

Parametric stiffness modeling of a variable cross-section link for a collaborative robot

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Abstract. This paper describes how a parametric stiffness model of the variable cross-section link of a collaborative robot is obtained. The variable cross-section link module is divided into three parts: the front connection part, the rear connection part, and the main part. The static condensation method is used to model the stiffness of the connecting parts. The parametric stiffness modeling of the main part is carried out using the structural mechanics method. The stiffness models of the three parts are integrated into the parametric stiffness model of the entire link. The parametric stiffness model of the variable cross-section link is related to the structural dimensions of the link. The elastic deformation errors in all directions obtained from the stiffness model are within 5%. The method establishes the mapping relationship between the link's structural dimensions and its stiffness model. Based on this, the structural optimization of a collaborative robot based on parametric stiffness model is made possible.

Keywords: Collaborative robot, Variable cross-section link, Parametric stiffness model.

1. Introduction

A parametric stiffness model can establish the mapping relationship between a link's structural dimensions and its stiffness model, and then between the link's structural dimensions and the robot's overall stiffness model. As a result, this type of model makes possible the structural optimization of a collaborative robot based on parametric stiffness model in the future. Since a collaborative robot mainly consists of a series of mechatronic joints and links and the stiffness of a mechatronic joint mainly depends on its harmonic drive unit [1], this paper is focused on the parameterized stiffness modeling method for link modules rather than joint modules. To solve the above problems, in this research, a parametric stiffness model of the variable cross-section link of the collaborative robot is obtained by using the structural mechanics method [2-4]. The parametric stiffness model can be fully expressed by a parametric stiffness matrix $\mathbf{K}$ [5].

The variable cross-section link is divided into three parts. One part is the main part of the link, and the other two parts are the connecting parts at the two ends of the link. The main part of the link can be regarded as a spatial beam model. Its parametric stiffness matrix can be directly obtained by the structural mechanics method. At the same time, the necessary integral operation is retained when the bending stiffness is modified, ensuring accuracy with the premise of saving the calculation time. In the parametric stiffness matrix of this part, all the non-zero elements are directly related to the structural dimensions of the link's main part. The connecting parts at the two ends of the link are modeled with static condensation. The stiffness matrix of these parts can be directly obtained with FEA software. Finally, the stiffness matrices of the three parts are integrated into one coordinate system to obtain the parametric stiffness matrix of the entire link module. The proposed model's accuracy is verified with commercial FEA software, and its error is quantitatively demonstrated as well.

2. The analyzed link model

The analyzed link in this paper comes from a 6-DOF collaborative robot shown in Fig. 1. Link I of the collaborative robot is a constant cross-section link. Link II and link III are variable cross-section links. The structures between joint 4, joint 5, and joint 6 are compact. Compared with other links, the

stiffnesses of link I, link II, and link III significantly affect the stiffness of the collaborative robot. However, the modeling method of an equal cross-section link has matured, so this paper does not discuss it. Therefore, variable cross-section link II and link III are the main research objects.

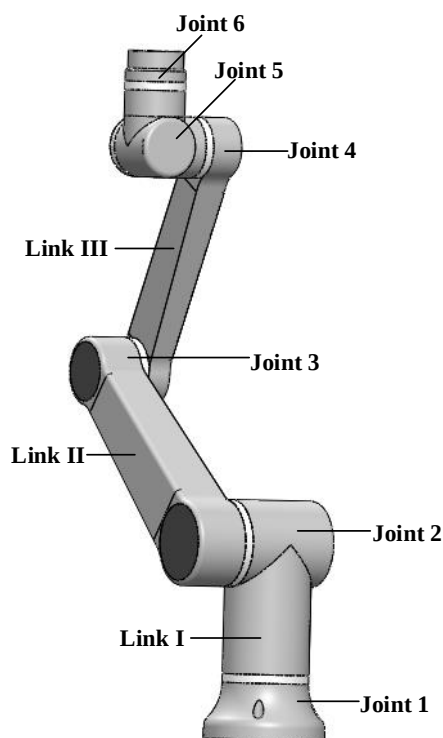


Figure 1. 6-DOF collaborative robot model

The variable cross-section link II and link III of the collaborative robot are shown in Fig. 2 below, divided into three parts. Substructure 1 and substructure 3 are the connecting parts at two ends of the link and are called the front connection part and the rear connection part, respectively. Substructure 2 is the main part of the link. The two links have the same structure and cross-section shape except for their different dimensions. The front connection part and the rear connection part are used to store some joint components of the robot, and substructure 2 is the main part of the link. The stiffness matrices of the front connection part and the rear connection part are extracted with the FEA method, and the stiffness matrices of the main part are calculated by the structural mechanics method. Then the stiffness matrices of the three parts are transformed into the coordinate system of the entire link for integration. Finally, the parametric stiffness matrix of the full link can be obtained.

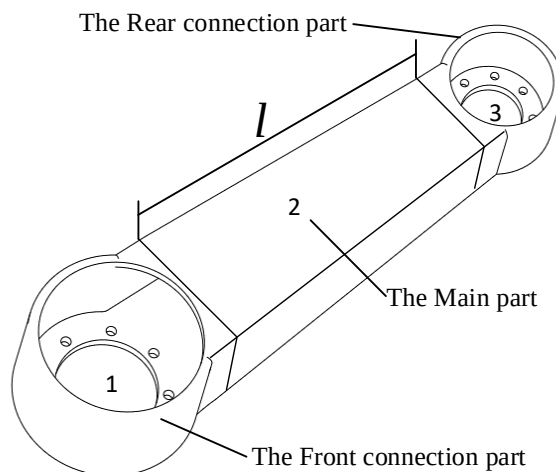


Figure 2. Variable cross-section link of the collaborative robot

3. Stiffness model

3.1. Stiffness matrix of the main part

Using the structural mechanics method, we regard substructure 2 of the link as a spatial beam model based on a Euler-Bernoulli beam. As shown in Fig. 3, the beam element has two nodes (node 1 and node 2). We use the right-hand rule to establish its local coordinate system, for which the x -axis coincides with the beam element axis, and the y -axis and z -axis are the main inertial axis directions of the beam element's cross-section.

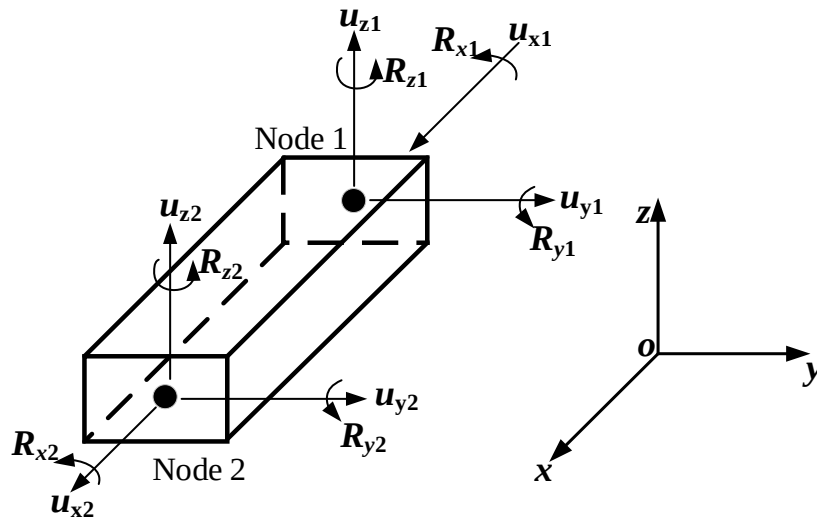


Figure 3. Spatial beam element and its coordinate system

The beam element is in the combined force state of axial tension and compression, plane bending, and torsion. Each element node has six degrees of freedom and bears the spatial six-dimensional node forces. Before proceeding to the next step, a coordinated transformation needs to be carried out. Instead of the absolute position x ($0 \leq x \leq l$), we use the relative position s ($-1 \leq s \leq 1$), which can be expressed as:

$$s = \frac{2x - l}{l} \quad (1)$$

Where x is the absolute coordinate of the element displacement field along the x -axis of the link, s is the relative coordinate of the element displacement field along the x -axis of the link, and l is the link's length. In this way, solving integral operations will be more convenient in the subsequent calculation process.

According to the force characteristics of the beam element, the force of the element is divided into three modes: tension-compression, torsion, and bending. These three modes can include the six-dimensional force borne by the beam element. According to the mechanical theory [6, 7], the stiffness matrix of the spatial beam model can be directly obtained.

$$\mathbf{K}_{(6 \times 6)} = \begin{bmatrix} \frac{EA}{l} & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{12EI_z}{l^3} & 0 & 0 & 0 & -\frac{6EI_z}{l^2} \\ 0 & 0 & \frac{12EI_y}{l^3} & 0 & \frac{6EI_y}{l^2} & 0 \\ 0 & 0 & 0 & \frac{GJ}{l} & 0 & 0 \\ 0 & 0 & \frac{6EI_y}{l^2} & 0 & \frac{4EI_y}{l} & 0 \\ 0 & -\frac{6EI_z}{l^2} & 0 & 0 & 0 & \frac{4EI_z}{l} \end{bmatrix} \quad (2)$$

Where A is the cross-section area, J is the torsional moment of inertia, I is the moment of inertia of the cross-section, E is Young's modulus, G is the shear modulus, and μ is Poisson's ratio.

It should be noted that $I_{z(y)}$ is the cross-section moment of inertia of the xoy plane or the xoz plane. The elements of the stiffness matrices in the bending force mode of the xoy plane and xoz plane are different only in the cross-section moment of inertia, so this paper does not list them repeatedly.

3.2. Correction of the stiffness matrix

Generally, the shear deformation of a beam caused by bending has little effect on the deflection of the beam and can be ignored [4]. However, when the beam height is more significant than 1/5 of the beam length, the influence of the shear deformation on the deflection cannot be ignored. When the beam section is thin-walled, the influence of the shear deformation is greater than that of other shapes [8]. Therefore, after we obtain the element stiffness matrix (see Eq. (2)), we add the shear deformation caused by bending to the stiffness matrix by adding a correction coefficient. The correction method is to add a force mode and pure shear force mode to the original three force modes. The parameters required to derive the matrix are shown in Table 1.

Table 1. Derivation of the stiffness matrix of the pure shear force mode

Force mode	Pure shear	
Shape function and its derivative	$\mathbf{N} = \left[\frac{1}{2}(1-s), \frac{1}{2}(1+s) \right]$ $\mathbf{N}' = \left[-\frac{1}{2}, \frac{1}{2} \right]$	
Node displacement vector	$\boldsymbol{\delta} = [v_{s1}, v_{s2}]^T$	
Strain transformation vector $\mathbf{B}$	$\mathbf{B} = \frac{d\mathbf{N}}{dx} = \frac{2}{l} \mathbf{N}'$	
Elastic matrix $\mathbf{D}$	$\mathbf{D} = \frac{GA}{k}$ $(k = \frac{A}{I} \int_A (\frac{s}{b})^2 dA)$	$\mathbf{D} = G \frac{A(s)}{k_i(s)} \quad (i = y, z)$ $(k_i(s) \text{ in section 3.6})$
Stiffness matrix calculation $\mathbf{k} = \mathbf{D} \iiint \mathbf{B}^T \mathbf{B} dV$	$\frac{GA}{kl} \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$	$\frac{2G}{l} \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \int_{-1}^1 \frac{A(s)}{k_i(s)} ds$
	Others' methods [6, 9]	Our method

Because the pure shear and bending modes are coupled, the stiffness matrices of the two modes can be linearly superimposed. In general, the modified element stiffness matrix for the bending mode can be represented by:

$$\begin{pmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{pmatrix} \quad (3)$$

$$\text{Where } k_{11} = \frac{\frac{12EI_Z(y)}{l^3} \times \frac{G}{2l} \int_{-1}^1 \frac{A(s)}{k_Y(Z)(s)} ds}{\frac{12EI_Z(y)}{l^3} + \frac{G}{2l} \int_{-1}^1 \frac{A(s)}{k_Y(Z)(s)} ds}, k_{12} = k_{21} = \frac{\frac{6EI_Z(y)}{l^2} \times \frac{G}{2l} \int_{-1}^1 \frac{A(s)}{k_Z(y)(s)} ds}{\frac{2}{l} \times \frac{6EI_Z(y)}{l^2} + \frac{G}{2l} \int_{-1}^1 \frac{A(s)}{k_Z(y)(s)} ds},$$

$$k_{22} = \frac{\frac{4EI_Z(y)}{l} \times \frac{G}{2l} \int_{-1}^1 \frac{A(s)}{k_Z(y)(s)} ds}{\frac{3}{l^2} \times \frac{4EI_Z(y)}{l} + \frac{G}{2l} \int_{-1}^1 \frac{A(s)}{k_Z(y)(s)} ds} + 3 \frac{(\frac{4EI_Z(y)}{l})^2}{4l^2 (\frac{3}{l^2} \times \frac{4EI_Z(y)}{l} + \frac{G}{2l} \int_{-1}^1 \frac{A(s)}{k_Z(y)(s)} ds)}.$$

Because the dimension parameters A , J , and I are functions of the independent variable s , it is more complex to solve the integral equations. An equivalent method [10] is used to deal with the variables A , J , and I to simplify the calculation. The equivalent variables can be moved directly outside the integral symbol as “fixed values” instead of variables. In this study, the modified element stiffness matrix for the bending force mode can be represented by:

$$\frac{EI_i}{l^3(1+\Phi_j)} \begin{pmatrix} 12 & -6l \\ -6l & (4+\Phi_j)l^2 \end{pmatrix} \quad (4)$$

$(i = z, j = y \text{ or } i = y, j = z)$

$$\text{Where } \Phi = \frac{12EI_k}{GA l^2}.$$

The difference between this study and other studies [11] is that this study does not use the equivalent values of structural parameters for the pure shear force mode but rather regards them as variables. Although the equivalent method reduces the amount of calculation, it causes a loss of accuracy. Therefore, our method ensures accuracy by reducing the amount of calculation.

The original bending stiffness matrix is replaced with the modified bending stiffness matrix for integration. Then the modified stiffness matrix of the variable cross-section link can be described as:

$$\mathbf{K}_2 = \begin{bmatrix} k_{11} & 0 & 0 & 0 & 0 & 0 \\ 0 & k_{22} & 0 & 0 & 0 & k_{26} \\ 0 & 0 & k_{33} & 0 & k_{35} & 0 \\ 0 & 0 & 0 & k_{44} & 0 & 0 \\ 0 & 0 & k_{53} & 0 & k_{55} & 0 \\ 0 & k_{62} & 0 & 0 & 0 & k_{66} \end{bmatrix} \quad (5)$$

The elements in the matrix are as follows:

$$k_{11} = \frac{EA}{l}, k_{22} = \frac{\frac{12EI_Z}{l^3} \times \frac{G}{2l} \int_{-1}^1 \frac{A(s)}{k_Y(s)} ds}{\frac{12EI_Z}{l^3} + \frac{G}{2l} \int_{-1}^1 \frac{A(s)}{k_Y(s)} ds}, k_{33} = \frac{\frac{12EI_Y}{l^3} \times \frac{G}{2l} \int_{-1}^1 \frac{A(s)}{k_Z(s)} ds}{\frac{12EI_Y}{l^3} + \frac{G}{2l} \int_{-1}^1 \frac{A(s)}{k_Z(s)} ds}, k_{44} = \frac{GJ}{l},$$

$$k_{55} = \frac{\frac{4EI_Y}{l} \times \frac{G}{2l} \int_{-1}^1 \frac{A(s)}{k_Y(s)} ds}{\frac{3}{l^2} \times \frac{4EI_Y}{l} + \frac{G}{2l} \int_{-1}^1 \frac{A(s)}{k_Y(s)} ds} + 3 \frac{(\frac{4EI_Y}{l})^2}{4l^2 (\frac{3}{l^2} \times \frac{4EI_Y}{l} + \frac{G}{2l} \int_{-1}^1 \frac{A(s)}{k_Y(s)} ds)},$$

$$k_{66} = \frac{\frac{4EI_Z}{l} \times \frac{G}{2l} \int_{-1}^1 \frac{A(s)}{k_Z(s)} ds}{\frac{3}{l^2} \times \frac{4EI_Z}{l} + \frac{G}{2l} \int_{-1}^1 \frac{A(s)}{k_Z(s)} ds} + 3 \frac{(\frac{4EI_Z}{l})^2}{4l^2 (\frac{3}{l^2} \times \frac{4EI_Z}{l} + \frac{G}{2l} \int_{-1}^1 \frac{A(s)}{k_Z(s)} ds)},$$

$$k_{26} = \frac{\frac{6EI_z}{l^2} \times \frac{G}{2l} \int_{-1}^1 \frac{A(s)}{k_z(s)} ds}{\frac{2}{l} \times \frac{6EI_z}{l^2} + \frac{G}{2l} \int_{-1}^1 \frac{A(s)}{k_z(s)} ds}, k_{35} = \frac{\frac{6EI_y}{l^2} \times \frac{G}{2l} \int_{-1}^1 \frac{A(s)}{k_y(s)} ds}{\frac{2}{l} \times \frac{6EI_y}{l^2} + \frac{G}{2l} \int_{-1}^1 \frac{A(s)}{k_y(s)} ds}, k_{53} = -k_{35}, k_{62} = -k_{26}.$$

After getting the element stiffness matrix, it is necessary to identify the geometric parameters of the variable cross-section link. Fig. 4 shows the specific dimensions of the main part of the variable cross-section links II and III.

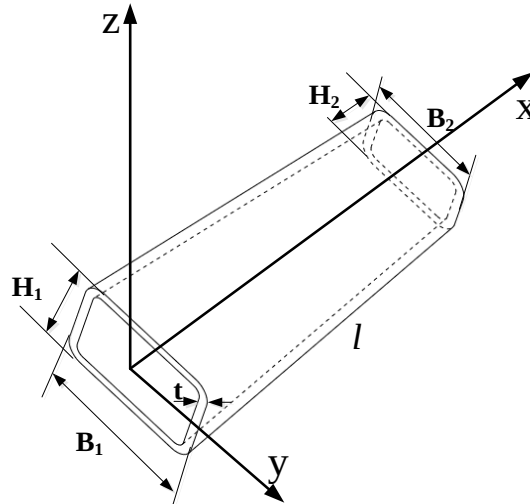


Figure 4. Specific dimensions of the main part

The expression related to the section parameters according to the actual shape of the link can be obtained by the equivalent method [6]. Before calculation, we simplify the link by removing the fillets of its outer and inner walls. After the values of $k_i(s)$, A , J and I are obtained, they are substituted into Eq. (5) to obtain the stiffness matrix $\mathbf{K}_2$ of the final required variable cross-section link.

3.3. Stiffness model of the link

The structures of the connecting parts are shown in Fig. 5. Using the static condensation technique, these parts are equivalent to a super-element with two nodes. Each substructure has two nodes, node 1 and node 2. Node 1 is fixed and the stiffness matrix of node 2 is extracted in FEA software. The extracted stiffness matrices represent the stiffness model of the connecting parts in the current coordinate system.

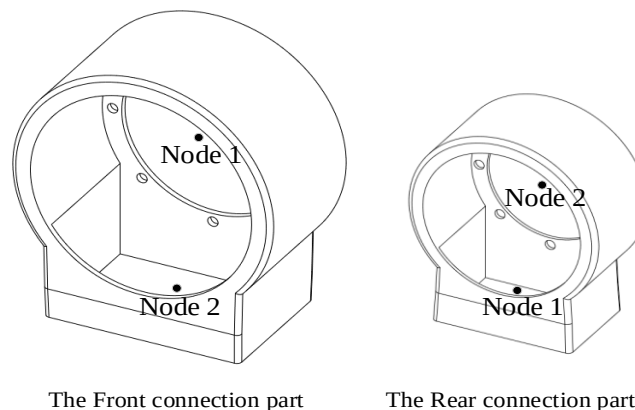


Figure 5. The structures of the front connection part and the rear connection part

After the stiffness matrices $\mathbf{K}_1$ and $\mathbf{K}_3$ in the local coordinate system of the link's connecting part are obtained, they need to undergo coordinate transformation. The relative position relations of the link's three substructures are shown in Fig. 6. According to the position relationships between them, the stiffness matrix of the three substructures in the local coordinate system is transformed.

Finally, the stiffness matrixes of the three substructures in the link coordinate system can be obtained as $\bar{\mathbf{K}}_1$, $\bar{\mathbf{K}}_2$, and $\bar{\mathbf{K}}_3$.

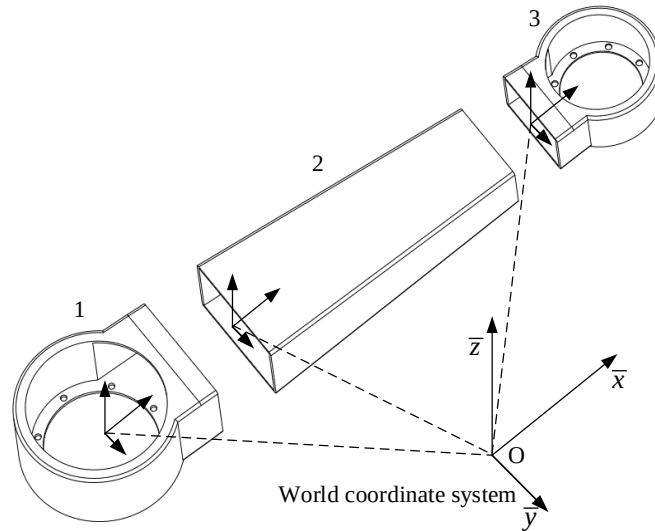


Figure 6. Relative position between the substructures and the entire link

With the linear superposition of the three stiffness matrices in the same coordinate system, the stiffness matrix of the entire variable cross-section link can be obtained from:

$$\bar{\mathbf{K}} = \bar{\mathbf{K}}_1 + \bar{\mathbf{K}}_2 + \bar{\mathbf{K}}_3 \quad (6)$$

The matrix $\bar{\mathbf{K}}$ in Eq. (6) is the stiffness matrix of the variable cross-section link module in the link coordinate system. The stiffness matrix of link II can be obtained by taking the dimensions of link II in the matrix, and the stiffness matrix of link III can be obtained by taking the dimensions of link III in the matrix.

4. Simulation

After obtaining the stiffness matrix of the link, it is necessary to design simulations to verify the accuracy of the stiffness matrix. The node can be fixed at one end of the link and a six-dimensional force can be applied to the node at the other end. According to Eq. (7), the node with an applied six-dimensional force will produce six-dimensional elastic deformation. By comparing the deformations of this method with those of the FEA method, the accuracy of the stiffness matrix obtained in this study can be compared. As before, the fixed node is called node 1, and the node applying force is called node 2. Then the following expression is obtained:

$$\underset{(6 \times 1)}{\bar{\boldsymbol{\delta}}} = \underset{(6 \times 6)}{\bar{\mathbf{K}}}^{-1} \cdot \underset{(6 \times 1)}{\bar{\mathbf{P}}} \quad (7)$$

Where $\bar{\mathbf{P}}$ is the external force vector acting on node 2 of the entire link, $\bar{\boldsymbol{\delta}}$ is the six-dimensional elastic deformation vector of node 2 ($\bar{\boldsymbol{\delta}}$ is also called the node displacement), and $\bar{\mathbf{P}} = [\bar{P}_u \ \bar{P}_v \ \bar{P}_w \ \bar{M}_x \ \bar{M}_y \ \bar{M}_z]^T = [60 \ 60 \ 60 \ 30 \ 30 \ 30]^T$, $\bar{\boldsymbol{\delta}} = [\bar{u}_x \ \bar{u}_y \ \bar{u}_z \ \bar{R}_x \ \bar{R}_y \ \bar{R}_z]^T$.

The material data for link II and link III are given in Table 2. The known sectional dimensions of link II and link III are shown in Table 3. The data in Table 2 and Table 3 are substituted into the stiffness matrix $\mathbf{K}_2$ to obtain the stiffness matrices of the link's main parts. Then the FEA software is used to extract the stiffness matrices of the connecting parts at two ends of the link. According to the procedure discussed in Section 3, the stiffness matrix of the entire link in the world coordinate system can be obtained. Then the six-dimensional elastic deformation can be obtained. Using the

same simulation settings, the deformations can be calculated by the FEA method, which are compared with the results obtained with our method. The results are shown in Table 4.

Table 2. Material properties

$\rho (kg / m^3)$	$E (GPa)$	$G (GPa)$	μ
2.81E-3	71	26.69	0.33

Table 3. Material properties

	H1(m)	B1(m)	H2(m)	B2(m)	t (m)	l (m)
Link II	0.085	0.120	0.062	0.088	0.003	0.33
Link III	0.072	0.098	0.056	0.076	0.003	0.30

Table 4. Simulation results

		$\bar{u}_x(mm)$	$\bar{u}_y(mm)$	$\bar{u}_z(mm)$	$\bar{R}_x(^{\circ})$	$\bar{R}_y(^{\circ})$	$\bar{R}_z(^{\circ})$
Link II	Our method	1.2435E-3	4.9752E-2	8.1831E-2	1.0168E-1	8.7605E-2	4.611E-2
	FEA Method	1.2382E-3	4.7875E-2	8.5517E-2	1.0994E-1	8.7772E-2	4.8085E-2
	Relative error	0.43%	3.92%	-4.31%	0.74%	-0.19%	-2.65%
Link III	Our method	1.4387E-3	7.4960E-2	1.1930E-1	1.6589E-1	1.2533E-1	7.3526E-2
	FEA Method	1.4345E-3	7.1670E-2	1.2092E-1	1.6599E-1	1.2694E-1	7.4555E-2
	Relative error	0.29%	4.59%	-1.34%	-0.06%	-1.27%	-1.38%

According to the above simulation results, the proposed method is satisfactory. The deformation errors in all directions are within 5% compared with those measured with the FEA method.

5. Summary

This paper proposes a parametric stiffness model of the link for a variable cross-section collaborative robot. The link is divided into three parts: the connection and the main parts. The parametric stiffness model of the main part is established using the structural mechanics method. The static condensation method is used to establish the stiffness model of the connection part. The stiffness model of the entire link is formed by integrating the stiffness model of the three parts. The parametric stiffness model of the main part of the link is directly related to the structural dimensions of the link, which is convenient for the dimension optimization of the robot. In establishing the parametric stiffness model of the main part, the existing methods are modified to make the stiffness model more accurate. The simulation results show that the relative error between the deformation calculated with our stiffness model and the deformation calculated with the FEA software is within 5%. In the structural design and optimization stage, the parametric stiffness model of the variable cross-section link obtained in this research can be applied instead of the FEA model.

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