\big)ds\right].$$

EI Karoui et al. [3] show the formula of recursive utility and studied recursive differential game problem. Wang and Yu [5, 6] proved the uniqueness and existence of the equilibrium point. Wei and Wu [7] defined the cost functional in terms of the solution of the following controlled backward stochastic differential equation

$$\begin{cases} -dY_s = C(s, x_s, Y_s, u_s, v_s)ds - Z_s dB_s^{u,v} \\ y_T = g(x_T) \end{cases},$$

In their paper, the problem for the existence of the saddle points for random zero-sum differential games is proved, and the best cost functions for specific solutions are given. The zero-sum game problems have a very wide application background, especially in the field of finance.

In our paper, the stochastic zero-sum differential game under model uncertainty is studied, the proposed model is as follows

$$\begin{cases} x_\theta(t) = x_0 + \\ \int_o^t b_\theta(s, x_\theta(s), u(s), v(s))ds + \int_0^t c_\theta(s, x(s), u(s), v(s))dB(s)\,, \\ y_\theta(t) = \varphi_\theta\big(x_\theta(T)\big) + \\ \int_t^T f_\theta(s, x_\theta(s), y_\theta(s), z_\theta(s), u(s), v(s))ds - \int_t^T z_\theta(s)dB(s), \end{cases}$$

which the cost function is described by the solution of a class of BSDE with uncertain parameters $\theta$,

$$J(u, v) = \sup_{Q\in\mathcal{Q}} \int_\Theta\ y_\theta(0)Q(d\theta).$$

This kind of stochastic zero-sum differential game problem has certain practical significance. For example, there is a growing awareness of the need for more advanced modeling in mathematical

finance, and the problem of model uncertainty becomes a focus of attention. In the financial market, uncertain factors always exist, but by discussing the game under the uncertain model, we can prove that the market will eventually reach Nash equilibrium, so as to ensure the fairness of the market.

The paper is organized as follows. In Section 2, we first introduce some notation and assumptions, then we formulate the stochastic recursive zero-sum differential game problem under model uncertainty. In Section 3, we define the payoff function and prove the existence of the saddle point.

## 2. Preliminaries

Throughout this paper, let $\{B(t): 0 \leq t \leq T\}$ be defined as an $m$-dimensional Brownian motion, which is on the space $(\Omega, \mathcal{F}, P)$.

Let us introduce some spaces: for any $p \geq 1$,

$L^p(\mathcal{F}_t; R^n)$ is the space of $R^n$-valued $\mathcal{F}_t$-measurable, where $\xi$ satisfying $\mathrm{E}[\,|\xi|^p] < \infty$;

$\mathcal{H}^p(0,T;R^n)$ is a space of $R^n$-valued $\mathcal{F}$-progressively measurable processes $z(t)$ on $[0,T]$ satisfying

$$\mathrm{E}\,[(\int_0^T |z(t)|^2\,\mathrm{dt})^{\frac{p}{2}}] < \infty;$$

$\mathcal{H}^{1,p}(0,T;R^n)$ is a space of $R^n$-valued $\mathcal{F}$-progressively measurable processes $z(t)$ on $[0,T]$ satisfying

$$\mathrm{E}\,[(\int_0^T |z(t)|\,\mathrm{dt})^p] < \infty;$$

$S^p(0,T;R^n)$ is a space of $R^n$-valued $\mathcal{F}$-adapted continuous processes $y(t)$ on $[0,T]$ satisfying

$$\mathrm{E}\,[\sup_{0\leq t\leq T} [y(t)|^p] < \infty;$$

$C(0,T;R^n)$ is a space of $R^n$-valued continuous functions on $[0,T]$;

$\mathcal{P}$ is the $\sigma$-algebra on $[0,T]\times\Omega$ of $\mathcal{F}_t$-progressively sets.

For each $\theta \in \Theta$, we consider a mapping $\sigma = (\sigma_{ij}): (t, x_\theta) \in [0,T]\times C \mapsto \sigma(t,x_\theta) \epsilon R^{m\times m}$ satisfying the following conditions:

(i) for every $1 \leq i,j \leq m$, $\sigma_{ij}$ is measurable;

(ii) there exists $K$ such that $|\sigma_\theta(t,x_\theta) - \sigma_\theta(t,x_\theta')| \leq K(1+|x_\theta|_t)$, where $K$ is a constant;

(iii) for any $(t,x_\theta)\epsilon[0,T]\times C$, the matrix $\sigma_\theta(t,x_\theta)$ is invertible and $|\sigma_\theta{}^{-1}(t,x_\theta)| \leq K(1+|x_\theta|_t^m)$ for some constants K and m>0.

Then, the equation

$$x_\theta(t) = x_\theta(0) + \int_0^t \sigma_\theta(s, x_\theta(s))dB(s)\,, t \leq T$$

has a unique solution $(x_\theta(t))$.

Next, let us consider a space $A(resp.B)$, and $\mathcal{U}(resp.\mathcal{V})$ is the space of $\mathcal{P}$-measurable $u := (u_t)_{t\leq T}(resp. v := (v_t)_{t\leq T})$ with values in $A(resp.B)$. For each $\theta \in \Theta$, let $\Phi: [0,T]\times C\times\mathcal{U}\times\mathcal{V}\to R^m$ be such that

(i)for any $(t,x_\theta)\epsilon[0,T]\times C$, the mapping $(u,v)\to\Phi_\theta(t,x_\theta,u,v)$ is consecutive;

(ii) for every $(u,v)\in A\times B$, $\Phi_\theta(\cdot, x_\theta(\cdot), u, v)$ is $\mathcal{P}$-measurable;

(iii) there exists $K$ such that $|\Phi_\theta(t,x_\theta,u,v)| \leq K(1+|x_\theta|_t)$ for any $t, x_\theta, u$ and $v$, where $K$ is a constant;

(iv) there exists $M$ such that $|\sigma_\theta{}^{-1}(t,x_\theta)\Phi_\theta(t,x_\theta,u,v)| \leq M$ for any $t, x_\theta, u$ and $v$, where $M$ is a constant;

Some of the assumptions of this article are as follows,

(H1) There exists $L$, for any $t\in[0,T]$, $x, x'\in R^n, y, y'\in R, z, z'\in R^d$, $u, u'\in U$, $v, v'\in U$, $\theta\in\Theta$,

$$\left|b_\theta(t,x,u,v)-b_\theta\left(t,x',u',v'\right)\right|+\left|c_\theta(t,x,u,v)-c_\theta\left(t,x',u',v'\right)\right|\le L(\left|x-x'\right|+\left|x-x'\right|)$$

$$\begin{gathered}\left|\varphi_\theta(x)-\varphi_\theta\left(x'\right)\right|+\left|f_\theta(t,x,y,z,u,v)-f_\theta\left(t,x',y',z',u',v'\right)\right|\\ \le L\left(1+|x|+\left|x'\right|+|u|+\left|u'\right|+|v|+\left|v'\right|\right)\left(\left|x-x'\right|+\left|u-u'\right|+\left|y-y'\right|+\left|z-z'\right|+\left|v-v'\right|\right)\\ \left|b_\theta(t,0,0,0)\right|+\left|c_\theta(t,0,0,0)\right|+\left|f_\theta(t,0,0,0,0,0)\right|\le L\end{gathered}$$

(H2) $b_\theta, c_\theta, \varphi_\theta, f_\theta$ are continuously differentiable in $(x,y,z,u,v)$ for any $\theta\in\Theta$.

(H3) There exists a $\bar{\omega}\colon[0,\infty)\to[0,\infty)$ such that

$$\left|l_\theta(t,x,y,z,u,v)-l_\theta\left(t,x',y',z',u',v'\right)\right|\le\bar{\omega}(\left|x-x'\right|+\left|y-y'\right|+\left|z-z'\right|+\left|u-u'\right|+\left|v-v'\right|)$$

for any $t\in[0,T]$, $x,x'\in R^n$, $y,y'\in R$, $z,z'\in R^d$, $u,u'\in U$, $v,v'\in U$, $\theta\in\Theta$, where $l_\theta$ is the derivative of $b_\theta, c_\theta, \varphi_\theta, f_\theta$ in $(x,y,z,u,v)$.

(H4) For every $N>0$, there exists a $\bar{\omega}_N(\mu(\theta,\theta')$, which ia a modulus of continuity;

for every $t\in[0,T]$, $|x|,|y|,|z|,|u|,|v|\le N$, $\theta,\theta'\in\Theta$, where $l_\theta$ is $b_\theta, c_\theta, \varphi_\theta, f_\theta$ in $(x,y,z,u,v)$.

(H5) $\mathcal{Q}$ is a convex and weakly compact set on $(\Theta,\mathcal{B}(\Theta))$.

For $(u,v)\in\mathcal{U}\times\mathcal{V}$, for every $\theta\in\Theta$, we define the measure $\mathcal{P}^{u,v}$ as

$$\frac{d\mathcal{P}^{u,v}}{d\mathcal{P}}=\exp\left\{\int_0^T\sigma_\theta{}^{-1}(s,x_\theta(s))\Phi_\theta(s,x_\theta(s),u(s),v(s))dB(s)-\frac{1}{2}\int_0^T\left|\sigma_\theta{}^{-1}(s,x_\theta(s))\Phi_\theta(s,x_\theta(s),u(s),v(s))\right|^2ds\right\}$$

Due to Girsanov's theorem, the process under the probability $\mathcal{P}^{u,v}$

$$B_t^{u,v}=B_t-\int_0^t\sigma_\theta{}^{-1}(s,x_\theta(s))\Phi_\theta(s,x_\theta(s),u(s),v(s))ds,t\le T$$

is a Brownian motion, and

$$x_\theta(t)=x_0+\int_o^t\Phi_\theta(s,x_\theta(s),u(s),v(s))ds+\int_0^t\sigma_\theta(s,x(s))dB(s)^{u,v},t\le T$$

$x_\theta(t)_{t\le T}$ is a weak solution.

Let us think about a system, whose evolution is described by the process $x_\theta(t)_{t\le T}$. There are two agents $c_1$ and $c_2$ intervene on the system. We define one control action for $c_1$(resp. $c_2$) is a process $u=(u_t)_{t\le T}(resp.\,v=(v_t)_{t\le T})$, which is belonging to $\mathcal{U}(resp.\mathcal{V})$.

SDE can be described by

$$x_\theta(t)=x_0+\int_o^t\Phi_\theta(s,x_\theta(s),u(s),v(s))ds+\int_0^t\sigma_\theta(s,x(s),u(s),v(s))dB(s),\qquad(1)$$

where $\theta\in\Theta$ and $\Theta$ is a locally compact, the relevant payoff function is given by the $y_\theta(0)$ term on [0, T]:

$$y_\theta(t)=\varphi_\theta(x_\theta(T))+\int_t^T f_\theta(s,x_\theta(s),y_\theta(s),z_\theta(s),u(s),v(s))ds-\int_t^T z_\theta(s)dB(s)\qquad(2)$$

## 3. The existence of the saddle point

Due to the uncertainty of the model, the payoff function is defined as

$$J(x_0,u,v)=\sup_{Q\in\mathcal{Q}}\int_\Theta y_\theta(0)Q(d\theta)\qquad(3)$$

where $\mathcal{Q}$ is a set of all possible probability measures on $(\Theta,\mathcal{B}(\Theta))$.

For the FBSDE game, a Nash equilibrium is a pair $(u^*,v^*)\in\mathcal{U}\times\mathcal{V}$ such that

$$J(u^*,v)\le J(u^*,v^*)\le J(u,v^*)\text{for all }u,v.\qquad(4)$$

From this we deduce that

$$\inf_{v\epsilon\mathcal{V}}\sup_{u\epsilon\mathcal{U}} J(u,v) \le \sup_{u\epsilon\mathcal{U}} J(u^*,v) \le J(u^*,v^*) \le \inf_{v\epsilon\mathcal{V}} J(u,v^*) \le \sup_{u\epsilon\mathcal{U}}\inf_{v\epsilon\mathcal{V}} J(u,v) \quad (5)$$

since we always have infsup $\geq$ supinf, we conclude that

$$\inf_{v\epsilon\mathcal{V}}\sup_{u\epsilon\mathcal{U}} J(u,v) = \sup_{u\epsilon\mathcal{U}} J(u^*,v) = J(u^*,v^*) = \inf_{v\epsilon\mathcal{V}} J(u,v^*) = \sup_{u\epsilon\mathcal{U}}\inf_{v\epsilon\mathcal{V}} J(u,v) \quad (6)$$

i.e., $(u^*,v^*) \in \mathcal{U}\times\mathcal{V}$ is a saddle point for $J(u,v)$.

In this case, only one Hamiltonian is needed, and only one set of adjoint equations.

We introduce two functions $C(t,x_\theta,y_\theta,u,v)$ and $g(x_\theta)$ satisfying the following assumption to define the payoff function: there exists $L>0$, for all $x_\theta, x_\theta{}' \in \mathcal{H}^{2,m}$, $y_\theta, y_\theta{}' \in S^2$, and each $\theta \in \Theta$, such that

$$|C_\theta(t,x_\theta(t),y_\theta(t),u,v) - C_\theta(t,x_\theta{}'(t),y_\theta(t),u,v)| \le L|x_\theta(t)-x_\theta{}'(t)| \quad (7)$$

$$\big(y_\theta(t)-y_\theta{}'(t)\big)\big(C_\theta(t,x_\theta(t),y_\theta(t),u,v) - C_\theta(t,x_\theta{}'(t),y_\theta(t),u,v)\big) \le L\big(y_\theta(t)-y_\theta{}'(t)\big)^2 \quad (8)$$

and $g_\theta(x_\theta) \in L^2$ is measurable.

In [8], let $\bar{u},\bar{v}$ be optimal controls and $(\bar{x}_\theta,\bar{y}_\theta,\bar{z}_\theta)$ be the corresponding stste process of (2.1) and (2.2) for every $\theta\in\Theta$. In particular, from lemma3.6 in [8], assume that the conditions (H1) -(H5) are sure. For every $u\in\mathcal{U}[0,T]$, there exists a probability $\bar{Q}\in\mathcal{Q}^{\bar{u}}$ so that

$$\sup_{Q\epsilon\mathcal{Q}^{\bar{u}}} \int_\Theta y_\theta(0)Q(d\theta) = \int_\Theta y_\theta(0)\bar{Q}(d\theta) \quad (9)$$

so, the payoff $J(u,v)$ can given by

$$J(u,v) = \sup_{Q\epsilon\mathcal{Q}} \int_\Theta y_\theta(0)Q(d\theta) = \int_\Theta y_\theta(0)\bar{Q}(d\theta) \quad (10)$$

where y satisfies the following BSDE:

$$-dy_\theta(s) = C_\theta\big(s,x_\theta(s),y_\theta(s),u(s),v(s)\big)ds - z_\theta(s)dB(s)^{u,v}, y_\theta(T) = g_\theta\big(x_\theta(T)\big) \quad (11)$$

We can draw the conclusion from [9], for $(u,v)$, there is a unique solution $(y_\theta,z_\theta)$. $c_1$ wants to minimize this payoff, however, $c_2$ wants to maximize the same payoff. We need to study the existence of the saddle point in this game, which is actually a pair $(u^*,v^*)$ of strategies, so that for each $u,v\in\mathcal{U}\times\mathcal{V}$each strategy, we can get $J(u^*,v)\le J(u^*,v^*)\le J(u,v^*)$.

For $(t,x_\theta,y_\theta,z_\theta,u,v)\in[0,T]\times C\times R\times R^m\times\mathcal{U}\times\mathcal{V}$, we introduced the Hamiltonian by

$$H_\theta(t,x_\theta,y_\theta,z_\theta,u,v) = z_\theta{\sigma_\theta}^{-1}(t,x_\theta)\Phi_\theta(t,x_\theta,u,v) + C_\theta(t,x_\theta,y_\theta,u,v) \quad (12)$$

the Isaacs' condition holds if for$(t,x_\theta,y_\theta,z_\theta) \in [0,T]\times C\times R\times R^m$,

$$\max_{v\epsilon\mathcal{V}}\min_{u\epsilon\mathcal{U}} H_\theta(t,x_\theta,y_\theta,z_\theta,u,v) = \min_{u\epsilon\mathcal{U}}\max_{v\epsilon\mathcal{V}} H_\theta(t,x_\theta,y_\theta,z_\theta,u,v). \quad (13)$$

Suppose that the Issacs' condition is satisfied, we can get there exists $u^*\colon[0,T]\times C\times R\times R^m\to\mathcal{U}$, $v^*\colon[0,T]\times C\times R\times R^m\to\mathcal{V}$, by the selection theorem (see Benes [10]), makes the following formula true

$$H_\theta(t,x_\theta,y_\theta,z_\theta,u^*,v) \le H_\theta(t,x_\theta,y_\theta,z_\theta,u^*,v^*) \le H_\theta(t,x_\theta,y_\theta,z_\theta,u,v^*) \quad (14)$$

Due to $\sigma,\Phi,$ and $C$, the function $H_\theta(t,x_\theta,y_\theta,z_\theta,u^*(t,x_\theta,y_\theta,z_\theta),v^*(t,x_\theta,y_\theta,z_\theta))$ is Lipschitz in $z_\theta$.

Then, the main result in this section can be given as follows.

Theorem 3.1. $(y_\theta^*,z_\theta^*)$ is the solution of the following BSDE

$$-dy_\theta^*(s) = H_\theta(s,x_\theta(s),y_\theta^*(s),z_\theta^*(s),u^*,v^*)ds - z_\theta^*(s)dB(s), y_\theta^*(T) = g_\theta\big(x_\theta(T)\big) \quad (15)$$

where $u^* = u^*\big(s,x_\theta(s),y_\theta^*(s),z_\theta^*(s)\big)$,
$v^* = v^*\big(s,x_\theta(s),y_\theta^*(s),z_\theta^*(s)\big)$.

Then, the pair $(u^*, v^*)$ is the saddle point for the game , and $\int_\Theta y_\theta^*(0)\bar{Q}(d\theta)$ is the optimal payoff $J(u^*, v^*)$.

Proof. Given the following BSDE:

$$y_\theta^*(t) = g_\theta\big(x_\theta(T)\big) + \int_t^T H_\theta(s, x_\theta(s), y_\theta^*(s), z_\theta^*(s), u^*, v^*)ds - \int_t^T z_\theta^*(s)dB(s). \quad (16)$$

Due to the theorem 2.1 in [9], the above equation gets a unique solution $(y_\theta^*, z_\theta^*)$. We know $y_\theta^*(0)$ is definite,

$$\begin{aligned} y_\theta^*(0) &= E^{u^*,v^*}[y_\theta^*(0)] \\ &= E^{u^*,v^*}\left[g_\theta\big(x_\theta(T)\big) + \int_0^T H_\theta(s, x_\theta(s), y_\theta^*(s), z_\theta^*(s), u^*, v^*)ds - \int_0^T z_\theta^*(s)dB(s)\right] \\ &= E^{u^*,v^*}\left[g_\theta\big(x_\theta(T)\big) + \int_0^T C_\theta(s, x_\theta(s), y_\theta^*(s), u^*, v^*)ds - \int_0^T z_\theta^*(s)dB^{u^*,v^*}(s)\right]. \end{aligned} \quad (17)$$

We can get $\int_\Theta y_\theta^*(0)\bar{Q}(d\theta) = J(u^*, v^*)$.

For any $u \in \mathcal{U}, v \in \mathcal{V}$, then we let

$$\begin{aligned} y_\theta(t) &= g_\theta\big(x_\theta(T)\big) + \int_t^T C_\theta(s, x_\theta(s), y_\theta(s), u^*, v)ds - \int_t^T z_\theta(s)dB^{u^*,v}(s) \\ &= g_\theta\big(x_\theta(T)\big) + \int_t^T H_\theta(s, x_\theta(s), y_\theta(s), z_\theta(s), u^*, v)ds - \int_t^T z_\theta(s)dB(s) \end{aligned} \quad (18)$$

$$\begin{aligned} y_\theta{}'(t) &= g_\theta\big(x_\theta(T)\big) + \int_t^T C_\theta(s, x_\theta(s), y_\theta'(s), u, v^*)ds - \int_t^T z_\theta(s)dB^{u^*,v}(s) \\ &= g_\theta\big(x_\theta(T)\big) + \int_t^T H_\theta(s, x_\theta(s), y_\theta'(s), z_\theta'(s), u, v^*)ds - \int_t^T z_\theta'(s)dB(s) \end{aligned} \quad (19)$$

By the the inequality (3.1) and the comparison theorem of BSDEs, we can compare the solutions of (3.14), (3.16) and (3.17), and get

$$y_\theta(t) \le y_\theta^*(t) \le y_\theta{}'(t), 0 \le t \le T \quad (20)$$

so for each $u \in \mathcal{U}, v \in \mathcal{V}$, there exists a probability $\bar{Q}' \in \mathcal{Q}$ so that

$$\int_\Theta y_\theta(0)\bar{Q}'(d\theta) \le \sup_{\bar{Q}' \in \mathcal{Q}} \int_\Theta y_\theta(0)\bar{Q}'(d\theta) = \int_\Theta y_\theta(0)\bar{Q}(d\theta) \le \int_\Theta y_\theta^*(0)\bar{Q}(d\theta) \quad (21)$$

that is

$$\int_\Theta y_\theta(0)\bar{Q}'(d\theta) = J(u^*, v) \le J(u^*, v^*) = \int_\Theta y_\theta^*(0)\bar{Q}(d\theta) \quad (22)$$

In a similar way, there exists a probability $\bar{Q}'' \in \mathcal{Q}$ so that

$$\int_\Theta y_\theta^*(0)\bar{Q}(d\theta) = J(u^*, v^*) \le J(u, v^*) = \int_\Theta y_\theta'(0)\bar{Q}''(d\theta) \quad (23)$$

So finally, we can get

$$J(u^*, v) \le J(u^*, v^*) \le J(u, v^*) \quad (24)$$

and the pair $(u^*, v^*)$ is a saddle point for the game.

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