Cubic equation and topology of complex numbers

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Abstract. The study of complex analysis is an intriguing and technical topic. Over the past few centuries, mathematicians have produced ground-breaking work in this field. Mathematicians initially defined complex numbers, and then they gradually investigated their algebraic and geometric properties. They stumbled upon and researched intricate calculations hundreds of years later. The solution of the cubic equation has motivated mathematicians to explore the study of complex numbers, but it has also advanced mathematics in many other ways by exposing previously undiscovered issues. First off, when there are multiple real roots to the equation, it is impossible to obtain all of them using Cardano's formula at that time without understanding the fundamentals of complex numbers. Therefore, when they test various examples, mathematicians occasionally discover that there are other roots that are quite clear but are impossible to pass via Cardano's formula. In this paper, it start from the research of cubic equaiton and then discuss in detail topology of complex numbers.

Keywords: Cubic equation, complex numbers, topology.

1. Introduction

The first imaginary number was found in the middle of the 1500s[1-10]. Complex numbers have been the focus of numerous discussions among mathematicians for many years. Early mathematicians were unaware of negative numbers, let alone fake ones. But when mathematicians looked at the answers to other cubic equations, they found something profound: numbers that weren't "possible." Since the idea was so unfamiliar to them, they believed they had discovered something unique. In essence, mathematicians had to work with a "linguistic barrier." Over the last four centuries, complex systems have been studied more and more since it has become accepted as a mathematical fact. Proof evolved when mathematicians continued to delve further and deeper into the issue. [1-10]. More abstract proofs emerged, giving mathematicians more confidence and allowing them to continue developing the mathematics of the complex system [1-10]. A complex number is defined as an member of the set $C:=\{s+it:s,\ t\in R \text{ and } i^2=1\}$. For $z=m+i\,n$, a person can define its real part as $Re(z):=x$, the imaginary part of it as $Im(z):=y$, and the length of it as $|z|:=\sqrt{x^2+y^2}$. The complex number set should be thought of as a coordinate plane R^2 where $z=x+iy$ can be thought of as the coordinate pair (x,y). Solving equation is one of goals in the field of mathematics. Take the equation $z^n=1+i$ as an example, where n is a natural number. To solve equation $z^n=1+i$,both geometric way and algebra way is feasible. By geometric way, since it's obvious that $|1+i|=\sqrt{2}$, $|z|=\sqrt[2n]{2}$. Since $\arg(1+\mathrm{i})=\frac{\pi}{4}$, $\arg(\mathrm{z})=\frac{\frac{\pi}{4}}{\mathrm{n}}\text{or}\frac{\frac{\pi}{4}+2\pi}{\mathrm{n}}\text{or}\frac{\frac{\pi}{4}+4\pi}{\mathrm{n}}\text{or}\cdots\text{or}\frac{\frac{\pi}{4}+2(\mathrm{n}-1)\pi}{\mathrm{n}}$. $\text{Thus, } \mathrm{z}_1=\sqrt[2n]{2}(\cos\frac{\frac{\pi}{4}}{\mathrm{n}}+i\sin\frac{\frac{\pi}{4}}{\mathrm{n}}), \mathrm{z}_2=\sqrt[2n]{2}(\cos\frac{\frac{\pi}{4}+2\pi}{\mathrm{n}}+i\sin\frac{\frac{\pi}{4}+2\pi}{\mathrm{n}}), \mathrm{z}_3=\sqrt[2n]{2}(\cos\frac{\frac{\pi}{4}+4\pi}{\mathrm{n}}+i\sin\frac{\frac{\pi}{4}+4\pi}{\mathrm{n}}),\ldots,\mathrm{z}_\mathrm{n}=\sqrt[2n]{2}(\cos\frac{\frac{\pi}{4}+2(\mathrm{n}-1)\pi}{\mathrm{n}}+i\sin\frac{\frac{\pi}{4}+2(\mathrm{n}-1)\pi}{\mathrm{n}})$. By algebraic way, w is defined as $\mathrm{w}=\frac{\mathrm{z}}{\sqrt[2n]{2}}$. It's obvious that $|w|=1$. So, let $\mathrm{w}=\mathrm{e}^{\mathrm{i}\theta}$. Since $z^n=1+i$, $(w\times\sqrt[2n]{2})^n=1+i$. So, $w^n=\frac{\sqrt{2}}{2}+\frac{\sqrt{2}}{2}i$, which is $e^{i\theta n}=\frac{\sqrt{2}}{2}+\frac{\sqrt{2}}{2}i$. Thus, $\cos n\theta+i\sin n\theta=\cos\frac{\pi}{4}+i\sin\frac{\pi}{4}$. So, $\theta=\frac{\frac{\pi}{4}}{\mathrm{n}}\text{or}\frac{\frac{\pi}{4}+2\pi}{\mathrm{n}}\text{or}\frac{\frac{\pi}{4}+4\pi}{\mathrm{n}}\text{or}\cdots\text{or}\frac{\frac{\pi}{4}+2(\mathrm{n}-1)\pi}{\mathrm{n}}$. $Thus, \mathrm{z}_1=\sqrt[2n]{2}(\cos\frac{\frac{\pi}{4}}{\mathrm{n}}+i\sin\frac{\frac{\pi}{4}}{\mathrm{n}}), \mathrm{z}_2=\sqrt[2n]{2}(\cos\frac{\frac{\pi}{4}+2\pi}{\mathrm{n}}+i\sin\frac{\frac{\pi}{4}+2\pi}{\mathrm{n}}), \mathrm{z}_3=\sqrt[2n]{2}(\cos\frac{\frac{\pi}{4}+4\pi}{\mathrm{n}}+i\sin\frac{\frac{\pi}{4}+4\pi}{\mathrm{n}}),\ldots,\mathrm{z}_\mathrm{n}=$

$\sqrt[2n]{2}(\cos\frac{\frac{\pi}{4}+2(n-1)\pi}{n}+i\sin\frac{\frac{\pi}{4}+2(n-1)\pi}{n})$. By these two way, z can be solved. In this paper, the development of solving cubic equation will be discussed in details and then topology of complex numbers will be discussed.

2. Solving cubic equation

From trying solving cubic equation, mathematicians started to notice that the calculating rule of square root of negative number seems important for the development of math. Here is a brief introduction about how mathematicians solve cubic equation.

For cubic equation $ax^3+bx^2+cx+d=0$, it will be easier to be solved if it's written as

$$x^3+px=q$$

First, let y=x+k. So, the former cubic equation $ax^3+bx^2+cx+d=0$ can be rewritten as $f(x)=f(k+y)=f(k)+\frac{f'(k)}{1!}y+\frac{f''(k)}{2!}y^2+\frac{f'''(k)}{3!}y^3=0$ by Taylor's expansion if f(x) is defined as $f(x)=ax^3+bx^2+cx+d$ as f(n)(x)=0 for n≧4(Of course,this procedure can be done by simply by substitute x by k+y and expanding the polynomial instead of Taylor's expansion). Second step is finding a k let f''(k)=0 in order to eliminate the term of y^2. The solution is k=-$\frac{a}{3}$. As a result, after solving the equation $f(-\frac{a}{3})+\frac{f'(-\frac{a}{3})}{1!}y+\frac{f'''(-\frac{a}{3})}{3!}y^3=0$, which is same with $y^3+\frac{f'(-\frac{a}{3})}{1!\times\frac{f'''(-\frac{a}{3})}{3!}}y=-f(-\frac{a}{3})$, x=y+k will be the solution to the original equation.

According to those, it's reasonable to assert that if solving a cubic equation forming like $x^3+px=q$ is possible, solving every cubic equation is possible as every equation can be transformed to that form.

Tartaglia had found a way to solve a cubic equation like $x^3+px=q$. Assuming that $\begin{cases}u^3-v^3=q\\ uv=\frac{p}{3}\end{cases}$, it's not difficult to found u^3 and v^3. Here is the solution to found it. Firstly, rewrite the simultaneous equations as $\begin{cases}u^3-v^3=q\\ u^3\times v^3=\left(\frac{p}{3}\right)^3\end{cases}$. Then, it's obvious that $(u^3+u^3)^2=q^2+4\times\left(\frac{p}{3}\right)^3$. So, $u^3+u^3=\pm\sqrt{q^2+4\times\left(\frac{p}{3}\right)^3}$. Further, $\begin{cases}u^3=\frac{q+\sqrt{q^2+4\times\left(\frac{p}{3}\right)^3}}{2}\\ v^3=\frac{-q+\sqrt{q^2+4\times\left(\frac{p}{3}\right)^3}}{2}\end{cases}$ or $\begin{cases}u^3=\frac{q-\sqrt{q^2+4\times\left(\frac{p}{3}\right)^3}}{2}\\ v^3=\frac{-q-\sqrt{q^2+4\times\left(\frac{p}{3}\right)^3}}{2}\end{cases}$. What Tartaglia have done that is really ingenious is that he found that x=u-v satisfy the equation. The reason is detailed as follows $uv=\frac{p}{3}$, substitute x by u-v, $x^3+px=(u-v)^3+3uv(u-v)=u^3-v^3=q$. In the days mathematicians can't accept imaginary number, there would be 2 problems. First, since $q^2+4\times\left(\frac{p}{3}\right)^3$ can't be guaranteed to be positive, this method of solving cubic equation may not work. Secondly, n-degree polynomial is considered to have n root. But, only one solution can get directly from here(although by getting one root we can change this cubic equation to quadratic equation by factor theorem, it's very difficult to calculate and the other root aren't necessarily real). In fact, if $q^2+4\times\left(\frac{p}{3}\right)^3>0$, the other two roots of this function are all complex number(not real). The imaginary number was later accepted by mathematicians and the

system of complex number was finally built up, which allows mathematicians to reconsider this question from a new perspective.

To begin with, when $q^2+4\times\left(\frac{p}{3}\right)^3\geq 0$, u^3 and v^3 are both real number. The 3 solution for u is $\sqrt[3]{\frac{q-\sqrt{q^2+4\times\left(\frac{p}{3}\right)^3}}{2}}$, $\sqrt[3]{\frac{q-\sqrt{q^2+4\times\left(\frac{p}{3}\right)^3}}{2}}\times\omega_1$, and $\sqrt[3]{\frac{q-\sqrt{q^2+4\times\left(\frac{p}{3}\right)^3}}{2}}\times\omega_2$, while the 3 solution for v is c, $\sqrt[3]{\frac{-q-\sqrt{q^2+4\times\left(\frac{p}{3}\right)^3}}{2}}\times\omega_1$, and $\sqrt[3]{\frac{-q-\sqrt{q^2+4\times\left(\frac{p}{3}\right)^3}}{2}}\times\omega_2, (\omega_1=\frac{-1+\sqrt{3}i}{2}, \omega_{12}=\frac{-1-\sqrt{3}i}{2})$. However, since $uv=\frac{p}{3}$, there are only 3 pair of (u,v), which is $(\sqrt[3]{\frac{q-\sqrt{q^2+4\times\left(\frac{p}{3}\right)^3}}{2}}, \sqrt[3]{\frac{-q-\sqrt{q^2+4\times\left(\frac{p}{3}\right)^3}}{2}}), (\sqrt[3]{\frac{q-\sqrt{q^2+4\times\left(\frac{p}{3}\right)^3}}{2}}\times\omega_1, \sqrt[3]{\frac{-q-\sqrt{q^2+4\times\left(\frac{p}{3}\right)^3}}{2}}\times\omega_2), (\sqrt[3]{\frac{q-\sqrt{q^2+4\times\left(\frac{p}{3}\right)^3}}{2}}\times\omega_2, \sqrt[3]{\frac{-q-\sqrt{q^2+4\times\left(\frac{p}{3}\right)^3}}{2}}\times\omega_1$. As a result, $x_1=\sqrt[3]{\frac{q-\sqrt{q^2+4\times\left(\frac{p}{3}\right)^3}}{2}}-\sqrt[3]{\frac{-q-\sqrt{q^2+4\times\left(\frac{p}{3}\right)^3}}{2}}$, $x_2=\sqrt[3]{\frac{q-\sqrt{q^2+4\times\left(\frac{p}{3}\right)^3}}{2}}\times\omega_1-\sqrt[3]{\frac{-q-\sqrt{q^2+4\times\left(\frac{p}{3}\right)^3}}{2}}\times\omega_2$, $x_3=\sqrt[3]{\frac{q-\sqrt{q^2+4\times\left(\frac{p}{3}\right)^3}}{2}}\times\omega_2-\sqrt[3]{\frac{-q-\sqrt{q^2+4\times\left(\frac{p}{3}\right)^3}}{2}}\times\omega_1$, where $\omega_1=\frac{-1+\sqrt{3}i}{2}, \omega_{12}=\frac{-1-\sqrt{3}i}{2}$). When $q^2+4\times\left(\frac{p}{3}\right)^3>0$, this equation has one real root and two complex roots which is not real. When $q^2+4\times\left(\frac{p}{3}\right)^3=0$, this equation has one real root and two same real roots(two distinct real roots)

All of those calculation is based on $\begin{cases}u^3=\frac{q-\sqrt{q^2+4\times\left(\frac{p}{3}\right)^3}}{2}\\ v^3=\frac{-q-\sqrt{q^2+4\times\left(\frac{p}{3}\right)^3}}{2}\end{cases}$. For $\begin{cases}u^3=\frac{q+\sqrt{q^2+4\times\left(\frac{p}{3}\right)^3}}{2}\\ v^3=\frac{-q+\sqrt{q^2+4\times\left(\frac{p}{3}\right)^3}}{2}\end{cases}$, the root of x will be exactly same.

This procedure can be done by both algebraic way and geometric way. The author will not introduce it since it is obvious.

Secondly, when $q^2+4\times\left(\frac{p}{3}\right)^3<0$, u^3 and v^3 are both complex number(not real number). Since the situation become more general and complex this time, the author is going to introduce how x can be found by both algebraic way and geometric way.

By geometric way, let $u^3=\frac{q-\sqrt{q^2+4\times\left(\frac{p}{3}\right)^3}}{2}=r^3\times(\cos\theta+i\sin\theta)$, for which r=$\sqrt[6]{\frac{1}{2}(q^2+2(\frac{p}{3})^3}$, $\theta=\arg(u^3)$, according to the rule of calculation of complex number, $u_1=r\times(\cos\frac{\theta}{3}+i\sin\frac{\theta}{3})$, $u_2=r\times(\cos\frac{\theta-2\pi}{3}+i\sin\frac{\theta-2\pi}{3})$, $u_3=r\times(\cos\frac{\theta+2\pi}{3}+i\sin\frac{\theta+2\pi}{3})$. Those solutions are not in order. Similarly, for $v^3=\frac{-q-\sqrt{q^2+4\times\left(\frac{p}{3}\right)^3}}{2}$, it's obvious that $v_1=r\times(\cos\frac{-\theta+\pi}{3}+i\sin\frac{-\theta+\pi}{3}), v_2=r\times(\cos\frac{-\theta-\pi}{3}+i\sin\frac{-\theta-\pi}{3}), v_3=r\times(\cos\frac{-\theta+3\pi}{3}+i\sin\frac{-\theta+3\pi}{3})$. Those solutions are not in order.

Since $uv=\frac{p}{3}$, there are only 3 pair of (u,v), which is $(u_1,v_3),(u_2,v_2),(u_3,v_1)$. Thus, it's possible to obtain x since x=u-v. By algebraic way, let u=x+yi. So, $u^3=(x^3-3xy^2)+(3x^2y-$

$y^3)i$. As a result, $\begin{cases} x^3 - 3xy^2 = \frac{q}{2} \\ 3x^2y - y^3 = \frac{-\sqrt{q^2+4\times\left(\frac{p}{3}\right)^3}}{2i} \end{cases}$. Let $x = m \times \cos\alpha, y = m \times \sin\alpha$. Then, $\begin{cases} m^3 \times \cos 3\alpha = \frac{q}{2} \\ m^3 \times \sin 3\alpha = \frac{-\sqrt{q^2+4\times\left(\frac{p}{3}\right)^3}}{2i} \end{cases}$. It's easy to solve m and α in this way x and y can be gotten after that. For v, the same method can be used and the result of x which finally can be gotten is same as the geometric way. When $q^2 + 4 \times \left(\frac{p}{3}\right)^3 < 0$, the equation has 3 distinct real roots.

All of those calculation is based on $\begin{cases} u^3 = \frac{q-\sqrt{q^2+4\times\left(\frac{p}{3}\right)^3}}{2} \\ v^3 = \frac{-q-\sqrt{q^2+4\times\left(\frac{p}{3}\right)^3}}{2} \end{cases}$ as well. For $\begin{cases} u^3 = \frac{q+\sqrt{q^2+4\times\left(\frac{p}{3}\right)^3}}{2} \\ v^3 = \frac{-q+\sqrt{q^2+4\times\left(\frac{p}{3}\right)^3}}{2} \end{cases}$, the root of x will be exactly same.

After all, it's sensible to sum up that if complex number and the rule of calculating it never exist, all those work is impossible to be done.

After solving quadratic equation successfully, mathematicians didn't realize that complex number exist since if $\Delta = b^2 - 4ac < 0$, the equation would just be considered as no root as for those equations, they truly don't have real roots . However, every cubic equations must have at least one real root, while it's not necessarily sure that we can get it if the rules of calculating complex number are not built up. When $q^2 + 4 \times \left(\frac{p}{3}\right)^3 < 0$ for equation $x^3 + px = q$, for those mathematicians who refused to accept imaginary number, the real roots is even unable to be get. So, even for those mathematicians who only thought real number as sensible, exploring imaginary world is what mathematicians must do. It is introduced that how cubic equation can be solved from a perspective of mathematicians live in now a days. However, in the days that mathematicians did not realize the necessity of accepting imaginary number for the developing of math or failed to imagine the existence of imaginary number, things didn't progress so favoring.

After Tartaglia solved a special kind of cubic equation forming like $x^3 + px = q$, Cardano published the solution and set about solving more general cubic equations with the square term. In his attempting, while he found a geometric way to solve the equation, he thought many equations was unsolvable as the the solution contained imaginary number, maybe was just called negative square root in that time. Specifically, since in that time they only researched the root $x = \sqrt[3]{\frac{q-\sqrt{q^2+4\times\left(\frac{p}{3}\right)^3}}{2}} - \sqrt[3]{\frac{-q-\sqrt{q^2+4\times\left(\frac{p}{3}\right)^3}}{2}}$ with no guarantee that $q^2 + 4 \times \left(\frac{p}{3}\right)^3 \geq 0$, x might not able to calculate whenever $q^2 + 4 \times \left(\frac{p}{3}\right)^3 < 0$. Bombelli was another great mathematician in the same period. After learnt that Cardano had published a general solution, as it may called, for cubic equations while he considered some of cubic equation could not be solved, Bombelli was appalled. What Bombelli did different than Cardano was that although he maybe still consider negative square root as impossible number, he realized negative square is a kind of quantity that is able to be calculated under some rules. He defined $\sqrt{-1}$ as pdm and $-\sqrt{-1}$ as mdm. He also found some properties for calculating imaginary numbers -- expressing in the notation we use nowadays, $i^2 = -1$ and $(i) \times (-i) = 1$.

Then, Bombelli claimed that x might be real number even it looked like a quantity that involved negative square root. Since $x = \sqrt[3]{\frac{q-\sqrt{q^2+4\times\left(\frac{p}{3}\right)^3}}{2}} - \sqrt[3]{\frac{-q-\sqrt{q^2+4\times\left(\frac{p}{3}\right)^3}}{2}}$ was the root they studied and $q^2 + 4 \times \left(\frac{p}{3}\right)^3 < 0$ was the condition they were confusing with, the root which Cardano thought unsolvable can be simplified to $x = \sqrt[3]{\frac{q-\sqrt{-m}}{2}} - \sqrt[3]{\frac{-q-\sqrt{-m}}{2}}$ for $-m = q^2 + 4 \times \left(\frac{p}{3}\right)^3 < 0$. Bombelli assumed that $\sqrt[3]{\frac{q-\sqrt{-m}}{2}} = a + bi\ (a, b \in R)$, which is equivalent to $\frac{q-\sqrt{-m}}{2} = a^3 + 3a^2bi - 3ab^2 - b^3i$ in real range (in order to make the progress more brief, the calculation will be written by modern notation, which is not what Bombelli used). So, $\begin{cases} a^3 - 3ab^2 = \frac{q}{2} \\ 3a^2b - b^3 = \frac{-\sqrt{m}}{2} \end{cases}$. After finding this, it is worth to be notice that $\sqrt[3]{\frac{-q-\sqrt{-m}}{2}} = -a + bi$. This is because, $(-a + bi)^3 = -a^3 + 3a^2bi + 3ab^2 - b^3i = -(a^3 - 3ab^2) + (3a^2b - b^3)i = \frac{-q-\sqrt{-m}}{2}$. Finally, it is acquired that $x = \sqrt[3]{\frac{q-\sqrt{-m}}{2}} - \sqrt[3]{\frac{-q-\sqrt{-m}}{2}} = (a + bi) - (-a + bi) = 2a$, which means, real root x is gotten through those operation based on complex number calculation. In fact, this is a very great step for human to accept imaginary number. Solving cubic equation not only has inspired mathematicians to dive into the research of complex number, but it also has promoted the development of mathematics in many other aspects as some new problems that have not been thought before had exposed. Firstly, using Cardano's formula in that time without the basic thought of complex number, it is impossible to get all the real roots of the equation when there are more than one real roots. So, sometimes mathematicians found there exist other roots which is very obvious but unable to get through Cardano's formula when they try some examples. Thus, mathematicians got confused about how many roots should a cubic equation have. Some visionary mathematicians also started to think about how many roots should a polynomial of n degree have. Although many mathematicians had give out the correct conclusion, as it may consider as in nowadays, no one had ever given out coherent proof. This problem was eventually solved by Gauss by a very ingenious way leading to the factor theorem. To briefly summarize it, the idea can be epitomized as that every polynomial $x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \cdots + a_1x^1 + a_0 = 0$can be rewritten as $(x - x_1)(x - x_2)\cdots(x - x_n) = 0$, in other word, the original polynomial has n linear factors. So, $x_i(i = 1,2,\cdots,n)$ is a root of the equation and x_i is not necessarily distinct from each other. This proof also bases on the acknowledgement of complex number as x_i is not necessarily real. Secondly, although this impetus caused by Cardano's formula is not very impressive and important for the development of math, it also encouraged mathematicians to make sparkles of imagination in some degree. Sometimes it was very easy to find some special rational root(s) of a equation, but the root received by Cardano's formula was awkward since it is expressed as a form of $\sqrt[3]{\frac{q-\sqrt{q^2+4\times\left(\frac{p}{3}\right)^3}}{2}} - \sqrt[3]{\frac{-q-\sqrt{q^2+4\times\left(\frac{p}{3}\right)^3}}{2}}$. This problem with coped by simplifying the expression by the method of undetermined coefficient, which will not be introduced at here. Solving quartic equation can use a way which is very similar with solving cubic equation. For any quartic equation forming like x4+ax3+bx2+cx+d=0, if it can be rewritten to a form of f(x)2+pf(x)+q=0 with f(x)=x2+αx+β, this equation is absolutely easy to solve. However, this progress have some difficulties need to face, which is, not every quartic equation can be rewritten to that new form directly.

To illustrate this, expanding f(x)2+pf(x)+q=0 where f(x)=x2+αx+β is a good idea. It is very obvious that the result should be $x^4 + 2\alpha x^3 + (\alpha^2 + 2\beta + p)x^2 + (2\alpha\beta + \alpha p)x + (\beta^2 + p\beta + q) = 0$.

So, $\begin{cases} a = 2\alpha \\ b = \alpha^2 + 2\beta + p \\ c = 2\alpha\beta + \alpha p \\ d = \beta^2 + p\beta + q \end{cases}$. After this step, the problem is exposed. In other word, this simultaneous equations only hold for special (a, b, c, d). This is because, the first equation $a = 2\alpha$ determine the value of α, and the third equation $c = 2\alpha\beta + \alpha p$ determine the value of $2\alpha\beta + \alpha p$. So, the second equation only hold when $b = \alpha^2 + \alpha c$, which is $b = \frac{a^2}{4} + \frac{a}{2}c$, but b cannot be guaranteed to be such value if the common condition is been discussed.

However, this problem can be coped by a way which is very similar with the way cubic equation is coped, which have been mention before. In general, the coefficients can be changed by finding another equation which have roots related to the roots of the original equation so that the new equation' s coefficients satisfy the requirement. If it is defined that g(x)=x4+ax3+bx2+cx+d and assume that x=k+y, $g(k + y) = g(k) + \frac{g'(k)}{1!}y + \frac{g''(k)}{2!}y^2 + \frac{g'''(k)}{3!}y^3 + \frac{g''''(k)}{4!}y^4$ (obviously that $\frac{g''''(k)}{4!}$=1). Then, a new equation for y is gotten and x can be gotten if y can be gotten for a given k. In order to get y through the method mention before, it is hoped that $\frac{g''(k)}{2!} = (\frac{g'''(k)}{3!\times 4})^2 + \frac{g'''(k)}{3!\times 2} \times \frac{g'(k)}{1!}$. By rewriting it, a cubic equation for k will be gotten and k can be solved through the way of solving cubic equation. Finally, a quartic equation for y will be received and it can be solved by rewriting it to a form of f(y)2+pf(y)+q=0 with f(y)=y2+αy+β and solving two quadratic equations. To sum up, solving a quartic equation can be solved by solving a cubic equation and two quadratic equations.

There are many other methods which have already exist for many years for solving quartic equation. Most of them include steeps of transferring the equation into some special form, like the equation without the term of x of 3 degree, and make some term into a form of square or cube. Ferrari's method is a very famous one in this field as it is the method been found most early closely after Cardano published the method of solving cubic equation.

3. Topology of the complex field

It's defined that Ω⊂C is open if for all $z \in \Omega$ there exist r>0 such that $D_r(z) \subset \Omega$ and Ω⊂C is closed if its complement Ωc is open. This can also be defined by stereographic projection using Riemann sphere. $\Omega \subset C$ is open if for all $P \in A$ there exist r>0 such that B:=$\{T||PT| < r\} \cap S_{Riemann\ sphere}$ and B⊂A, where P is the correspond point of z on Riemann sphere for $z \in \Omega$ and A is the set for all P. Ω⊂C is closed if for all $Q \in S_{Riemann\ sphere}\backslash A$ there exist r>0 such that B:=$\{T||QT| < r\} \cap S_{Riemann\ sphere}$ and $B \subset S_{Riemann\ sphere}\backslash A$. So, the point at infinite can be visualized. In this way, it's easy to understand that C and $\emptyset$ are both open and closed.

There is a very ingenious way for testing if a set Ω⊂C is closed. A set Ω⊂C is closed iff for any convergent sequence $\{z_n\}_{n\in N} \subset \Omega$, $\lim_{n\to\infty} z_n \in \Omega$.

Some notations give every set, which may not closed nor open, a form of closed or open, which is associated with the original set. It's defined that for Ω⊂C, $z \in C$ is called an interior point of Ω if there exist r>0 such that $D_r(z) \subset \Omega$, and Ωo is the set of all interior point of Ω. Besides, z is a limit point of Ω if there exit a sequence such that $\{z_n\}_{n\in N} \subset \Omega\backslash\{z\}$ with $\lim_{n\to\infty} z_n = z$, and $\bar{\Omega}$ is the set of all limit point of Ω. Finally, it is defined that the boundary of Ω is $\partial\Omega := \bar{\Omega}\backslash\Omega^o$. Thus, it is obvious that Ω is open iff $\Omega=\Omega^o$, and Ω is closed iff $\Omega=\bar{\Omega}$.

The author is going to introduce anther definition which illustrates if a set is bounded. $\Omega\backslash subset$ C is bounded if there exist r>0 such that $|z|$<r for every $z \in \Omega$. In other word, $\Omega \subset$Dr(0).

If Ω is bounded, its diameter is the quantity $\text{diam}(\Omega) := \max_{z,w\in\Omega} |z - w|$. Ω is labeled as compact if it is closed and bounded.

4. Summary

Complex analysis research is a fascinating and sophisticated field. Mathematicians have created groundbreaking work in this area during the past few centuries. Complex numbers were first defined by mathematicians, who subsequently gradually looked into their algebraic and geometric features. After hundreds of years, they discovered and studied complex computations. In addition to spurring interest in the study of complex numbers, the solution of the cubic equation has helped mathematics in many other ways by bringing to light problems that had not yet been resolved. First off, without having a firm grasp of the basics of complex numbers, it is difficult to find all of the real roots of an equation when there are several of them. Because of this, mathematicians occasionally find that there are alternative roots that are extremely evident but cannot be verified using Cardano's method when they examine numerous situations. This essay begins with a study of cubic equations before delving further into the topology of comp

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