

Calculating Specific Integrals by Using Residue Theorem

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Abstract. The aim of this paper is to solve the some specific integrals such as $\int_C \frac{a_1 z^{n-1} + a_2 z^{n-2} + \dots + a_n}{b z^n + b_1 z^{n-1} + b_2 z^{n-2} + \dots + b_n} dz$, $\int_Y \frac{g(z)}{b z^a + n} dz$, and $\int_Y \frac{g(z)}{b z^a + b_1 z^{a-1} + b_2 z^{a-2} + \dots + c} dz$. The traditional method is that people normally decompose this polynomial into several partial fractions first. This process involves adding it all up, expanding brackets, and doing matrices computation, which takes too many steps of calculation. The partial fraction part requires using Euler's formula and large amounts of expanding brackets to prove that the multiplication of those partial denominators is equal to the denominator given in the original equation. Once making one little mistake, the next process will be all nonsense. Therefore, introducing complex analysis and residue theorem can result in considerably fewer calculation steps than the traditional method does in order to calculate integral, as opposed to completing hundreds of steps of partial fraction decomposition and substitution. To solve the first integral: $\int_C \frac{a_1 z^{n-1} + a_2 z^{n-2} + \dots + a_n}{b z^n + b_1 z^{n-1} + b_2 z^{n-2} + \dots + b_n} dz$, we can rewrite Cauchy's residue theorem in a new form, by considering residue at infinity, $\int_C f(z) dz = 2\pi i \operatorname{Res}_{z=0} \left[\frac{1}{z^2} f\left(\frac{1}{z}\right) \right]$. Then, we can get the coefficient of $\frac{1}{z}$ and hence compute the integral. For the second and third integral: $\int_Y \frac{g(z)}{b z^a + n} dz$, and $\int_Y \frac{g(z)}{b z^a + b_1 z^{a-1} + b_2 z^{a-2} + \dots + c} dz$, we exploit the reciprocal function of Taylor polynomials, writing $\frac{g(z)}{b z^a + n}$ and $\frac{g(z)}{b z^a + b_1 z^{a-1} + b_2 z^{a-2} + \dots + c}$ in a new form to get the coefficient of $\frac{1}{z}$. Then we can compute the integrals by using the residue theorem.

Keywords: Cauchy Residue Theorem, contour integrals, Taylor series, analytic functions, Cauchy-Goursat Theorem.

1. Introduction

Complex analysis and contour integrals are used widely among professional scientists and university students, and it is included in many textbooks [1, 2]. However, many contour integrals are always complicated to calculate especially for those who have just begun to learn complex numbers.

$\int_C \frac{a_1 z^{n-1} + a_2 z^{n-2} + \dots + a_n}{b z^n + b_1 z^{n-1} + b_2 z^{n-2} + \dots + b_n} dz$, $\int_Y \frac{g(z)}{b z^a + n} dz$, and $\int_Y \frac{g(z)}{b z^a + b_1 z^{a-1} + b_2 z^{a-2} + \dots + c} dz$. The particular reason for this circumstance is that people normally decompose this polynomial into several partial fractions first. Then, the rest of the process involves adding it all up, expanding brackets, and doing matrices computation, and many arduous calculations are involved in this process. The partial fraction part requires using large amounts of algebraic operations to make sure that the sum of the several partial fractions, whose denominators are concise, is equal to the original equation. A single mistake will lead to a vain calculation in the following calculation steps. As a result, people have to be extremely careful and always check if their calculations are correct. Therefore, introducing complex analysis and residue theorem can avoid hundreds of steps of partial fraction decomposition and substitution and result in considerably fewer calculation steps than the traditional method does in the process of

calculating integrals. The residue theorem not only simplifies the process but also provides an alternative thought for researchers. Calculating integrals by using the residue theorem will be displayed in this article, and some technical results of certain integrals are proved. The contour integral calculation, some involving residues, has a wide range of applications, like large-scale eigenvalue problems applied in the area of engineering [3] and using contour integrals in reduced basis methods [4]. The previous study has already provided important information on the residue theorem. A person called Cauchy must be mentioned here. “Cauchy was a revolutionary in mathematics and a highly original founder of modern complex function theory.” He made a great contribution to complex function theory in the 19th century [5]. Cauchy's Integral Theorem, which is regarded as the most significant theorem in complex analysis, is the foundation of the residue theorem. It states that: let $f(z)$ be holomorphic in a simply connected region, then $\int_C f(z) dz = 0$ for every closed contour C contained in the region [6]. And estimating the line integrals of analytic functions of closed curves is the final purpose of using Cauchy Residue Theorem [7]. People can write functions of z in the form of partial fractions, but transcendental numbers will appear when the polynomial in the denominator has power higher than 5. Thus, the traditional decomposing method doesn't work anymore. The residue theorem is helpful because it is often possible to find singular points and the accompanying residues even for reasonably intricate functions [6]. In this paper, calculations of three specific kinds of contour integral problems, and the ultimate formulae to calculate them, are shown in detail, with the first one being a specific case, the second a generalization, and the third a further generalization. The first approach contains the Laurent series' simple topology and infinite contour. By considering infinite contour, the residue theorem can be expressed in a new form so the coefficient of $\frac{1}{z}$ can be computed more easily. The second approach exploits the Taylor's polynomials of some reciprocal function. The third approach expands on the second with the help of Taylor polynomials and ultimately gives a more generalized notion.

2. Method

In this section, the theorems and formulae that are used to calculate $\int_C \frac{a_1 z^{n-1} + a_2 z^{n-2} + \dots + a_n}{b z^n + b_1 z^{n-1} + b_2 z^{n-2} + \dots + b_n} dz$, $\int_\gamma \frac{g(z)}{b z^a + n} dz$, and $\int_\gamma \frac{g(z)}{b z^a + b_1 z^{a-1} + b_2 z^{a-2} + \dots + c} dz$ are listed.

2.1. Rewriting the Cauchy Residue Theorem

By considering residue at infinity, the Cauchy residue theorem can be written in this new form:

$$\int_C f(z) dz = 2\pi i \underbrace{\text{Res}}_{z=0} \left[\frac{1}{z^2} f\left(\frac{1}{z}\right) \right] \quad (1)$$

2.2. Laurent expansion

The Laurent expansion formula, which is the series expansion of a function $f(z)$ when $f(z)$ is not analytic at z_0 , tells us that:

$$f(z) = \sum_{n=-\infty}^{\infty} C_n (z - z_0)^n \quad (2)$$

2.3. The Taylor polynomials and the Maclaurin polynomials in complex form

According to the formula of the Taylor series, the Taylor polynomials in complex form is

$$f(z) = \sum_{\lambda=0}^{\infty} \frac{f^{(\lambda)}(z_0)}{\lambda!} (z - z_0)^\lambda \quad (3)$$

According to the definition of the Maclaurin polynomials, when $z = 0$ in formula (3), the Maclaurin polynomials in complex form can be written as follows:

$$f(z) = \sum_{\lambda=0}^{\infty} \frac{f^{(\lambda)}(0)}{\lambda!} (z)^\lambda \quad (4)$$

2.4. The Cauchy Residue Theorem

According to Cauchy Residue Theorem, for functions $f(z)$ that are holomorphic inside and on the simple closed contour, C , described in the positive sense except for a finite number of singularities z_k ($k = 1, 2, \dots, n$) inside that contour, the contour integral of such functions can be computed as

$$\int_C f(z) dz = 2\pi i \sum_{k=1}^n \text{Res } f(z) \quad (5)$$

2.5. The Cauchy-Goursat Theorem

According to Cauchy-Goursat Theorem, for functions $f(z)$ that are holomorphic inside and on the simple closed contour, C , described in the positive sense, the contour integral of such functions can be computed as

$$\int_C f(z) dz = 0 \quad (6)$$

2.6. Writing the Taylor polynomial of the reciprocal of a function

Suppose $\frac{1}{h(z)} = \frac{1}{h(z_k)} + b_1(z - z_k) + b_2(z - z_k)^2 + \dots$, where on the left hand side, $h(z)$ can be written in the form of Taylor polynomials at $z = z_k$ as $h(z_k) + a_1(z - z_k) + a_2(z - z_k)^2 + \dots$, and on the right hand side, $\frac{1}{h(z_k)} + b_1(z - z_k) + b_2(z - z_k)^2 + \dots$ is the Taylor polynomials at $z = z_k$ of $\frac{1}{h(z)}$ (assume that $b_1, b_2, \dots$ exist). Then, $b_1, b_2, \dots$ can be computed by multiplying the right-hand side with the denominator of the left-hand side:

$$\begin{aligned} 1 &= \left(\frac{1}{h(z_k)} + b_1(z - z_k) + b_2(z - z_k)^2 + \dots \right) (h(z_k) + a_1(z - z_k) + a_2(z - z_k)^2 + \dots) \\ &= 1 + (b_1 + a_1)(z - z_k) + (b_2 + a_2 + b_1 a_1)(z - z_k)^2 + \dots \end{aligned}$$

Therefore,

$$\begin{cases} b_1 + a_1 = 0 \\ b_2 + a_2 + b_1 a_1 = 0 \end{cases}$$

And

$$\begin{cases} b_1 = -a_1 \\ b_2 = a_1^2 - a_2 \\ \dots \end{cases} \quad (7)$$

This means that $b_1, b_2, \dots$ do exist, and can be computed with $a_1, a_2, \dots$.

3. Results

3.1. Computing $\int_C \frac{a_1 z^{n-1} + a_2 z^{n-2} + \dots + a_n}{b z^n + b_1 z^{n-1} + b_2 z^{n-2} + \dots + b_n} dz$

In order to solve the contour integral problem like $\int_C \frac{a_1 z^{n-1} + a_2 z^{n-2} + \dots + a_n}{b z^n + b_1 z^{n-1} + b_2 z^{n-2} + \dots + b_n} dz$, it is of necessity to first prove the theorem in Part 2.1, formula (1). The proof of the theorem is as follows.

When function $f(z)$ is an analytic function in an anticlockwise close contour C except for finite singularities. And we denote R , the radius of the circle $|z| = R$ (Fig.1), large enough that the circle $|z|$ completely encircles the contour C .

It is obvious that function $f(z)$ is an analytic function in the domain $0 < |z| < \infty$, and we can say that the point at infinity is a singularity of function $f(z)$.

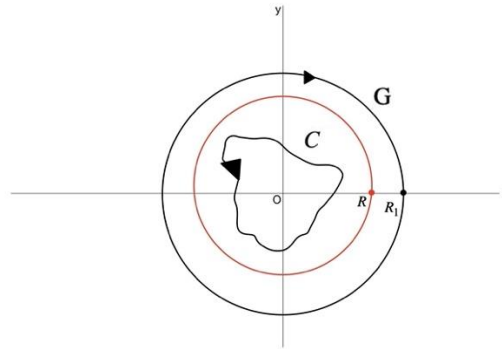


Fig.1: a sketch of the contours and circle

Now, we denote the radius of circle G , which is a clockwise oriented contour, R_1 , where $R_1 > R$.

Then the following means of the equation can define the residue of $f(z)$ at infinity

$$\int_G f(z) dz = 2\pi i \underbrace{\text{Res}}_{z=\infty} f(z)$$

Since $f(z)$ is analytic in the closed domain bounded by C and G , we can get that

$$\int_C f(z) dz = \int_{-G} f(z) dz = - \int_G f(z) dz$$

Therefore

$$\int_C f(z) dz = - \int_G f(z) dz = -2\pi i \underbrace{\text{Res}}_{z=\infty} f(z)$$

To find the residue in the equation above, we can write the Laurent series as

$$f(z) = \sum_{n=-\infty}^{\infty} C_n z^n \quad (R < |z| < \infty) \quad (8)$$

Where

$$C_n = \frac{1}{2\pi i} \int_{-C_0} \frac{f(z)}{z^{n+1}} dz \quad (9)$$

Replacing z in the equation (8) by $\frac{1}{z}$ and multiplying $\frac{1}{z^2}$ to the equation (8), we can get that

$$\frac{1}{z^2} f\left(\frac{1}{z}\right) = \sum_{n=-\infty}^{\infty} \frac{C_{n-2}}{z^n}$$

To find the function's residues, we need to find the coefficient of $\frac{1}{z}$. Obviously, in the Laurent expansion of $\frac{1}{z^2} f\left(\frac{1}{z}\right)$, we can see

$$C_{-1} = \underbrace{\text{Res}}_{z=0} \left[\frac{1}{z^2} f\left(\frac{1}{z}\right) \right] \quad (10)$$

Let $n = -1$ in equation (9), we can then get

$$C_{-1} = \frac{1}{2\pi i} \int_{-C_0} \frac{f(z)}{z^{1-1}} dz = \frac{1}{2\pi i} \int_{-C_0} f(z) dz \quad (11)$$

Combining equation (10) and equation (11), we are able to get

$$\int_{C_0} f(z) dz = -2\pi i \underbrace{\text{Res}}_{z=0} \left[\frac{1}{z^2} f\left(\frac{1}{z}\right) \right] = - \int_C f(z) dz \quad (12)$$

Then, we can rewrite the equation (12) as follows:

$$\int_C \frac{a_1 z^{n-1} + a_2 z^{n-2} + \dots + a_n}{b z^n + b_1 z^{n-1} + b_2 z^{n-2} + \dots + b_n} dz = 2\pi i \underbrace{\text{Res}}_{z=0} \left[\frac{1}{z^2} \frac{a_1 z^{1-n} + a_2 z^{2-n} + \dots + a_n}{b z^{-n} + b_1 z^{-n+1} + b_2 z^{-n+2} + \dots + b_n} \right]$$

This equation is equal to the following expression:

$$2\pi i \underbrace{\text{Res}}_{z=0} \left[\frac{1}{z} \frac{a_1 z^{1-n} + a_2 z^{2-n} + \dots + a_n}{b z^{1-n} + b_1 z^{2-n} + b_2 z^{3-n} + \dots + b_n z} \right] \quad (13)$$

Multiplying z^{n-1} to the nominator and denominator of $\frac{a_1 z^{1-n} + a_2 z^{2-n} + \dots + a_n}{b z^{1-n} + b_1 z^{2-n} + b_2 z^{3-n} + \dots + b_n z}$, we are able to get the following expression:

$$2\pi i \underbrace{\text{Res}}_{z=0} \left[\frac{1}{z} \frac{a_1 + a_2 z^1 + \dots + a_n z^{n-1}}{b + b_1 z^1 + b_2 z^2 + \dots + b_n z^n} \right]$$

When $z = 0$, the coefficient of $\frac{1}{z}$ is $\frac{a_1}{b}$.

Therefore:

$$\int_C \frac{a_1 z^{n-1} + a_2 z^{n-2} + \dots + a_n}{b z^n + b_1 z^{n-1} + b_2 z^{n-2} + \dots + b_n} dz = 2\pi i \frac{a_1}{b} \quad (14)$$

3.2. Generalization: computing $f(z) = \frac{g(z)}{b z^{a+n}}$, $a \in \mathbb{N}_+$

In the previous part, in order to use the formula of $\int_C f(z) dz = 2\pi i \frac{a_1}{b}$, it is required that $f(z)$ satisfy many requirements regarding the position and the size of the contour γ and the form of the numerator of $f(z)$. However, in the next section, the Taylor series is exploited, so that besides being in the form of $f(z) = \frac{g(z)}{b z^{a+n}}$, $a \in \mathbb{N}_+$, $n \neq 0$, $f(z)$ is only required that the numerator of $f(z)$, namely $g(z)$, does not have any singularities inside the contour γ . Therefore, it is the generalization of the method described in the previous section. (Note: the contour γ should be a simple closed contour described in the positive sense.)

Functions in the form of $f(z) = \frac{g(z)}{bz^a+n}$, $a \in \mathbb{N}_+$ with a finite number of singularities $z = z_k$, where $k = 1, 2, 3, \dots$, $\int_{\gamma} f(z)dz$ can be computed by exploiting the Taylor polynomials of $\frac{1}{bz^a+n}$ and $g(z)$. It is required that $g(z)$ does not have singularities in the domain of $f(z)$ to guarantee that all the singularities, z_k , come from the rational function $\frac{1}{bz^a+n}$. When $n \neq 0$ and $b \neq 0$, it is recommended to first compute the Taylor polynomials of $\frac{1}{bz^a+n}$ at $z = z_k$ and then get the reciprocal of that Taylor polynomial in order to calculate the Taylor polynomials of $\frac{1}{bz^a+n}$ at $z = z_k$; when $n = 0$ and $b \neq 0$, it is recommended to directly calculate the Maclaurin expansion of $g(z)$; when $b = 0$, use Cauchy-Goursat Theorem directly. The calculation is as follows.
If $n \neq 0$ and $b \neq 0$:

According to the formula of Taylor polynomials of complex numbers at $z = z_0$ (formula (3) in Method 2.3), the Taylor polynomials of $\frac{1}{bz^a+n}$ at $z = z_k$ can be computed as

$$bz^a + n = (bz_k^a + n) + (abz_k^{a-1})(z - z_k) + \frac{(ab(a-1)z_k^{a-2})(z - z_k)^2}{2!} + \dots \quad (15)$$

Because all the singularities, z_k , come from the rational function $\frac{1}{bz^a+n}$, then we can know that

$$bz_k^a + n = 0 \quad (16)$$

According to equation (16), it is evident that

$$bz_k^a = -n \quad (17)$$

Then equation (17) can be integrated into equation (15), and then equation (15) can be rewritten as follows:

$$bz^a + n = 0 + (abz_k^{a-1})(z - z_k) + \frac{(ab(a-1)z_k^{a-2})(z - z_k)^2}{2!} + \dots \quad (18)$$

By extracting the common factor of equation (18) (* denotes some unimportant coefficients, and * may always have different values for different terms), we can get

$$bz^a + n = \frac{a(z - z_k)}{z_k} (bz_k^a + * (z - z_k) + \dots) \quad (19)$$

Combining equation (19) and equation (18), we can rewrite formula (19) as following:

$$bz^a + n = \frac{a(z - z_k)}{z_k} ((-n) + * (z - z_k) + \dots) \quad (20)$$

Extracting the common factor of equation (20), this time $\frac{-an(z-z_k)}{z_k}$, we can rewrite the above as follows:

$$bz^a + n = \frac{-an(z - z_k)}{z_k} (1 + * (z - z_k) + \dots) \quad (21)$$

Then, it is simple to compute $\frac{1}{bz^a+n}$ by interchanging the nominator and denominator of the common factor $\frac{-an(z-z_k)}{z_k}$ and the polynomial $(1 + * (z - z_k) + \dots)$ and get

$$\frac{1}{bz^a + n} = \frac{z_k}{-an(z - z_k)} \left(\frac{1}{1 + * (z - z_k) + \dots} \right) \quad (22)$$

According to Method 2.7 and equation (7), the Taylor's polynomial of $\frac{1}{1+*(z-z_k)+\dots}$ can also be written in the form of $1+*(z-z_k)+\dots$ (Note that * denotes some unimportant coefficient, and * may always have different values for different terms.), so that equation (22) can be rewritten as

$$\frac{1}{bz^a + n} = \frac{z_k}{-an(z-z_k)} (1+*(z-z_k)+\dots) \quad (23)$$

Again, according to the formula of Taylor polynomial, the Taylor polynomial of $g(z)$ at $z = z_k$ can also be computed as follows:

$$g(z) = g(z_k) + g'(z_k)(z-z_k) + \dots \quad (24)$$

Therefore, by combining equation (23) and equation (24), we can get

$$\begin{aligned} f(z) &= \frac{1}{bz^a + n} g(z) = \frac{z_k}{-an(z-z_k)} (1+*(z-z_k)+\dots)(g(z_k) + g'(z_k)(z-z_k) + \dots) \\ f(z) &= \frac{z_k}{-an(z-z_k)} (g(z_k) + *(z-z_k) + \dots) \\ f(z) &= \frac{z_k g(z_k)}{-an(z-z_k)} (1+*(z-z_k)+\dots) \end{aligned} \quad (25)$$

It is obvious that in this equation (25), the coefficient of $\frac{1}{z-z_k}$ is $\frac{z_k g(z_k)}{-an}$, so according to the definition of residues,

$$Res(f(z), z_k) = \frac{z_k g(z_k)}{-an} \quad (26)$$

Finally, according to Cauchy Residue Theorem (in formula (5) in Part 2.4), the contour integral can be calculated with equation (26) as

$$\int_{\gamma} \frac{g(z)}{bz^a + n} dz = 2\pi i \sum_{k=1}^m \frac{z_k g(z_k)}{-an} \quad (27)$$

Where m denotes the number of singularities inside the contour γ .

If $n = 0$, the function then becomes $f(z) = \frac{g(z)}{bz^a} = \frac{g(z)}{bz^a}$, $a \in \mathbb{N}_+$, and the only singularity of $f(z)$ is at $z = 0$.

To calculate the contour integral of $f(z)$, it is recommended to first compute the Maclaurin expansion of $g(z)$ according to formula (4) in Method 2.3:

$$g(z) = \sum_{\lambda=0}^{\infty} \frac{g^{(\lambda)}(0)}{\lambda!} (z)^\lambda \quad (28)$$

With equation (28), $f(z)$ can then be rewritten as:

$$f(z) = \frac{g(z)}{bz^a} = \frac{\sum_{\lambda=0}^{\infty} \frac{g^{(\lambda)}(0)}{\lambda!} (z)^\lambda}{bz^a} = \frac{\sum_{\lambda=0}^{\infty} \frac{g^{(\lambda)}(0)}{\lambda!} (z)^{\lambda-a}}{b} \quad (29)$$

To find the coefficient of $\frac{1}{z}$, we have to find the term when $\lambda - a = -1$, namely $\lambda = a - 1$. The λ th term of Maclaurin's series of $f(z)$ can be computed to be:

$$\frac{\frac{g^{(a-1)}(0)}{(a-1)!} (z)^{-1}}{b}$$

Therefore, with the fact that the only singularity of $f(z)$ is at $z = 0$, the coefficient of $\frac{1}{z}$, or the residue of $f(z)$, can be written as:

$$\operatorname{Res}_{z=0} f(z) = \frac{g^{(a-1)}(0)}{b(a-1)!} \quad (30)$$

Finally, according to Cauchy Residue Theorem (in formula (5) in Method 2.4), the contour integral can be calculated with equation (30) as

$$\int_{\gamma} \frac{g(z)}{bz^a} dz = \frac{g^{(a-1)}(0)2\pi i}{b(a-1)!} \quad (31)$$

In the special case when $g(z)$ is a constant and $a > 1$, for example when $g(z)$ equals 2 or 3 (note that $g(z)$ can either be a real constant or a complex constant), $g^{(a-1)}(0)$ will always equal to 0, so that for a simple closed contour described in the positive sense γ that has the singularity $z = 0$ inside, $\int_{\gamma} \frac{g(z)}{bz^a} dz = \frac{0}{b(a-1)!} = 0$; when $g(z)$ is a constant and $a = 1$, $g^{(a-1)}(0)$ will always equal to $g(z)$, so that for a simple closed contour described in the positive sense γ has the singularity $z = 0$ inside, $\int_{\gamma} \frac{g(z)}{bz^a} dz = \frac{g(z)2\pi i}{b(a-1)!} = \frac{g(z)2\pi i}{b}$.

If $b = 0$, $\int_{\gamma} f(z) dz = \int_{\gamma} \frac{g(z)}{n} dz = 0$ because $g(z)$ is holomorphic everywhere inside the contour γ , which makes $f(z) = \frac{g(z)}{n}$ also a holomorphic function. Cauchy-Goursat Theorem guarantees that $\int_{\gamma} \frac{g(z)}{n} dz = 0$.

To summarize the results in 3.2, Table 1 lists all the integral formulae and requirements for all the functions discussed above (it is assumed that the contour γ is simple, closed, described in the positive direction, and contains at least one singularity).

Table 1: the formulae deduced in Results 3.2

Contour integral	Requirement of $g(z)$	Requirement of a	Requirement of b	Requirement of n	Function then becomes	Integral formula
$\int_{\gamma} \frac{g(z)}{bz^a + n} dz$	Holomorphic inside γ	$a \in \mathbb{N}_+$	$b \neq 0$	$n \neq 0$	$\int_{\gamma} \frac{g(z)}{bz^a + n} dz$	$2\pi i \sum_{k=1}^m \frac{z_k g(z_k)}{-an}$
$\int_{\gamma} \frac{g(z)}{bz^a + n} dz$	Holomorphic inside γ	$a \in \mathbb{N}_+$	$b \neq 0$	$n = 0$	$\int_{\gamma} \frac{g(z)}{bz^a} dz$	$\frac{g^{(a-1)}(0)2\pi i}{b(a-1)!}$
$\int_{\gamma} \frac{g(z)}{bz^a + n} dz$	constant	$a \in \mathbb{N}_+, a > 1$	$b \neq 0$	$n = 0$	$\int_{\gamma} \frac{g(z)}{bz^a} dz$	0
$\int_{\gamma} \frac{g(z)}{bz^a + n} dz$	constant	$a = 1$	$b \neq 0$	$n = 0$	$\int_{\gamma} \frac{g(z)}{bz} dz$	$\frac{g(z)2\pi i}{b}$
$\int_{\gamma} \frac{g(z)}{bz^a + n} dz$	Holomorphic inside γ	No requirement	$b = 0$	$n \neq 0$	$\int_{\gamma} \frac{g(z)}{n} dz$	0

These formulae can apply to more general conditions than the formula in Result 3.1, and it is therefore a generalization.

3.3. Further generalization: from $f(z) = \frac{g(z)}{bz^a + n}$ to $f(z) = \frac{g(z)}{bz^a + b_1z^{a-1} + b_2z^{a-2} + \dots + c}$

In fact, by expanding the denominator, the function $f(z) = \frac{g(z)}{bz^a + n}$ can be generalized into $\frac{g(z)}{bz^a + b_1z^{a-1} + b_2z^{a-2} + \dots + c}$, and $\int_{\varepsilon} \frac{g(z)}{bz^a + b_1z^{a-1} + b_2z^{a-2} + \dots + c} dz$ can be computed by using the similar method above.

The Taylor polynomials of $bz^a + b_1z^{a-1} + b_2z^{a-2} + \dots + c$ at $z = z_x$ can be written as

$$bz^a + b_1z^{a-1} + b_2z^{a-2} + \dots + c = (bz_x^a + b_1z_x^{a-1} + b_2z_x^{a-2} + \dots + c) + (abz_x^{a-1} + (a-1)b_1z_x^{a-2} + \dots)(z - z_x) + \dots \quad (32)$$

Since,

$$bz^a + b_1z^{a-1} + b_2z^{a-2} + \dots + c = 0 \quad (33)$$

So,

$$bz^a + b_1z^{a-1} + b_2z^{a-2} + \dots + c = (z - z_x)((abz_x^{a-1} + (a-1)b_1z_x^{a-2} + \dots) + (z - z_x) + \dots) \quad (34)$$

$$\frac{1}{bz^a + b_1z^{a-1} + b_2z^{a-2} + \dots + c} = \frac{1}{(z - z_x)(abz_x^{a-1} + (a-1)b_1z_x^{a-2} + \dots) + (z - z_x) + \dots} \quad (35)$$

(* denotes some unimportant coefficients)

According to Part 2.7 and equation (35), we can write

$$\frac{1}{(abz_x^{a-1} + (a-1)b_1z_x^{a-2} + \dots) + (z - z_x) + \dots} = A(z_x) + (z - z_x) + \dots \quad (36)$$

Since

$$((abz_x^{a-1} + (a-1)b_1z_x^{a-2} + \dots) + (z - z_x) + \dots)(A(z_x) + (z - z_x) + \dots) = 1 \quad (37)$$

We can let $(abz_x^{a-1} + (a-1)b_1z_x^{a-2} + \dots)A(z_x) = 1$ and the rest of the product of these two polynomials equal to 0

Then

$$A(z) = \frac{1}{abz^{a-1} + (a-1)b_1z^{a-2} + \dots} = \frac{1}{\frac{d(bz^a + b_1z^{a-1} + b_2z^{a-2} + \dots + c)}{dz}}, z = z_x \quad (38)$$

Therefore,

$$\frac{1}{bz^a + b_1z^{a-1} + b_2z^{a-2} + \dots + c} = \frac{1}{(z - z_k) \left(\frac{1}{\frac{d(bz^a + b_1z^{a-1} + b_2z^{a-2} + \dots + c)}{dz}} + * (z - z_k) + \dots \right)}, z = z_x \quad (39)$$

Also, according to the formula of Taylor's polynomial, the Taylor polynomial of $g(z)$ at $z = z_x$ can be computed as follows:

$$g(z) = g(z_x) + g'(z_x)(z - z_x) + \dots$$

Therefore, by combining these formulae, we can get

$$\begin{aligned} f(z) &= \frac{1}{bz^a + b_1z^{a-1} + b_2z^{a-2} + \dots + c} g(z) \\ &= \frac{1}{(z - z_k) \left(\frac{1}{\frac{d(bz^a + b_1z^{a-1} + b_2z^{a-2} + \dots + c)}{dz}} + * (z - z_k) + \dots \right)} (g(z_x) + g'(z_x)(z - z_x) + \dots) \\ &= \frac{\frac{1}{\frac{d(bz^a + b_1z^{a-1} + b_2z^{a-2} + \dots + c)}{dz}}}{(z - z_k)} (g(z_x) + * (z - z_x) + \dots), z = z_x \end{aligned} \quad (40)$$

It is obvious that in this expression, the coefficient of $\frac{1}{z - z_k}$ is $\frac{1}{\frac{d(bz^a + b_1z^{a-1} + b_2z^{a-2} + \dots + c)}{dz}} g(z_x), z = z_x$

z_x

According to the definition of residues,

$$Res(f(z), z_x) = \frac{1}{\frac{d(bz^a + b_1z^{a-1} + b_2z^{a-2} + \dots + c)}{dz}} g(z), z = z_x \quad (41)$$

Finally, according to Cauchy Residue Theorem,

$$\int_{\varepsilon} \frac{g(z)}{bz^a + b_1z^{a-1} + b_2z^{a-2} + \dots + c} dz = 2\pi i \sum_{k=1}^h \frac{1}{\frac{d(bz^a + b_1z^{a-1} + b_2z^{a-2} + \dots + c)}{dz}} g(z), \quad z = z_x, n = 0, b \neq 0, a \in \mathbb{N}_+$$

$0, a \in \mathbb{N}_+$

Where h denotes the number of singularities in the contour ε .

4. Discussion

With the introduction of the residue theorem, the three cases of contour integral problems can be computed in a simpler way by using the neat formulae suggested in the results, offering students, especially those who major in math, applied math, physics and engineering, an easier way to solve the problems than the traditional methods. For example, consider the case when $f(z) =$

$\frac{az^3+bz^2+cz+d}{z^4-1}$ where $a = 10$, $b = -2i - 2$, $c = -2$, $d = 2i - 2$, and contour $\gamma(t) = 2e^{it}$ ($0 \leq t \leq 2\pi$). To calculate the contour integral of $f(z)$ with the traditional method, one may struggle with extensive matrix manipulation, including using Fourier matrix transformation methods (which includes calculating the conjugates of many complex numbers [9]) or the Cayley-Hamilton theorem (a trivial method to find the inverse matrix compared with Fourier matrix transformation), to solve the system of equations to find the coefficients of $\frac{1}{z-1}$, $\frac{1}{z-i}$, $\frac{1}{z+1}$, $\frac{1}{z+i}$, or struggle with Taylor expansion to find the coefficients. But with the formulae $\int_C \frac{a_1 z^{n-1} + a_2 z^{n-2} + \dots + a_n}{bz^n + b_1 z^{n-1} + b_2 z^{n-2} + \dots + b_n} dz = 2\pi i \frac{a_1}{b}$, which is deduced in method 1, the contour integral of $f(z)$ can be simply calculated as $\int_\gamma f(z) dz = 2\pi i \frac{a_1}{b} = 2\pi i \frac{10}{1} = 20\pi i$.

Some of the calculations of the contour integrals in the previous sections can still use the basic formula $\int_\gamma f(z) dz = \int_a^b f(\gamma(t))\gamma'(t) dt$ [10], but through the formulae offered in this paper, the problems can be solved in a novel and easy way, and in some special cases, the contour integrals can even be calculated without the use of draft paper.

5. Conclusions

With the formulae deduced, once a function satisfies the requirements of these formulae, one can compute the contour integral rather quickly, resulting in efficiency in calculation.

However, while stating the advantages, it should be conceded that these methods also have some defaults: for the first equation (14), $\int_C \frac{a_1 z^{n-1} + a_2 z^{n-2} + \dots + a_n}{bz^n + b_1 z^{n-1} + b_2 z^{n-2} + \dots + b_n} dz = 2\pi i \frac{a_1}{b}$, the contour must encompass all the singularities of $f(z)$, otherwise, the formula will not work; for the second group of formulae to compute $\int_\gamma \frac{g(z)}{bz^a + n} dz$ and $\int_\gamma \frac{g(z)}{bz^a + b_1 z^{a-1} + b_2 z^{a-2} + \dots + c} dz$, $g(z)$ should always be holomorphic at any point inside the contour γ , while there are still many other cases where these conditions cannot be guaranteed. Further studies can investigate calculation methods for other general forms of contour integrals with the integration of more theorems and lemmas, and the studies can also look for simpler ways to compute and prove the deduced formulae with fewer requirements.

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