Exploration of the distillability problem

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Abstract. Quantum mechanics has a history more than 100 years, and there are still many challenging open-ended questions. The problems encountered in modern physics such as matrix mechanics and quantum information often involve both quantum theory and open-ended problems in modern mathematics such as linear algebra. Distillability problem is an important open-ended problem in the field of quantum information and linear algebra. This article recalls the development history and application of linear algebra, summarizes the background and progress of distillability conjecture, and attempts to explore this problem by summarizing and using the knowledge of linear algebra and matrix theory learned in the course.

Keywords: Linear algebra; distillability problem; quantum information; normal matrix; Kronecker product.

1. Introduction

Linear algebra, one of the first two mathematics courses in college, is the foundation of algebra. Linear algebra is concerned with defining the properties of the linear spaces of addition and multiplication, two operations with linear properties, and this is fundamental to many of the good properties in linear algebra.

As a discipline closely related to physics and engineering, linear algebra also highlights its importance in the field of application. For example, in mechanics, matrices are often used to describe the motion of rigid bodies, and this method is also applied to computer graphics, automation, and other engineering practices that need to quantify the processing of object positions.

At the same time, linear algebra is inseparable from other areas of mathematics. For example, in multivariate function calculus, where the Jacobian determinant is required to express multiple parametric degrees of freedom in the same equation, in ordinary differential equations, Jordan standard forms and diagonalizations in linear algebra play a role in solving the system of constant-coefficient differential equations by converting coupled systems of equations into orthogonal bases for solving.

In addition, with the introduction of linear algebra in the field of social sciences, its quantitative predictions also have a mathematical basis. For example, in securities forecasting, the introduction of linear algebra is conducive to in-depth study of the intrinsic relationship between various parameters, and scientific and objective empowerment algorithms are inseparable from the application of linear algebra.

Here are some application cases of linear algebra.

Image compression based on singular value decomposition:

In the case of a square matrix, if the nth-order solid square matrix A has n linearly independent eigenvectors, the eigenvalue decomposition $A = CDC^T$ can be performed, where D is a diagonal array and C is an orthogonal array.

In the case of A non-square matrix, set to m*n order matrix, singular value decomposition can make$A = UMV^T$, where M is only diagonal elements non-zero m*n order matrix, U, V are orthogonal squares.

In practice, the original image is read, the RGB three-color primary color feature is decomposed using singular values to select and extract the main features, and some singular value pairs are used according to different clarity The image is reassembled to produce a compressed picture.

Singular value decomposition retains the characteristics of the original matrix to a large extent, that is, effectively reduces the space required for image storage without losing image quality as much as possible, and retains the main information of the picture.

Application of clustering algorithms in market forecasting:

Clustering is a method of establishing a classification that divides a dataset into groups or classes with the expectation that the objects represented by the data within the same group have a high degree of similarity. The measure of similarity is determined by quantifying the description of the data object.

The process of classifying a group of figurative or abstract objects according to the degree of similarity, in which similar objects form a group is called a clustering process.

In the market forecast, it is necessary to classify and screen the research object according to its characteristics and indicators. The clustering process can find the connections that exist between the characteristics of the object data from a given dataset.

Therefore, linear algebra plays an extremely critical role in today's mathematical science research, social science prediction, practical application of engineering and many other fields, its importance is becoming increasingly prominent, it is widely opened among various majors, and it has gradually become an indispensable knowledge skill in the information age of data processing with mathematical models today.

2. Mainbody

2.1. Background and history

Linear algebra is a large branch of algebra, and algebras that discuss linear equations and linear operations are called linear algebra. The most important things in linear algebra are determinants and matrices. Determinants and matrices received a great deal of attention in the nineteenth century, and thousands of articles were written on both subjects. Vectors also use force or velocity as a direct physical meaning, giving physical quantities a good mathematical expression. Vectors are also used for gradient divergence, and curl operations. Similarly, determinants allow us to directly and creatively imagine what is happening physically. Thus, while determinants and matrices are nothing more than a language or shorthand, most of their vivid concepts provide the key to new areas of thought. These two concepts are highly useful tools in mathematical physics.

The discipline of linear algebra and matrix theory were introduced and developed along with the study of the coefficients of equations in linear systems. The concept of determinant was first proposed by the seventeenth-century Japanese mathematician Takahazu Seki, who in 1683 wrote a work called "The Method of Solving Volt Problems"[2], which means "method of solving the determinant problem", which has a clear description of the concept of determinant and it’s unfolding. The first european to propose the concept of determinant was leibnitz (1693), a German mathematician and one of the founders of calculus. 1750 Cramer published an important fundamental formula for solving the equations of linear systems, the familiar Cramer Clem's law, in his Introduction d l'analyse des lignes courbes alge'briques [11]. In 1764, Bezout systematized the procedure for determining the symbol for each term of the determinant, given n homogeneous linear equations with n unknowns, Bezout proved that the coefficient determinant equal to zero is a condition for the system of equations to have a non-zero solution. Vandermonde was the first to systematically expound the theory of determinants, i.e. to separate the theory of determinants from the solution of systems of linear equations. And gives a rule to expand the determinant with second-order subforms and their courtendons. He was the founder of the theory in terms of the study of the determinant itself. Laplace, in his 1772 paper Recherches sur le calcul intégral ET sur le système du monde [12], proved some of Vandermonde's rules and generalized his method of unfolding determinants with the set of sub-forms contained in the r-line and their remainders, which is still named after him today. The German mathematician Jacobi also summed up and proposed a systematic theory of determinants in 1841[13]. Another researcher of determinants was the greatest French mathematician, Cauchy, who greatly developed the theory of determinants, in which he arranged elements into squares and for the first

time adopted a new notation of double foot scales, while discovering a formula for multiplying two determinants and improving and proving laplace's expansion theorem [14]. The first use of the matrix concept was demonstrated by Lagrange in bilinear work after 1700. Lagrange expects to understand the problem of the maximum and minimum values of multivariate functions, and its method is known as lagrangian iterative method. To accomplish this, he first needs a first-order partial derivative of 0, and a second-order partial derivative matrix. This condition is what is known today as a positive or negative definition. Although Lagrange did not explicitly propose the use of matrices. Gauss proposed gaussian elimination method around 1800 and used it to solve the problem of least squares in celestial calculations and later calculations of the Earth's surface [15]. This branch of applied mathematics, which involves measuring, finding the shape of the Earth, or the precise location of the local area, is called geodesy. Although Gauss is known for his successful elimination of variables from linear equations by this technique, manuscripts Chinese centuries ago have appeared explaining how to use gaussian elimination to solve triorboor systems with three unknown amounts. For several years at the time, Gaussian elimination was considered part of the development of geodesy, rather than mathematics. The Gaussian-Jordan elimination law originally appeared in a geodetic manual written by Wilhelm Jordan [16].

For the rich development of matrix algebra, one needs to have suitable symbols and suitable matrix multiplication definitions. In 1848, J.J. Sylvester of England first coined the term matrix [17], which is derived from Latin and stands for a row of numbers. In 1855, with Cayley's work, which studied the composition of linear transformations and proposed a definition of matrix multiplication so that the coefficient matrix of the composite transformation ST became the product of matrix S and matrix T, matrix algebra was nurtured. He also further studied algebraic problems, such as the famous Cayley-Hamilton theory, which asserts that the square of a matrix is the root of its characteristic polynomial, was proposed by Cayley in his treatise on matrix theory in 1858 [18]. It is essential for matrix algebra development that people started using letters to represent matrices. Then a connection between matrix algebra and determinant was built by the formula det(AB) = det(A)det(B). Mathematician Cauchy first gave the terms of the eigen equation and proved that matrices with orders greater than 3 have eigenvalues and any order real symmetry determinant have real eigenvalues; gave the concept of similar matrices, and proved that similar matrices have the same eigenvalues; studied substitution theory. Mathematicians have tried to study vector algebras, but the fact that in arbitrary dimensions there is no natural definition of the product of two vectors makes it hard to do. The first involves a non-commutative vector product, the vector algebra of $v \times w$ is different from$w \times v$, proposed in the book Die lineale Ausdehnungslehre by Hermann Grassmann [19]. His idea was also introduced into the product of column matrix and row matrix, which means a matrix with a rank of 1, or a simple matrix. At the end of the 19th century, a famous treatise on Elements of Vector Analysis published by Willard Gibbs, the American mathematical physicist. Physicist P. A.M. Dirac later proposed that the product of row and column vectors is scalar. The column matrices and vectors, which we are now accustomed to, were given by morden physicists.

The development of matrices is closely related to linear transformations. By the 19th century it had only a limited space in the formation of linear transformation theory. In 1888, the modern definition of the vector space was proposed by Peano [20]. After 1950s, with the development of high-performance computers, matrices have new meanings, especially in the numerical analysis of matrices. Due to the rapid development and wide application of computers, many practical problems can be quantitatively solved through discrete numerical calculations. Therefore, as a linear algebra for dealing with discrete problems, it has become an indispensable mathematical foundation for scientific researchers and engineering design.

The background problem of the distillability problem is the mathematical equivalent form of determining whether a quantum event is an entangled state, proposed by Horodecki et al and Divincenzo et al in 1999 [1] In quantum information science, two-way shared quantum systems, often spatially separated and called Alice and Bob. If Alice's subsystem can be described by Hilbert space H_A while Bob's subsystem can be described byH_B, then the pure state in the composite system that

can be described by the tensor product of two spaces$H_A \otimes H_B$. It is called a dichotomous quantum state or separable quantum state, which implies the independence of this quantum state in two subsystems. Pure state that cannot be described by the tensor products of two spaces $H_A \otimes H_B$ are called entangled states, which are important resources in quantum information processing. Therefore, an important problem in the field of quantum information is to determine whether a given quantum state in a composite system is entangled.

2.2. Research progress of the distillability problem

In 2007, the problem was shown that the mathematical expression under 2-copy is equivalent to the distillability problem, [1] and proved the condition that A, B is diagonal matrix.

The distillability problem is

If$A, B, I \in C^{d \times d}, d \geq 4$

$$X = A \otimes I + I \otimes B \tag{1}$$

When TrA = TrB = 0,

$$TrA^H A \ + \ TrB^H B \ = \ \frac{1}{d} \sup \left(\sigma_1{}^2 + \sigma_2{}^2\right) \leq \frac{1}{2} \tag{2}$$

And $\sigma_1, \cdots, \sigma_{d^2}$is the singular value of X.

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In 2018, Yi Shen and Lin Chen's article proved that when d=4, A, B belong to 3 specific types of matrix. [3]

In 2021, Lin Chen · Huixia He · Xian Shi · Li-Jun Zhao's article proves that when d=4, A and B are single-term matrixes, A, B is at most two non-zeros and A rank is 1, B rank is not greater than 2. [4]

2.3. The application of linear algebra in quantum mechanics

In April 1925, physicist Heisenberg began to study the spectral lines of the hydrogen atom. Although Bohr's theory has explained the spectral line problem when the hydrogen atom is radiated. Heisenberg thought Bohr's theory cannot accurately predict the specific value of the spectral line intensity. Heisenberg's view is that since the interior of an atom cannot be observed, only the spectrum of light emitted by electrons is considered and calculated. Heisenberg discovered that this problem could be solved by introducing an ineffectual observable quantity, and thus established matrix mechanics, which was later matured by Jordan, Born, Pauli, and Bohr.

In the matrix mechanics formulation any physical quantity is represented by a Hermitian matrix. The Hamiltonian quantities of the system are also represented by a Hermitian matrix and are a function of the coordinates and momentum matrices.

$|0>, |1>$ Is a group of orthonormal bases in$\mathbb{C}^2$, which is also known as qubit?

And remember$< x| = |x >^H$, called brax and ketx.

That is

$$< 0|1 >=< 1|0 >= 0, < 0|0 >=< 1|1 >= 1 \tag{3}$$

Call a, b∈ $\mathbb{C}$, $a|1> + b|0>$ as a superposition, where $|a|^2 + |b|^2 = 1$

2.4. Kronecker product

If *A* is a matrix of m x n and *B* is a matrix of p x q, the Kronecker product is a Matrix of mp x nq, and

$$A \otimes B = \begin{bmatrix} a_{11}B & \cdots & a_{1n}B \\ \vdots & \ddots & \vdots \\ a_{m1}B & \cdots & a_{mn}B \end{bmatrix} \tag{4}$$

$$A\otimes(B\otimes C) = (A\otimes B)\otimes C \tag{5}$$

$$A\otimes(B + C) = A\otimes B + A\otimes C \tag{6}$$

$$aA\otimes bB = ab(A\otimes B) \tag{7}$$

$$(A\otimes B)^H = A^H\otimes B^H \tag{8}$$

$$(A\otimes B)^{-1} = A^{-1}\otimes B^{-1} \tag{9}$$

$$tr(A\otimes B) = trA * trB \tag{10}$$

$$r(A\otimes B) = rA * rB \tag{11}$$

$$\det(A\otimes B) = \det(A)^n * \det(B)^m, (A)m * m, (B)n * n \tag{12}$$

$$[A1, A2]\otimes B = [A1\otimes B, A2\otimes B] \tag{13}$$

[5] [9]

2.5. Normal matrix

Normal matrix include unitary matrix and Hermitian matrix.
A is the unitary matrix if and only if $AA^H = A^HA = I, and\ \ A|0>= |1>, A|1>= |0>$ in$\mathbb{C}^2$.
A is Hermitian matrix if and only if$A = A^H$
A is a Hermitian matrix is a sufficient condition for the spectral decomposition of A, that A=$\sum_j P_j|a_j><a_j|$, where is the eigenvalue$P_j, |a_j>$ is a group of orthogonal base.
If A is both a unitary matrix and a Hermitian matrix, then A is called a pauli matrix
Denote $\sigma_1 = \begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}, \sigma_2 = \begin{bmatrix}0 & i\\ -i & 0\end{bmatrix}, \sigma_3 = \begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}$,
$\sigma_1, \sigma_2, \sigma_3$ And I are all Pauli matrix.

$$\text{And there are } [\sigma_1, \sigma_2]_+ = [\sigma_2, \sigma_3]_+ = [\sigma_3, \sigma_1]_+ = 0 \tag{14}$$

[5] [9]

2.6. Pure state

$\Pi(\mathrm{n}) = \frac{1}{2}(I + \sum_{j=1}^{3} n_j\sigma_j)$, where n= $[n_1\ n_2\ n_3]^H$, |n|=1
By$\sigma_1, \sigma_2, \sigma_3$ and I are all Hermitian matrix

$$\Pi(\mathrm{n})^H = \Pi(\mathrm{n}) \tag{15}$$

By tr(σ_1) = tr(σ_2) = tr(σ_3) = 0 and tr(I) = 2

$$tr\big(\Pi(\mathrm{n})\big) = 1 \tag{16}$$

$\Pi(\mathrm{n})^2 = \frac{1}{4}(I + \sum_{j=1}^{3} n_j\sigma_j)^2 = \frac{1}{4}I + \frac{1}{2}\sum_{j=1}^{3} n_j\sigma_j + \frac{1}{4}\sum_{j=1\ k=1}^{3\ 3} n_jn_k\,\sigma_j\sigma_k$

And by $[\sigma_1, \sigma_2]_+ = [\sigma_2, \sigma_3]_+ = [\sigma_3, \sigma_1]_+ = 0$

$$= \frac{1}{4}I + \frac{1}{2}\sum_{j=1}^{3} n_j\sigma_j + \frac{1}{4}\sum_{j=1}^{3} nj^2 + \frac{1}{4}\sum_{\substack{j=1\\ j<k}}^{3} n_jn_k\big[\sigma_j, \sigma_k\big]_+ = \Pi(\mathrm{n}) \tag{17}$$

That $\Pi(\mathrm{n})$ is an idempotent matrix.
Then by the Hermitian matrix can be spectrally decomposed, there is,$P_1{}^2 = P_1, P_2{}^2 = P_2,\ P_1$and P_2 is 0 or 1

And by tr(Π(n) = 1, only one of $P_1$and P_2 is 1
Π(n) = $|a><a|$, Where $|a>$ satisfiedΠ(n)$|a>=|a>$
[6]

2.7. Proof of distillability problem in special condition

A=0, B^HB semi-positive matrix and the maximum eigenvalue ofB^HB is less than the sum of the other eigenvalues ofB^HB

Then

$$X = I \otimes B \tag{18}$$

By $(A \otimes B)^H = A^H \otimes B^H$ and $(A \otimes B)(C \otimes D) = AC \otimes BD$

$$\begin{aligned} X^H X &= (I \otimes B)^H (I \otimes B) \\ &= (I \otimes B^H)(I \otimes B) \\ &= I \otimes B^H B \end{aligned} \tag{19}$$

Therefore, the square of the X singular value is the same as the eigenvalue of B^HB
by $TrA^HA + TrB^HB = TrB^HB = \frac{1}{d}$, $d \geq 4$
and TrB^HB is the sum of the eigenvalue of B^HB [7]

$$\sup(\sigma_1{}^2 + \sigma_2{}^2) \leq \frac{2}{2d} = \frac{1}{d} \tag{20}$$

3. Conclusion

This article summarizes the development process of linear algebra and its important role in today's technological development, explores the physical background and progress of an important conjecture of linear algebra in the field of quantum information, and proves the special case of the distillability problem at A=0, B^HB semi-positive matrix and the maximum eigenvalue of B^HB is less than the sum of the other eigenvalues ofB^HB.

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