Stochastic Recursive Zero-sum Differential Games with Payoff Functional in BDSDEs under Model Uncertainty

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Abstract. In this paper, problems related to stochastic recursive 0-sum differential games under model uncertainty are studied. In this model, the cost function is defined by $\int_\Theta E[y_\theta(0)]Q(d\theta)$, in which $y_\theta(0)$ is the solution of a family of forward and backward doubly stochastic differential equations. We deduced the existence of the saddle points for this robust problem with the selection theorem and comparison theorem of BDSDEs. The specific form of the cost function has also been given.

Keywords: Stochastic recursive 0-sum differential game; Saddle point; Model uncertainty; Backward doubly stochastic differential equations; Compare theorem.

1. Introduction

Bismut [1] introduced the linear BSDEs, with the consider of stochastic optimal control problems in 1978. Then Pardoux and Peng [2] formulated basic frameworks of nonlinear BSDEs and the existence and uniqueness of their solutions were proved.

The classical form of BSDES is

$$y(t) = \xi + \int_t^T f\big(\omega, s, y(s), z(s)\big)ds - \int_t^T z(s)dW(s) \tag{1}$$

where $\{W(t)\}_{0\le t\le T}$ is a naturally standard Brownian motion, f is a given function under some assumptions and ξ is a given random variable. Thus, the problems related to reaching the given terminal of systems in an environment with some stochastic disturbance can be described as a BSDE. Then, adding a backward Itô's integral to respect policies and other inside information, Peng [3] introduces BSDEs with double Brownian motion processes (BDSDEs):

$$y(t) = \xi + \int_t^T f\big(\omega, s, y(s), z(s)\big)ds + \int_t^T g\big(\omega, s, y(s), z(s)\big)dB(s) - \int_t^T z(s)dW(s).$$

In recent decades, the theories of backward doubly stochastic differential equations have been extensively used in financial fields, especially in option pricing and the derivative security market. The comparison theorem of BDSDEs can be referred to Shi and Gu [4], then Auguste introduced a new kind of BSDESs with reflection in [5]:

$$y_t = \xi + \int_t^T f\big(\omega, s, y(s), z(s)\big)dr + \int_t^T g\big(\omega, s, y(s), z(s)\big)dB(r) - \int_t^T z(s)dW(s) + K_t,$$

and proved that the solution of this kind of RBSDEs exists and is unique.

When the solution of (1) is used to describe the cost functional of a stochastic optimal control problem, the problem can be called a recursive optimal control problem. It is different from classical problems about stochastic optimal control, since the generator f is now dependent of the parameters y and z.

If there are two agents c_1 and c_2 who can both control the system through the accessible control u and v, let the cost function $J(u, v)$ be the payoff of c_1 and the reward of c_2, then this problem is called a 0-sum stochastic differential game (ZSDG). There are also many research achievements on this kind of games, especially on this kind of stochastic differential games with the cost function defined through the solutions of BSDEs with controls. Hamane and Lepeltier [6] consider the cost function:

$$J(u,v) = E^{(u,v)}\left[g(x_T) + \int_t^T h(s, x_s, u_s, v_s)ds\right],$$

and proved the existence of the saddle point. Wang and Wu [7] gave the uniqueness of saddle point with the help of the maximum principle. El Karoui [8] introduced the formula of recursive utility and considered the recursive differential game problems. Wei and Wu [9] proved the uniqueness of saddle point when the solution of the following BSDE is defined as the cost function:

$$\begin{cases} -dy(t) = C(t, x(t), y(t), u, v)dt - z(t)dW^{u,v}(t), \\ y(T) = g\big(x(T)\big), \end{cases}$$

and deduced the optimal cost function.

In this paper, the stochastic 0-sum differential game problems with payoff function in BDSDEs under model uncertainty are considered. The original cost function is $J(u,v) = y(0)$, where

$$\begin{cases} x(t) = x_0 + \int_0^t b\big(s, x(s), u(s), v(s)\big)ds + \int_0^t \sigma\big(s, x(s)\big)dW(s), \\ y(t) = \varphi\big(x(T)\big) + \int_t^T f(s, x(s), y(s), z(s), u, v)ds \\ \qquad + \int_t^T g\big(s, x(s), y(s)\big)dB(s) - \int_t^T z(s)dW(s). \end{cases}$$

Hu and Wang [10] introduced the stochastic recursive optimal problems under model uncertain. Combining with the ideas of robust problems, they assume that the coefficients can be different under different market conditions. If respect the different market conditions with parameter $\theta \in \Theta$, then there should be a random variable, which has uncertain possible probability distribution $Q \in \mathcal{Q}$, on $\big(\Theta, \mathcal{B}(\Theta)\big)$. Note that Θ should be a Polish space which is locally compact. Thus, we can formulate the model with uncertainty:

$$\begin{cases} x_\theta(t) = x_0 + \int_0^t \Phi_\theta\big(s, x_\theta(s), u(s), v(s)\big)ds + \int_0^t \sigma_\theta\big(s, x_\theta(s)\big)dW(s), \\ y_\theta(t) = g_\theta\big(x_\theta(T)\big) + \int_t^T C_\theta\big(s, x_\theta(s), y_\theta(s), u(s), v(s)\big)ds \\ \qquad + \int_t^T D_\theta\big(s, x_\theta(s), y_\theta(s)\big)dB(s) - \int_t^T z_\theta(s)dW^{u,v}(s). \end{cases}$$

with the cost function:

$$J(u,v) = \sup_{Q\in\mathcal{Q}} \int_\Theta E[y_\theta(0)]Q(d\theta).$$

In this paper, we consider the existence of the saddle point of this kind of 0-sum differential games under model uncertainty with the cost function defined as solution of BDSDEs and give the form of the optimal cost function. For this point, we have to use some theorems about backward double stochastic differential equations, including the selection theorem when Isaacss condition is satisfying and comparison theorem of BDSDEs, with the consideration of some idea related to the robust problem, like some weakly convergence methods.

This paper is divided into 3 sections and is organized as follows. In Section 1, the development process of BDSDEs, zero-sum games and model uncertainty are given, and then we introduce the problems we considering in this paper. In Section 2, we give the fundamental nations and some assumptions for dealing with our problems. In Section 3, the proof of the existence of saddle point and the form of the optimal cost function are given.

2. Preliminaries

In this section, we at first give some normal definitions and then make some proper assumptions for the convenience of considering our problems.

There are two standard Brownian motion processes $\{W_t, 0 \le t \le T\}$ and $\{B_t, 0 \le t \le T\}$, which are independent naturally, defined on $(\Omega, \mathcal{F}, P)$. Their values are respectively in $\mathbb{R}^m$ and $\mathbb{R}^d$. Then let $\mathcal{N}$ denote the subset of $\mathcal{F}$ with all of P-null sets in which. For each $t \in [0,T]$, define

$$\mathcal{F}_t \triangleq \mathcal{F}_t^W \bigvee \mathcal{F}_{t,T}^B,$$

in which for any process $\{X_t\}$, define $\mathcal{F}_{s,t}^X = \sigma\{X_r - X_s; s \le r \le t\} \vee \mathcal{N}, \mathcal{F}_t^X = \mathcal{F}_{0,t}^X$.

Then for any p ≥ 1, define the following Banach spaces:

$\mathcal{L}^p$ is the space of $\mathcal{F}_T$-measurable random variable ξ satisfying $E[|\xi|^p] < \infty$;

$\mathcal{M}^p$ is the space of $\mathcal{F}_t$ progressively measurable process $\{u(t)\}_{0 \le t \le T}$ satisfying $E\left(\int_0^T |u(t)|^p dt\right) < +\infty$;

$\mathcal{M}^\infty$ is the space of $\mathcal{F}_t$ progressively measurable process $\{u(t)\}_{0 \le t \le T}$ satisfying $ess \sup_{(t,\omega)} |u(t)| < +\infty$;

$\mathcal{H}^p$ is the space of $\mathcal{F}_t$ progressively measurable process $\{z(t)\}_{0 \le t \le T}$ satisfying $E\left[\left(\int_0^T |z(t)|^2 dt\right)^{\frac{p}{2}}\right] < +\infty$;

$\mathcal{H}^{1,p}$ is the space of $\mathcal{F}_t$ progressively measurable process $\{z(t)\}_{0 \le t \le T}$ satisfying $E\left[\left(\int_0^T |z(t)|\, dt\right)^p\right] < +\infty$;

$\mathcal{S}^p$ is the space of $\mathcal{F}_t$ progressively measurable process $\{y(t)\}_{0 \le t \le T}$ satisfying $E(\sup_{0 \le t \le T} |y(t)|^p) < +\infty$;

$\mathcal{C}$ is the space of continuous functions on $[0,T]$;

$\mathcal{P}$ is the σ-algebra on $[0,T] \times \Omega$ of $\mathcal{F}_t$-progressively sets.

In the FBDSDEs introduced above, the process $x(t)$ is used to describe the initial evolution of a controlled system. For examples, in option pricing, $x(t)$ can describe the price process of a stock. Then based on $x_\theta(t)$ and the given Brown motion dW_t, we can build a new measure $\mathcal{P}_\theta^{u,v}$.

For each $\theta \in \Theta$, consider a mapping $\sigma_\theta\left(\sigma_{\theta_{ij}}\right): (t, x_\theta) \in [0,T] \times \mathcal{C} \to \sigma(t, x_\theta) \in \mathbb{R}^{m \times m}$ satisfying several assumptions following:

For any $1 \le i, j \le m, \sigma_{\theta_{ij}}$ is progressively measurable;

There exists a constant K such that $|\sigma_\theta(t, x_\theta) - \sigma_\theta(t, x'_\theta)| \le K(1 + |x_\theta|_t)$;

For any $(t, x_\theta) \in [0,T] \times \mathcal{C}$, the matrix $\sigma_\theta(t, x_\theta)$ is invertible and $\left|\sigma_\theta^{-1}(t, x_\theta)\right| \le K(1 + |x_\theta|_t^m)$ for some constants K and $m > 0$.

Now the equation

$$x_\theta(t) = x_\theta(0) + \int_0^t \sigma_\theta\big(s, x_\theta(s)\big) dW(s), t \le T$$

has the unique solution $x_\theta(t)$.

Let A(resp. B) be a compact space of metric. Then we define $\mathcal{U}$(resp. $\mathcal{V}$) as the space of $\mathcal{P}$-measurable processes $u := (u_t)_{t \le T}$(resp. $v := (v_t)_{t \le T}$) with values in A(resp. B). For each $\theta \in \Theta$, $\Phi_\theta: [0,T] \times \mathcal{C} \times \mathcal{U} \times \mathcal{V} \to \mathbb{R}^m$ is such a function satisfying that

(1) for any $(t, x_\theta) \in [0,T] \times \mathcal{C}$, the mapping $(u, v) \to \Phi_\theta(t, x_\theta, u, v)$ is continuous;

(2) for any $(u, v) \in A \times B$, the function $\Phi_\theta(\cdot, x_\theta(\cdot), u, v)$is $\mathcal{P}$-measurable;

(3) there exists a constant K, for any t, x_θ, u and v, has $|\Phi_\theta(t, x_\theta, u, v)| \le K(1 + |x_\theta|_t)$;

(4) there exists a constant H, for any t, x_θ, u and v, has $\left|\sigma_\theta^{-1}(t, x_\theta)\Phi_\theta(t, x_\theta, u, v)\right| \le H$.

Then, for $(u, v) \in \mathcal{U} \times \mathcal{V}$, we define the new measure $\mathcal{P}^{u,v}$ as:

$$\frac{d\mathcal{P}_\theta^{u,v}}{d\mathcal{P}} = \exp\left\{\int_0^T \sigma_\theta^{-1}\big(t, x_\theta(t)\big)\Phi_\theta\big(t, x_\theta(t), u(t), v(t)\big) dW(t) - \frac{1}{2}\int_0^T \left|\sigma_\theta^{-1}\big(t, x_\theta(t)\big)\Phi_\theta\big(t, x_\theta(t), u(t), v(t)\big)\right|^2 dt\right\}.$$

Based on Girsnov's Theorem, under the probability $\mathcal{P}_\theta^{u,v}$, the process

$$W_\theta^{u,v}(t) = W(t) - \int_0^t \sigma_\theta^{-1}\big(s, x_\theta(s)\big)\Phi_\theta\big(s, x_\theta(s), u(s), v(s)\big)ds\,, t \le T$$

is a Brownian motion process? Let $x_\theta(s)_{t\le T}$ be a weak solution o this stochastic equation

$$x_\theta(t) = x_0 + \int_0^t \Phi_\theta\big(s, x_\theta(s), u(s), v(s)\big)ds + \int_0^t \sigma_\theta\big(s, x(s)\big)dW_\theta^{u,v}(s)\,, t \le T.$$

Thus, we have a new measure $\mathcal{P}_\theta^{u,v}$.

Definition 2.1 $u, v: [0,T]\times\Omega \to U, V$ are said to be admissible controls if $u, v \in \mathcal{M}^p$. The sets of admissible controls are donated by $\mathcal{U}[0,T]$ and $\mathcal{V}[0,T]$.

The SDE of this 0-sum game problem under model uncertain is described by

$$x_\theta(t) = x_0 + \int_0^t \Phi_\theta\big(s, x_\theta(s), u(s), v(s)\big)ds + \int_0^t \sigma_\theta\big(s, x_\theta(s)\big)dW^{u,v}(s), \tag{2}$$

where $\theta \in \Theta$, and Θ is a metric space with distance μ. Besides, we assume that Θ is a locally compact, complete separable space.

Then we try to define the cost function. Let us introduce three functions: $C_\theta: [0,T]\times\mathbb{R}^m\times\mathbb{R}\times U\times V\to\mathbb{R}$, $D_\theta: [0,\mathrm{T}]\times\mathbb{R}^{\mathrm{m}}\times\mathbb{R}\to\mathbb{R}^d$ and $g: \mathbb{R}^{\mathrm{m}}\to\mathbb{R}$ satisfying the following assumptions: for any $\theta\in\Theta$, there exists $L>0$, for all $x_\theta, x'_\theta \in \mathcal{H}^p$ and $y_\theta, y'_\theta \in \mathcal{S}^2$ such that

$$\begin{aligned}
&|C_\theta(s, x_\theta(s), y_\theta(s), u(s), v(s)) - C_\theta(s, x'_\theta(s), y_\theta(s), u(s), v(s))| \le L|x_\theta(s) - x'_\theta(s)|\\
&\big(y_\theta(s) - y'_\theta(s)\big)\big(C_\theta(s, x_\theta(s), y_\theta(s), u(s), v(s)) - C_\theta(s, x_\theta(s), y'_\theta(s), u(s), v(s))\big)\\
&\quad\le L\big(y_\theta(s) - y'_\theta(s)\big)^2\\
&\quad\big|D_\theta\big(s, x_\theta(s), y_\theta(s)\big) - D_\theta\big(s, x'_\theta(s), y_\theta(s)\big)\big| \le L|x_\theta(s) - x'_\theta(s)|\\
&\big(y_\theta(s) - y'_\theta(s)\big)\Big(D_\theta\big(s, x_\theta(s), y_\theta(s)\big) - D_\theta(s, x_\theta(s), y'_\theta(s))\Big) \le L\big(y_\theta(s) - y'_\theta(s)\big)^2,
\end{aligned}$$

$g_\theta(x_\theta)\in\mathcal{L}^2$ is a measurable function satisfying Lipschitz conditions with respect to x_θ. Thus, the corresponding cost function can be given by $y_\theta(t)$, which solve the following BDSDE on $[0,T]$:

$$\begin{aligned}
y_\theta(t) &= g_\theta\big(x_\theta(T)\big) + \int_t^T C_\theta\big(s, x_\theta(s), y_\theta(s), u(s), v(s)\big)ds\\
&+ \int_t^T D_\theta(s, x_\theta(s), y_\theta(s))dB(s) - \int_t^T z_\theta(s)dW^{u,v}(s)
\end{aligned}\tag{3}$$

Then the cost function $J(x_0, u, v)$ of this problem under model uncertain is given by:

$$J(u,v) = \sup_{Q\in\mathcal{Q}}\int_\Theta E[y_\theta(0)]Q(d\theta), \tag{4}$$

where $\mathcal{Q}$ is a set of probability measures defined on $\big(\Theta, \mathcal{B}(\Theta)\big)$.

Finally, we give some assumptions related to smoothness and boundary:

(H1) There exists a constant $L>0$, for any $t\in[0,T], x, x'\in\mathbb{R}^m, y, y'\in\mathbb{R}, z, z'\in\mathbb{R}^m, u, u'\in\mathcal{U}, v, v'\in\mathcal{V}$ and $\theta\in\Theta$, have:

$$\begin{aligned}
&|\Phi_\theta(x,u,v) - \Phi_\theta(x',u',v')| + |\sigma_\theta(x) - \sigma_\theta(t,)|\\
&\le L(|x-x'| + |u-u'| + |v-v'|),\\
&|g_\theta(x) - g_\theta(x')| + |C_\theta(x,y,u,v) - C_\theta(x',y',u',v')| + |D_\theta(x,y) - D_\theta(x',y')|\\
&\le L(|x-x'| + |y-y'| + |z-z'| + |u-u'| + |v-v'|)(1 + |x| + |x'| + |u|\\
&+ |u'| + |v| + |v'|),\\
&|\Phi_\theta(0,0,0)| + |\sigma_\theta(0)| + |g_\theta(0)| + |C_\theta(0,0,0,0)| + |D_\theta(0,0)| \le L
\end{aligned}$$

(H2) For any $\theta\in\Theta$, $\Phi_\theta, \sigma_\theta, g_\theta, C_\theta, D_\theta$ are continuously differentiable in (x,y,z,u).

(H3) There exists a mapping $\bar{\omega}: [0,\infty)\to[0,\infty)$, which is a continuous modulus, and has

$$\begin{aligned}
&|l_\theta(t,x,y,z,u,v) - l_\theta(t,x',y',z',u',v')|\\
&\le \bar{\omega}(|x-x'| + |y-y'| + |z-z'| + |u-u'| + |v-v'|)
\end{aligned}$$

for any $t\in[0,T], u, u'\in\mathcal{U}, v, v'\in\mathcal{V}$, $\theta\in\Theta$, $x, x'\in\mathbb{R}^m, y, y'\in\mathbb{R}$, and $z, z'\in\mathbb{R}^m$, where l_θ is the derivatives of $\Phi_\theta, \sigma_\theta, g_\theta, C_\theta, D_\theta$ in (x,y,z,u,v).

(H4) For each $N > 0$, there exists a mapping $\bar{\omega}_N: [0,\infty) \to [0,\infty)$, which is a continuous modulus, and has

$$|l_\theta(t,x,y,z,u,v) - l_{\theta'}(t,x,y,z,u,v)| \le \bar{\omega}_N\big(\mu(\theta,\theta')\big)$$

for any $t \in [0,T], |u|,|v|,|x|,|y|,|z| \le N, \theta,\theta' \in \Theta$, where l_θ is $\Phi_\theta, \sigma_\theta, g_\theta, C_\theta, D_\theta$ and their derivatives in (x,y,z,u,v).

(H5): $\mathcal{Q}$ is a set of probability measures Q on $\big(\Theta, \mathcal{B}(\Theta)\big)$, and we assume that $\mathcal{Q}$ is weakly compact and convex.

3. Existence of the saddle point

In this section, we will consider the saddle point for a stochastic recursive 0-sum differential game under model uncertainty and give the uniqueness of it.

From the result in [3], for any $\theta \in \Theta$, there exist a unique solution (y_θ, z_θ) for (u,v) of the BDSDE (3), which ensure that the inner part of our cost function is well-defined. Then we have to note that in a 0-sum differential game, the agent c_1 find their optimal control u^* to minimum their payoff $J(x_0,u,v)$, on the contrary, the other agent c_2 have to find their optimal control v^* to maximum their income $J(x_0,u,v)$. So, in this problem, we want to find a point (u^*,v^*) satisfying

$$J(x_0,u^*,v) \le J(x_0,u^*,v^*) \le J(x_0,u,v^*)$$

for any $(u,v) \in \mathcal{U} \times \mathcal{V}$. In other words, there is a Nash equilibrium for the FBDSDE game.

Then we can deduce that:

$$\inf_{u\in\mathcal{U}} \sup_{v\in\mathcal{V}} J(u,v) \le \sup_{v\in\mathcal{V}} J(u^*,v) \le J(u^*,v^*) \le \inf_{u\in\mathcal{u}} J(u,v^*) \le \sup_{v\in\mathcal{V}} \inf_{u\in\mathcal{u}} J(u,v).$$

Since we always have infsup $\ge$ supinf, we can deduce that

$$\inf_{u\in\mathcal{U}} \sup_{v\in\mathcal{V}} J(u,v) = \sup_{v\in\mathcal{V}} J(u^*,v) = J(u^*,v^*) = \inf_{u\in\mathcal{u}} J(u,v^*) = \sup_{v\in\mathcal{V}} \inf_{u\in\mathcal{u}} J(u,v)$$

i.e., $(u^*,v^*) \in \mathcal{U} \times \mathcal{V}$ is a saddle point of $J(u,v)$.

Then let $(\bar{u},\bar{v})$ be optimal control and $(\bar{x}_\theta, \bar{y}_\theta, \bar{z}_\theta)$ be the state process of (2) and (3) correspondingly, for each $\theta \in \Theta$. Assume that the condition (H1), (H4) - (H5) are satisfied, then from lemma 3.5 in [8], the set $\mathcal{Q}^{u,v}$ is nonempty for each $(u,v) \in \mathcal{U} \times \mathcal{V}$, in which $\mathcal{Q}^{u,v}$ is defined as :

$$\mathcal{Q}^{u,v} = \big\{Q \in \mathcal{Q} \big| J(u,v) = \textstyle\int_\Theta E[y_\theta(0)]Q(d\theta)\big\}.$$

In other words, there exists a probability $Q^{u,v} \in \mathcal{Q}$, which has

$$\sup_{Q\in\mathcal{Q}} \int_\Theta E[y_\theta(0)]Q(d\theta) = \int_\Theta E[y_\theta(0)]Q^{u,v}(d\theta).$$

Thus, the cost function $J(u,v)$ can be given by

$$J(u,v) = \sup_{Q\in\mathcal{Q}} \int_\Theta E[y_\theta(0)]Q(d\theta) = \int_\Theta E[y_\theta(0)]Q^{u,v}(d\theta),$$

Thus, in the remain part of this section, we can investigate the existence of the saddle point. Using a normal method, we at first give a Hamiltonian function. For

$$\theta \in \Theta \text{ and } (t,x,y,z,u,v) \in [0,T] \times \mathcal{C} \times \mathbb{R} \times \mathbb{R}^m \times \mathcal{U} \times \mathcal{V},$$

let

$$H_\theta(t,x,y,z,u,v) = z\sigma_\theta^{-1}(t,x)\Phi_\theta(t,x,u,v) + C_\theta(t,x,y,u,v). \tag{5}$$

We say that the Isaacs' condition is satisfying, if for any $(t,x,y,z) \in [0,T] \times \mathcal{C} \times \mathbb{R} \times \mathbb{R}^m$,

$$\min_{u\in\mathcal{U}} \max_{v\in\mathcal{V}} H(t,x,y,z,u,v) = \max_{v\in\mathcal{V}} \min_{u\in\mathcal{U}} H(t,x,y,z,u,v).$$

Now, assume that the Isaacs' condition is satisfied, we can deduce that there exist $u^*: [0,T] \times \mathcal{C} \times \mathbb{R} \times \mathbb{R}^m \to U$ and $v^*: [0,T] \times \mathcal{C} \times \mathbb{R} \times \mathbb{R}^m \to V$ satisfying

$$H_\theta(t,x_\theta,y_\theta,z_\theta,u^*,v) \le H_\theta(t,x_\theta,y_\theta,z_\theta,u^*,v^*) \le H_\theta(t,x_\theta,y_\theta,z_\theta,u,v^*), \tag{6}$$

from t selection theorem in Benes [11],

Besides, for the assumptions about $\Phi_\theta, \sigma_\theta, g_\theta, C_\theta, D_\theta$ given above, the function $H_\theta(t,x_\theta,y_\theta,z_\theta,u^*(t,x_\theta,y_\theta,z_\theta),v^*(t,x_\theta,y_\theta,z_\theta))$ is Lipschitz in z_θ and monotone in y_θ, like the function C_θ.

Now, we have completed all the preparations and can deduce the main results in this paper.

Theorem 3.1 (y_θ^*, z_θ^*) is the solution of the following BDSDE:

$$\begin{cases} -dy_\theta^*(t) = H_\theta\big(t, x_\theta(t), y_\theta^*(t), z_\theta^*(t), u^*(t, x_\theta, y_\theta^*, z_\theta^*), v^*(t, x_\theta, y_\theta^*, z_\theta^*)\big)dt \\ \qquad\qquad +D\big(t, x_\theta(t), y_\theta^*(t)\big)dB(t) - z_\theta^*(t)dW(t), \\ \qquad y_\theta^*(T) = g_\theta\big(x_\theta(T)\big). \end{cases}$$

Then $\int_\Theta E[y_\theta^*(0)]\bar{Q}(d\theta)$ is the optimal cost $J(u^*, v^*)$, and this recursive 0-sum game has the unique saddle point, the pair (u^*, v^*).

Proof. We consider the integral form of $y_\theta^*(t)$:

$$y_\theta^*(t) = g_\theta\big(x_\theta(T)\big) + \int_t^T D_\theta\big(s, x_\theta(s), y_\theta^*(s)\big)dB(s) - \int_t^T z_\theta^*(s)dW(s) \tag{7}$$
$$+ \int_t^T H_\theta\left(s, x_\theta(s), y_\theta^*(s), z_\theta^*(s), u^*\big(t, x_\theta(t), y_\theta^*(t), z_\theta^*(t)\big), v^*\big(t, x_\theta(t), y_\theta^*(t), z_\theta^*(t)\big)\right)ds.$$

Thanks for the assumptions above, there is a unique solution (y^*, z^*). We have

$$\begin{aligned} E[y_\theta^*(0)] &= E^{u^*,v^*}[y_\theta^*(0)] \\ &= E^{u^*,v^*}[g_\theta\big(x_\theta(T)\big) + \int_0^T D_\theta\big(s, x_\theta(s), y_\theta^*(s)\big)dB(s) - \int_0^T z_\theta^*(s)dW(s) \\ &\quad + \int_0^T H_\theta\big(s, x_\theta(s), y_\theta^*(s), z_\theta^*(s), u^*(s, x_\theta, y_\theta^*, z_\theta^*), v^*(s, x_\theta, y_\theta^*, z_\theta^*)\big)\,ds] \\ &= E^{u^*,v^*}[g_\theta\big(x_\theta(T)\big) + \int_0^T C_\theta\big(s, x_\theta(s), y_\theta^*(s), u^*(s), v^*(s)\big)\,ds \\ &\quad + \int_0^T D_\theta\big(s, x_\theta(s), y_\theta^*(s)\big)dB(s) - \int_0^T z_\theta^*(s)dW^{u^*,v^*}(s)]. \end{aligned}$$

We get $\int_\Theta y_\theta^*(0)Q^{u^*,v^*}(d\theta) = J(u^*, v^*)$. For any $u \in \mathcal{U}, v \in \mathcal{V}$, we let

$$\begin{aligned} y_\theta(t) &= g_\theta\big(x_\theta(T)\big) + \int_t^T C_\theta\big(s, x_\theta(s), y_\theta(s), u^*(s), v(s)\big)\,ds \\ &\quad + \int_t^T D_\theta\big(s, x_\theta(s), y_\theta(s)\big)dB(s) - \int_t^T z_\theta(s)dW^{u^*,v}(s) \\ &= g_\theta\big(x_\theta(T)\big) + \int_t^T H_\theta\big(s, x_\theta(s), y_\theta(s), z_\theta(s), u^*(s), v(s)\big)\,ds \\ &\quad + \int_t^T D_\theta\big(s, x_\theta(s), y_\theta(s)\big)dB(s) - \int_t^T z_\theta(s)dW(s), \\ y_\theta'(t) &= g_\theta\big(x_\theta(T)\big) + \int_t^T C_\theta\big(s, x_\theta(s), y_\theta'(s), u(s), v^*(s)\big)\,ds \\ &\quad + \int_t^T D_\theta\big(s, x_\theta(s), y_\theta'(s)\big)dB(s) - \int_t^T z_\theta'(s)dW^{u,v^*}(s) \\ &= g_\theta\big(x_\theta(T)\big) + \int_t^T H_\theta\left(s, x_\theta(s), y'_\theta(s), z_\theta'(s), u(s), v^*(s)\right)ds \\ &\quad + \int_t^T D_\theta\big(s, x_\theta(s), y_\theta'(s)\big)dB(s) - \int_t^T z_\theta'(r)dW(r), \end{aligned} \tag{8}$$

From the comparison theorem of BDSDEs and inequality (6), the solution off (7) and (8) can be compared and we deduce that $y_\theta(t) \le y_\theta^*(t) \le y_\theta'(t), 0 \le t \le T$. Then we find that

$$J(u^*, v) = \int_\Theta E[y_\theta(0)]Q^{u^*,v}(d\theta) \le \int_\Theta E[y_\theta^*(0)]Q^{u^*,v}(d\theta) \le \sup_{Q\in\mathcal{Q}} \int_\Theta E[y_\theta^*(0)]Q(d\theta) = J(u^*, v^*)$$
$$J(u^*, v^*) = \int_\Theta E[y_\theta^*(0)]Q^{u^*,v^*}(d\theta) \le \int_\Theta E[y_\theta'(0)]Q^{u^*,v^*}(d\theta) \le \sup_{Q\in\mathcal{Q}} \int_\Theta E[y_\theta'(0)]Q(d\theta) = J(u, v^*)$$

i.e.

$$J(u^*, v) \leq J(u^*, v^*) \leq J(u, v^*).$$

Tus the pair (u^*, v^*) is the saddle point.

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